When $\pi(n)$ divides $n$ and when it does not

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Abstract

Let $\pi(n)$ denote the prime-counting function. In this paper we work with explicit formulas for $\pi(n)$ that are valid for infinitely many positive integers $n$, and we prove that if $n \geq 60184$ and $\frac{\ln n}{\pi(n)} = \ln n - \lfloor \ln n \rfloor > 0.5$, then $\pi(n)$ does not divide $n$. Based on this result, we show that if $e$ is the base of the natural logarithm, $a$ is a fixed integer $\geq 11$ and $n$ is any integer in the interval $[e^{a+0.5}, e^{a+1}]$, then $\pi(n) \mid n$. In addition, we prove that if $n \geq 60184$ and $n/\pi(n)$ is an integer, then $n$ is a multiple of $[\ln n - 1]$ located in the interval $[e^{[\ln n-1]+1}, e^{[\ln n-1]+1.5}]$.

Keywords: bounds on the prime-counting function, explicit formulas for the prime-counting function, intervals, prime numbers

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0 Notation

Throughout this paper the number $n$ is always a positive integer. Moreover, we use the following symbols:

• $\lfloor \cdot \rfloor$ (floor function)
• $\lceil \cdot \rceil$ (ceiling function)
• $\nmid$ (does not divide)
• frac() (fractional part)

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1 Introduction

Determining how prime numbers are distributed among natural numbers is one of the most difficult mathematical problems. This explains why the prime-counting function \( \pi(n) \), which counts the number of primes less than or equal to a given number \( n \), has been one of the main objects of study in Mathematics for centuries.

In [2] Gaitanas obtains an explicit formula for \( \pi(n) \) that holds infinitely often. His proof is based on the fact that the function \( f(n) = n/\pi(n) \) takes on every integer value greater than 1 (as proved by Golomb [3]) and on the fact that \( x/(\ln x - 0.5) < \pi(x) < x/(\ln x - 1.5) \) for \( x \geq 67 \) (as shown by Rosser and Schoenfeld [4]). In this paper we find alternative expressions that are valid for infinitely many positive integers \( n \), and we also prove that for \( n \geq 60184 \) if \( \ln n - \lfloor \ln n \rfloor > 0.5 \), then \( n/\pi(n) \) is not an integer.

We will place emphasis on the following three theorems, which were proved by Golomb, Dusart, and Gaitanas respectively:

**Theorem 1.1** [3]. The function \( f(n) = n/\pi(n) \) takes on every integer value greater than 1.

**Theorem 1.2** [1]. If \( n \) is an integer \( \geq 60184 \), then

\[
\frac{n}{\ln n - 1} < \pi(n) < \frac{n}{\ln n - 1.1}.
\]

**Remark 1.3.** Dusart’s paper states that for \( x \geq 60184 \) we have \( x/(\ln x - 1) \leq \pi(x) \leq x/(\ln x - 1.1) \), but since \( \ln n \) is always irrational when \( n \) is an integer \( > 1 \), we can state his theorem the way we did.

**Theorem 1.4** [2]. The formula

\[ \pi(n) = \frac{n}{\lfloor \ln n - 0.5 \rfloor} \]

is valid for infinitely many positive integers \( n \).

2 Main theorems

We are now ready to prove our main theorems:

**Theorem 2.1.** The formula

\[ \pi(n) = \frac{n}{\lfloor \ln n - 1 \rfloor} \]

holds for infinitely many positive integers \( n \).
Proof. According to Theorem 1.2, for \( n \geq 60184 \) we have

\[
\frac{n}{\ln n - 1} < \pi(n) < \frac{n}{\ln n - 1.1} \Rightarrow \frac{\ln n - 1.1}{n} < \frac{1}{\pi(n)} < \frac{\ln n - 1}{n}.
\]

If we multiply by \( n \), we get

\[
\ln n - 1.1 < \frac{n}{\pi(n)} < \ln n - 1. \tag{1}
\]

Since \( \ln n - 1.1 \) and \( \ln n - 1 \) are both irrational (for \( n > 1 \)), inequality (1) implies that when \( n/\pi(n) \) is an integer we must have

\[
\frac{n}{\pi(n)} = [\ln n - 1] = [\ln n - 1.1] + 1 = [\ln n - 1.1] = [\ln n] - 1. \tag{2}
\]

Taking Theorem 1.2 and equality (2) into account, we can say that for every \( n \geq 60184 \) when \( n/\pi(n) \) is an integer we must have

\[
\pi(n) = \frac{n}{\lceil \ln n - 1 \rceil} = n - 1.
\]

Since Theorem 1.1 implies that \( n/\pi(n) \) is an integer infinitely often, it follows that there are infinitely many positive integers \( n \) such that \( \pi(n) = n/\lceil \ln n - 1 \rceil \).

