When $\pi(n)$ does not divide $n$

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Abstract

Let $\pi(n)$ denote the prime-counting function and let

$$f(n) = \left| \left\lfloor \log n - \lfloor \log n \rfloor - 0.1 \right\rfloor \right| \frac{\lfloor n/\lfloor \log n - 1 \rfloor \rfloor \lfloor \log n - 1 \rfloor}{n}.$$

In this paper we prove that if $n$ is an integer $\geq 60184$ and $f(n) = 0$, then $\pi(n)$ does not divide $n$. We also show that if $n \geq 60184$ and $\pi(n)$ divides $n$, then $f(n) = 1$. In addition, we prove that if $n \geq 60184$ and $n/\pi(n)$ is an integer, then $n$ is a multiple of $\lfloor \log n - 1 \rfloor$ located in the interval $[e^{\lfloor \log n - 1 \rfloor + 1}, e^{\lfloor \log n - 1 \rfloor + 1.1}]$. This allows us to show that if $c$ is any fixed integer $\geq 12$, then in the interval $[e^c, e^{c+0.1}]$ there is always an integer $n$ such that $\pi(n)$ divides $n$.

Let $S$ denote the sequence of integers generated by the function

$$d(n) = n/\pi(n)$$

where $n \in \mathbb{Z}$ and $n > 1$ and let $S_k$ denote the $k$th term of sequence $S$. Here we ask the question whether there are infinitely many positive integers $k$ such that $S_k = S_{k+1}$.

Keywords: bounds on the prime-counting function, explicit formulas for the prime-counting function, intervals, prime numbers, sequences

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0 Notation

Throughout this paper the number $n$ is always a positive integer. Moreover, we use the following notation:

- $|\cdot|$ (absolute value)
- $\lceil \cdot \rceil$ (ceiling function)
- $|$ (divides)
- $\nmid$ (does not divide)
- $\lfloor \cdot \rfloor$ (floor function)
- $\text{frac}(\cdot)$ (fractional part)
- $\log$ (natural logarithm)

1 Introduction

Determining how prime numbers are distributed among natural numbers is one of the most difficult mathematical problems. This explains why the prime-counting function $\pi(n)$ (which counts the number of primes less than or equal to a given number $n$) has been one of the main objects of study in Mathematics for centuries.

In [2] Gaitanas obtains an explicit formula for $\pi(n)$ that holds infinitely often. His proof is based on the fact that the function $d(n) = n/\pi(n)$ takes on every integer value greater than 1 (as proved by Golomb [3]) and on the fact that $x/(\log x - 0.5) < \pi(x) < x/(\log x - 1.5)$ for $x \geq 67$ (as shown by Rosser and Schoenfeld [4]). In this paper we find alternative expressions that are valid for infinitely many positive integers $n$, and we also prove, among other results, that if $n \geq 60184$ and

$$\left|\left[\log n - \left[\log n\right] - 0.1\right]\right| \left[\frac{n/\left[\log n - 1\right]}{\left[\log n - 1\right]}\right]$$

equals 0, then $\pi(n)$ does not divide $n$.

We will place emphasis on the following three theorems, which were proved by Golomb, Dusart, and Gaitanas respectively:
Theorem 1.1 [3]. The function $d(n) = n/\pi(n)$ takes on every integer value greater than 1.

Theorem 1.2 [1]. If $n$ is an integer $\geq 60184$, then

$$\frac{n}{\log n - 1} < \pi(n) < \frac{n}{\log n - 1.1}.$$

Remark 1.3. Dusart’s paper states that for $x \geq 60184$ we have $x/(\log x - 1) \leq \pi(x) \leq x/(\log x - 1.1)$, but since $\log n$ is always irrational when $n$ is an integer $> 1$, we can state his theorem the way we did.

Theorem 1.4 [2]. The formula

$$\pi(n) = \left\lfloor \frac{n}{\log n - 0.5} \right\rfloor$$

is valid for infinitely many positive integers $n$.

2 Main results

We are now ready to prove our main results:

Theorem 2.1. The formula

$$\pi(n) = \left\lfloor \frac{n}{\log n - 1} \right\rfloor$$

holds for infinitely many positive integers $n$.

