Introducing Unitary Circles

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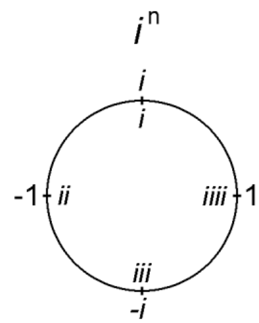
Abstract

Unitary circles model the property of spin, and are based on imaginary numbers.



The imaginary constant *i* is not a real number, and it has not been apparent how it relates to real numbers. With unitary circles, the role of imaginary numbers can be clearly understood. And the definition of *i* changes.

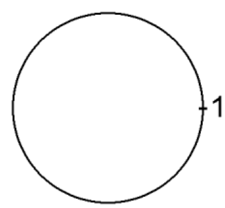
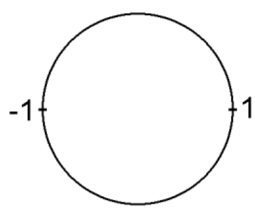
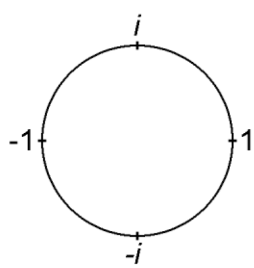
Background

Imaginary numbers do not appear on the real number line, so they have been difficult to conceptualize. *i* is currently defined as the square root of -1, but it is not clear how a number squared can yield a negative number.

Arvand diargrams illustrate how imaginary numbers can be viewed in two-dimensional space (one dimension representing real numbers and the other representing imaginary numbers). However, imaginary numbers have no relevance to two-dimensional space. Imaginary numbers pertain to circles, and to the property of spin.

Unitary Circles

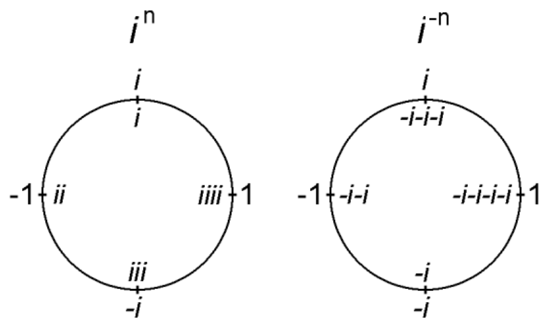
Unitary circles model the property of spin. They spin around in one, two, or four multiplicative steps. What makes them all unitary circles, is that their step unit raised to the number of steps, equals one.

Number of Steps	Unitary Circle	Step Unit	Unitary Formula
1		1	$1^1 = 1$
2		-1	$-1^2 = 1$
4		*i*	$i^4 = 1$

The last circle, with four steps of *i*, is called the imaginary unit circle.

Imaginary Unit Circle

The imaginary unit circle is a graphical representation of the expressions i^n and i^{-n}..

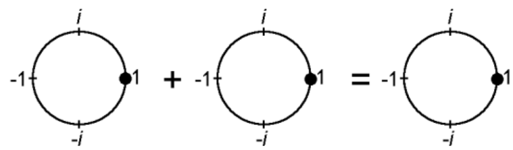
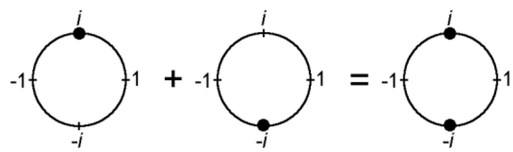
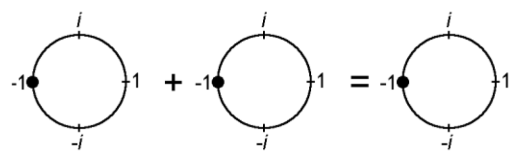
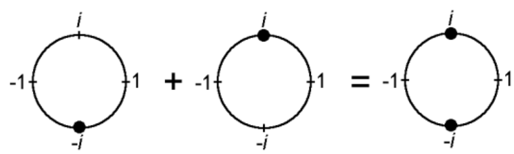
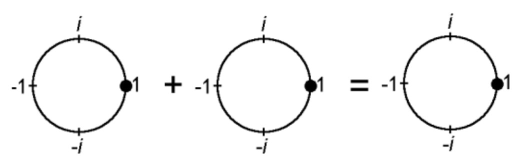
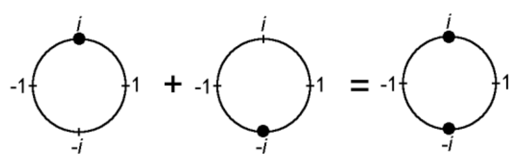


As n grows larger, i^n steps around the circle in a counter-clockwise direction. Similarly, the expression i^{-n} steps around the circle in a clockwise direction.