In fact, the following theorem follows from Theorems 1.1, from Gaitana’s proof of Theorem 1.4, and from the proof of Theorem 2.1:

**Theorem 2.2.** For every \( n \geq 60184 \) when \( n/\pi(n) \) is an integer we must have

\[
\frac{n}{\pi(n)} = [\ln n - 0.5] = [\ln n - 1] = [\ln n - 1.1] + 1 = [\ln n - 1.1] = [\ln n] - 1.
\]

In other words, for \( n \geq 60184 \) when \( n/\pi(n) \) is an integer we must have

\[
\pi(n) = \frac{n}{[\ln n - 0.5]} = \frac{n}{[\ln n - 1]} = \frac{n}{[\ln n - 1.1] + 1} = \frac{n}{[\ln n - 1.1]} = \frac{n}{[\ln n - 1]} - 1.
\]

**Theorem 2.3.** Let \( n \) be an integer \( \geq 60184 \). If \( \ln n - [\ln n] \geq 0.5 \), then \( \pi(n) \nmid n \) (that is to say, \( n/\pi(n) \) is not an integer).
Proof. According to Theorem 2.2, for $n \geq 60184$ if $n/\pi(n)$ is an integer, then

$$\frac{n}{\pi(n)} = \lfloor \ln n - 0.5 \rfloor = \lfloor \ln n - 1 \rfloor.$$  

In other words, for $n \geq 60184$ when $n/\pi(n)$ is an integer we have

$$\lfloor \ln n - 0.5 \rfloor = \lfloor \ln n - 0.5 - 0.5 \rfloor$$

$$\frac{\ln n - 0.5}{0.5} \geq 0.5$$

$$\ln n - 0.5 - \lfloor \ln n - 0.5 \rfloor \geq 0.5$$

$$\ln n - \lfloor \ln n - 0.5 \rfloor \geq 1$$

$$\frac{\ln n}{0.5} < 0.5$$

$$\ln n - \lfloor \ln n \rfloor < 0.5.$$  

Suppose that $P$ is the statement ‘$n/\pi(n)$ is an integer’ and $Q$ is the statement ‘$\ln n - \lfloor \ln n \rfloor < 0.5$’. According to propositional logic, the fact that $P \rightarrow Q$ implies that $\neg Q \rightarrow \neg P$.  

Similar theorems can be proved by using Theorem 2.2 and equality (3).

Remark 2.4. When we mentioned Theorem 2.3 in the abstract, we replaced the expression $\ln n - \lfloor \ln n \rfloor \geq 0.5$ with $\ln n - \lfloor \ln n \rfloor > 0.5$ due to the fact that $\ln n$ is irrational when $n > 1$.

Remark 2.5. Because $\ln n$ is irrational for $n > 1$, another way of stating Theorem 2.3 is by saying that if $n \geq 60184$ and the first digit to the right of the decimal point of $\ln n$ is 5, 6, 7, 8, or 9, then $\pi(n) \nmid n$.

Or we could also say that for $n \geq 60184$ if

$$n > e^{0.5+\lfloor \ln n \rfloor},$$

then $\pi(n) \nmid n$.

The following theorem follows from Theorem 2.3.

Theorem 2.6. Let $e$ be the base of the natural logarithm. If $a$ is any integer $\geq 11$ and $n$ is any integer contained in the interval $[e^{a+0.5}, e^{a+1}]$, then $\pi(n) \nmid n$. (The number $e^r$ is irrational when $r$ is a rational number $\neq 0$.)

Example 2.7. Take $a = 18$. If $n$ is any integer in the interval $[e^{18.5}, e^{19}]$, then $\pi(n) \nmid n$.  