Proof. According to Theorem 1.2, for $n \geq 60184$ we have

$$\frac{n}{\log n - 1} < \pi(n) < \frac{n}{\log n - 1.1} \Rightarrow \frac{\log n - 1.1}{n} < \frac{1}{\pi(n)} < \frac{\log n - 1}{n}.$$

If we multiply by $n$, we get

$$\log n - 1.1 < \frac{n}{\pi(n)} < \log n - 1. \tag{1}$$

Since $\log n - 1.1$ and $\log n - 1$ are both irrational (for $n > 1$), inequality (1) implies that when $n/\pi(n)$ is an integer we must have

$$\frac{n}{\pi(n)} = \lfloor \log n - 1 \rfloor = \lfloor \log n - 1.1 \rfloor + 1 = \lfloor \log n - 1.1 \rfloor = \lfloor \log n - 1 \rfloor - 1. \tag{2}$$
Taking Theorem 1.2 and equality (2) into account, we can say that for every 
\( n \geq 60184 \) when \( \frac{n}{\pi(n)} \) is an integer we must have

\[
\frac{n}{\pi(n)} = [\log n - 1] \Rightarrow \pi(n) = \frac{n}{[\log n - 1]}
\]

Since Theorem 1.1 implies that \( \frac{n}{\pi(n)} \) is an integer infinitely often, it follows that there are infinitely many positive integers \( n \) such that \( \pi(n) = n/[\log n - 1] \). ■

In fact, the following theorem follows from Theorems 1.1, from Gaitana’s proof of Theorem 1.4, and from the proof of Theorem 2.1:

**Theorem 2.2.** For every \( n \geq 60184 \) when \( \frac{n}{\pi(n)} \) is an integer we must have

\[
\frac{n}{\pi(n)} = [\log n - 1.5] = [\log n - 0.5] = [\log n - 1] = [\log n - 1.1] + 1 = [\log n - 1] - 1.
\]

In other words, for \( n \geq 60184 \) when \( \frac{n}{\pi(n)} \) is an integer we have

\[
\frac{n}{\pi(n)} = [\log n - 1.5] = [\log n - 0.5] = [\log n - 1] = [\log n - 1.1] + 1 = [\log n - 1] - 1.
\]

**Theorem 2.3.** Let \( n \) be an integer \( \geq 60184 \). If \( \text{frac}(\log n) = \log n - [\log n] > 0.1 \), then \( \pi(n) \nmid n \) (that is to say, \( n/\pi(n) \) is not an integer). ■

**Proof.** According to Theorem 2.2, if \( n \geq 60184 \) and \( \frac{n}{\pi(n)} \) is an integer, then

\[
\frac{n}{\pi(n)} = [\log n - 1] = [\log n - 1.1].
\]

In other words, for \( n \geq 60184 \) when \( \frac{n}{\pi(n)} \) is an integer we have

\[
[\log n - 1] = [\log n - 1.1] \\
[\log n - 1] = [\log n - 1 - 0.1] \\
\text{frac}(\log n - 1) \leq 0.1 \\
\log n - 1 - [\log n - 1] \leq 0.1
\]
Suppose that $P$ is the statement ‘$n/\pi(n)$ is an integer’ and $Q$ is the statement ‘$\log n - \lfloor \log n \rfloor \leq 0.1$’. According to propositional logic, the fact that $P \rightarrow Q$ implies that $\neg Q \rightarrow \neg P$. $\blacksquare$

Similar theorems can be proved by using Theorem 2.2 and equality (3).

**Remark 2.4.** We can also say that if $n \geq 60184$ and $\pi(n)$ divides $n$, then $\pi(n) \nmid n$.

**Remark 2.5.** Because $\log n$ is irrational for $n > 1$, another way of stating Theorem 2.3 is by saying that if $n \geq 60184$ and the first digit to the right of the decimal point of $\log n$ is 1, 2, 3, 4, 5, 6, 7, 8, or 9, then $\pi(n) \nmid n$.

**Example:**

$\log_{10} 31 = 7.138...$

The first digit after the decimal point of $\log_{10} 31$ (in red) is 3. This implies that $\pi(10^{31})$ does not divide 1031. We can also say that if $n \geq 60184$ and $\pi(n)$ divides $n$, then the first digit after the decimal point of $\log n$ can only be 0.

Now, if $y$ is a positive noninteger, then the first digit after the decimal point of $y$ is equal to $\lfloor 10 \frac{y}{10} - 10 \lfloor y \rfloor \rfloor$. So, we can say that if $n \geq 60184$ and $\lfloor 10 \log n - 10 \lfloor \log n \rfloor \rfloor \neq 0$, then $\pi(n) \nmid n$. On the other hand, if $n \geq 60184$ and $\pi(n)$ divides $n$, then $\lfloor 10 \log n - 10 \lfloor \log n \rfloor \rfloor = 0$. $\blacksquare$

The following theorem follows from Theorem 2.3:

**Theorem 2.6.** Let $e$ be the base of the natural logarithm. If $a$ is any integer $\geq 11$ and $n$ is any integer contained in the interval $[e^{a+0.1}, e^{a+1}]$, then $\pi(n) \nmid n$. (The number $e^r$ is irrational when $r$ is a rational number $\neq 0$.) $\blacksquare$