The expressions i^n and i^{-n} generate both real and complex terms. It is, therefore, problematic to use these expressions in formulas, because real world problems require only real world answers.

However, i^n can be “paired” with i^{-n}, so that the complex terms appear with their conjugate and cancel out. “Pairing” is mathematically equivalent to addition, and i^n and i^{-n} are bound together and treated as one mathematical entity.

i^n Paired With i^{-n}

n	i^n + i^{-n} = Pair		Value
0		$= 1 + 1$	2
1		$= i + -i$	0
2		$= -1 + -1$	-2
3		$= -i + i$	0
4		$= 1 + 1$	2
5		$= i + -i$	0
⋮	⋮		⋮

Unitary Circles and Pi

Unitary circles appear to have a natural connection with the constant pi. The following pi formula is based on the imaginary unit circle, and the pairing of i^n and i^{-n}.

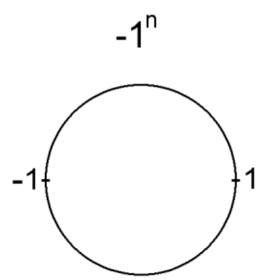
$$\sum_{n=0}^{\infty} \frac{i^{n}+i^{-n}}{n+1} = \frac{\pi}{2}$$

Another formula based on a unitary circle is the Leibniz pi formula.

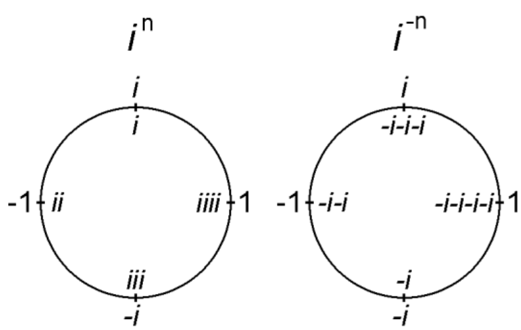
$$\sum_{n=0}^{\infty} \frac{-1^{n}}{2n+1} = \frac{\pi}{4}$$

These formulas are represented graphically with unitary circles. -1^n spins counter-clockwise 180^o for each step of n. Similarly, i^n and i^{-n} spin 90^o counter-clockwise and clockwise respectively.

Unitary Circle **Pi Formula**



$$\sum_{n=0}^{\infty} \frac{-1^{n}}{2n+1} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \cdots = \frac{\pi}{4}$$



$$\sum_{n=0}^{\infty} \frac{i^{n}+i^{-n}}{n+1} = \frac{2}{1} - \frac{2}{3} + \frac{2}{5} - \frac{2}{7} \cdots = \frac{\pi}{2}$$

A Symmetric Pi Formula

The imaginary pi formula (just shown) can be rewritten in an equivalent form, so that it starts at 1 instead of 0. The formula must be adjusted so that each occurrence of n is lowered by 1, and each occurrence of –n is raised by 1.

$$\sum_{n=0}^{\infty} \frac{i^{n}+i^{-n}}{n+1} = \sum_{n=1}^{\infty} \frac{i^{n-1}+i^{-n+1}}{n}$$

The new formula, with n starting at 1, is symmetric. That is, running the formula backwards from -1 to –infinity also yields pi divided by 2.

$$\sum_{n=1}^{\infty} \frac{i^{n-1}+i^{-n+1}}{n} = \sum_{n=-1}^{-\infty} \frac{i^{n-1}+i^{-n+1}}{n} = \frac{\pi}{2}$$

Summary

Unitary circles model the property of spin, and have the characteristic that the step unit, raised to the number of steps, equals 1.

$$\text{step unit}^{\text{number of steps}} = 1$$

The imaginary unit circle is a unitary circle, with *i* as the step unit, and the number of steps is four.

$$i^4 = 1$$

The new definition of *i*:

> The step unit in the imaginary unit circle; equal in value to the fourth root of 1.

The elegant connections between unitary circles and pi formulas suggest that unitary circles play an important role in the mathematical world.