**Example 2.7.** Take $a = 18$. If $n$ is any integer in the interval $[e^{18.1}, e^{19}]$, then $\pi(n) \nmid n$. $\blacksquare$

**Corollary 2.8.** If $a$ is any positive integer $> 1$, then $\pi([e^a]) \nmid [e^a]$. $\blacksquare$

**Proof.** For $a \geq 12$ the proof follows from Theorem 2.6. On the other hand, $[e^a]/\pi([e^a])$ is not an integer whenever $2 \leq a \leq 11$, as shown in the following table:
In other words, if \( a \in \mathbb{Z}^+ \), then \( \pi([e^a]) \mid [e^a] \) only when \( a = 1 \). \( \blacksquare \)

**Theorem 2.9.** Let \( n \) be an integer \( \geq 60184 \) and let

\[
f(n) = ||\log n - [\log n] - 0.1|| \left\lfloor \frac{n/|\log n - 1|}{|\log n - 1|} \right\rfloor.
\]

If \( f(n) = 0 \), then \( \pi(n) \nmid n \). On the other hand, if \( \pi(n) \mid n \), then \( f(n) = 1 \). \( \blacksquare \)

**Proof.**

- **Part 1**

Suppose that

\[ f(n) = g(n)h(n), \]

where

\[ g(n) = ||\log n - [\log n] - 0.1|| \]

and

\[ h(n) = \left\lfloor \frac{n/|\log n - 1|}{|\log n - 1|} \right\rfloor. \]

To begin with, if \( n \geq 60184 \), then \( \log n - [\log n] \) can never be equal to 0.1. Now, when \( \log n - [\log n] < 0.1 \) we have \(-1 < \log n - [\log n] - 0.1 < 0 \) and hence \( ||\log n - [\log n] - 0.1|| = 1 \). On the other hand, when \( \log n - [\log n] > 0.1 \) we have \( 0 < \log n - [\log n] - 0.1 < 1 \) and hence \( ||\log n - [\log n] - 0.1|| = 0 \). This means that if \( n \) is any integer \( \geq 60184 \), then...
then \( g(n) \) equals either 0 or 1. We can also say that if \( n \geq 60184 \) and \( g(n) = 0 \), then \( \log n - \lfloor \log n \rfloor > 0.1 \), which implies that \( \pi(n) \not| n \) (according to Theorem 2.3). (This means that if \( n \geq 60184 \) and \( \pi(n) \not| n \), then \( g(n) = 1 \).)

**Part 2**

If \( n \geq 60184 \), then

\[
\left\lfloor \frac{n}{\log n - 1} \right\rfloor \leq \frac{n}{\log n - 1},
\]

which means that

\[
\left\lfloor \frac{n}{\log n - 1} \right\rfloor / \frac{n}{\log n - 1} = \left\lfloor \frac{n}{\lfloor \log n - 1 \rfloor} \right\rfloor / \left\lfloor \log n - 1 \right\rfloor = h(n)
\]

equals either 0 or 1. If \( h(n) = 0 \), then \( n \) is not divisible by \( \lfloor \log n - 1 \rfloor \), which implies that \( \pi(n) \not| n \) (according to Theorem 2.2). In other words, if \( n \geq 60184 \) and \( h(n) = 0 \), then \( \pi(n) \not| n \). (This means that if \( n \geq 60184 \) and \( \pi(n) \not| n \), then \( h(n) = 1 \).)

**Part 3**

There are two possible outputs for \( g(n) \) (0 or 1) as well as two possible outputs for \( h(n) \) (0 or 1). This means that for \( n \geq 60184 \) we have either

\[
g(n)h(n) = 0 \cdot 0 = 0,
\]

or

\[
g(n)h(n) = 0 \cdot 1 = 0,
\]

or

\[
g(n)h(n) = 1 \cdot 0 = 0,
\]

or

\[
g(n)h(n) = 1 \cdot 1 = 1.
\]

If \( f(n) = g(n)h(n) = 0 \), then at least one of the factors \( g(n) \) and \( h(n) \) equals 0, which implies that \( \pi(n) \not| n \) (see Part 1 and Part 2). This means that if \( n \geq 60184 \) and \( f(n) = 0 \), then \( \pi(n) \not| n \). Consequently, if \( n \geq 60184 \) and \( \pi(n) \not| n \), then \( f(n) = 1 \).

**Theorem 2.10.** If \( n \geq 60184 \) and \( n/\pi(n) \) is an integer, then \( n \) is a multiple of \( \lfloor \log n - 1 \rfloor \) located in the interval \( [e^{\lfloor \log n - 1 \rfloor + 1}, e^{\lfloor \log n - 1 \rfloor + 1.1}] \).
Proof. According to Theorems 2.2 and 2.3, if \( n \geq 60184 \) and \( n/\pi(n) \) is an integer, then

\[
\frac{n}{\pi(n)} = \lfloor \log n - 1 \rfloor \Rightarrow n = \pi(n)\lfloor \log n - 1 \rfloor
\]

and

\[
\text{frac}(\log n) = \log n - \lfloor \log n \rfloor \leq 0.1.
\]

The fact that \( \text{frac}(\log n) \leq 0.1 \) implies that \( n \) is located in the interval

\[
[e^k, e^{k+0.1}]
\]

for some positive integer \( k \). In other words, we have

\[
e^k < n < e^{k+0.1} \Rightarrow k < \log n < k + 0.1 \Rightarrow k - 1 < \log n - 1 < k - 0.9,
\]

which means that

\[
k - 1 = \lfloor \log n - 1 \rfloor
\]

\[
k = \lfloor \log n - 1 \rfloor + 1.
\]

Remark 2.11. Suppose that \( b \) is any fixed integer \( \geq 12 \). Theorem 2.10 implies that if \( n \) is an integer in the interval \([e^b, e^{b+0.1}]\) and at the same time \( n \) is not a multiple of \( b-1 \), then \( \pi(n) \nmid n \). This means that if \( n \geq 60184 \) and \( \pi(n) \) divides \( n \), then \( n \) is located in the interval \([e^b, e^{b+0.1}]\) for some positive integer \( b \) and \( n \) is a multiple of \( b-1 \). ▶

The following theorem follows from Theorems 1.1 and 2.10 and from the fact that \( n/\pi(n) < 11 \) for \( n \leq 60183 \) (this fact can be checked using software):

Theorem 2.12. Let \( c \) be any fixed integer \( \geq 12 \). In the interval \([e^c, e^{c+0.1}]\) there is always an integer \( n \) such that \( \pi(n) \) divides \( n \). In other words, in the interval \([e^c, e^{c+0.1}]\) there is always an integer \( n \) such that \( \pi(n) = n/(c-1) \). □

3 Conclusion and Further Discussion

The following are the main theorems of this paper:

Theorem 2.9. Let \( n \) be an integer \( \geq 60184 \) and let

\[
f(n) = \lfloor \lfloor \log n - \lfloor \log n \rfloor - 0.1 \rfloor \rfloor \left| \frac{n/\lfloor \log n - 1 \rfloor}{n} \right| \frac{\lfloor \log n - 1 \rfloor}{n}.
\]
If \( f(n) = 0 \), then \( \pi(n) \nmid n \). On the other hand, if \( \pi(n) \mid n \), then \( f(n) = 1 \).

**Theorem 2.10.** If \( n \geq 60184 \) and \( n/\pi(n) \) is an integer, then \( n \) is a multiple of \( \lfloor \log n - 1 \rfloor \) located in the interval \([e^{\lfloor \log n - 1 \rfloor + 1}, e^{\lfloor \log n - 1 \rfloor + 1.1}]\).

**Theorem 2.12.** Let \( c \) be any fixed integer \( \geq 12 \). In the interval \([e^c, e^{c+0.1}]\) there is always an integer \( n \) such that \( \pi(n) \) divides \( n \). In other words, in the interval \([e^c, e^{c+0.1}]\) there is always an integer \( n \) such that \( \pi(n) = n/(c-1) \).

We recall that Golomb [3] proved that for every integer \( n > 1 \) there exists a positive integer \( m \) such that \( m/\pi(m) = n \). Suppose now that \( R \) is the sequence of numbers generated by the function \( d(n) = n/\pi(n) \) (\( n \in \mathbb{Z} \) and \( n > 1 \)). In other words,

\[
R = (2, \ 1.5, \ 2, \ 1.66..., \ 2, \ 1.75, \ 2, \ 2.25, \ 2.5, \ \ldots).
\]

Suppose also that \( S \) is the sequence of integers generated by the function \( d(n) = n/\pi(n) \). In other words,

\[
S = (2, \ 2, \ 2, \ 3, \ 3, \ 3, \ 4, \ 4, \ \ldots).
\]

Motivated by Golomb’s result and Theorem 2.12 we ask the following question:

**Question 3.1.** Are there infinitely many positive integers \( a \) such that in the interval \([e^a, e^{a+0.1}]\) there are at least two distinct positive integers \( n_1 \) and \( n_2 \) such that \( \pi(n_1) \mid n_1 \) and \( \pi(n_2) \mid n_2 \)? In other words, are there infinitely many positive integers \( n \) that can be expressed as \( m/\pi(m) \) in more than one way?

Now, let \( S_k \) denote the \( k \)th term of sequence \( S \). Clearly, Question 3.1 is equivalent to the following question:

**Question 3.2.** Are there infinitely many positive integers \( k \) such that \( S_k = S_{k+1} \)?

**References**