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Theorem 2.8. If \( n \geq 60184 \) and \( n/\pi(n) \) is an integer, then \( n \) is a multiple of \( \lfloor \ln n - 1 \rfloor \) located in the interval \([e^{\lfloor \ln n - 1 \rfloor + 1}, e^{\lfloor \ln n - 1 \rfloor + 1.5}]\).

Proof. According to the proof of Theorem 2.3 if \( n \geq 60184 \) and \( n/\pi(n) \) is an integer, then
\[
\frac{n}{\pi(n)} = \lfloor \ln n - 1 \rfloor \Rightarrow n = \pi(n)\lfloor \ln n - 1 \rfloor
\]
and
\[
\text{frac}(\ln n) = \ln n - \lfloor \ln n \rfloor < 0.5.
\]
The fact that \( \text{frac}(\ln n) < 0.5 \) implies that \( n \) is located in the interval
\[
[e^k, e^{k+0.5}]
\]
for some positive integer \( k \). In other words, we have
\[
e^k < n < e^{k+0.5} \Rightarrow k < \ln n < k + 0.5 \Rightarrow k - 1 < \ln n - 1 < k - 0.5.
\]
This means that
\[
k - 1 = \lfloor \ln n - 1 \rfloor
\]
\[
k = \lfloor \ln n - 1 \rfloor + 1,
\]
which proves the theorem.

Remark 2.9. In other words, if \( n \geq 60184 \) and \( n/\pi(n) \) is an integer, then \( n \) is located in the interval \([e^k, e^{k+0.5}]\) for some positive integer \( k \) and \( n \) is divisible by \( k - 1 \).

Question 2.10. Let \( b_0 \) be a sufficiently large positive integer and let \( b \) be any integer \( \geq b_0 \). In the interval \([e^b, e^{b+0.5}]\), is there always an integer \( n \) that is divisible by \( \pi(n) \)?

A Better results

Theorem A.1. Let \( n \) be an integer \( \geq 60184 \). If \( \text{frac}(\ln n) = \ln n - \lfloor \ln n \rfloor > 0.1 \), then \( \pi(n) \nmid n \) (that is to say, \( n/\pi(n) \) is not an integer).

Proof. According to Theorem 2.2 for \( n \geq 60184 \) if \( n/\pi(n) \) is an integer, then
\[
\frac{n}{\pi(n)} = \lfloor \ln n - 1 \rfloor = \lfloor \ln n - 1.1 \rfloor.
\]
In other words, for \( n \geq 60184 \) when \( n/\pi(n) \) is an integer we have

\[
\lfloor \ln n - 1 \rfloor = \lceil \ln n - 1 - 0.1 \rceil
\]

\[
\frac{\ln n - 1}{\ln n} \leq 0.1
\]

\[
\ln n - \lfloor \ln n - 1 \rfloor \leq 1.1
\]

\[
\frac{\ln n}{\ln n} \leq 0.1
\]

\[
\ln n - \lfloor \ln n \rfloor \leq 0.1
\]

Suppose that \( P \) is the statement ‘\( n/\pi(n) \) is an integer’ and \( Q \) is the statement ‘\( \ln n - \lfloor \ln n \rfloor \leq 0.1 \)’. The fact that \( P \to Q \) implies that \( \neg Q \to \neg P \).

\[ \blacksquare \]

**Remark A.2.** Because \( \ln n \) is irrational for \( n > 1 \), Theorem [A.1] implies that if \( n \geq 60184 \) and \( \pi(n) \) divides \( n \), then the first digit to the right of the decimal point of \( \ln n \) can only be 0. In other words, if \( n \geq 60184 \) and the first digit to the right of the decimal point of \( \ln n \) is 1, 2, 3, 4, 5, 6, 7, 8, or 9, then \( \pi(n) \) does not divide \( n \).

\[ \blacksquare \]

The following theorem follows from Theorem [A.1]

**Theorem A.3.** Let \( e \) be the base of the natural logarithm. If \( a \) is any integer \( \geq 11 \) and \( n \) is any integer contained in the interval \( [e^a, e^{a+0.1}] \), then \( \pi(n) \nmid n \).

\[ \blacksquare \]

**Example A.4.** Take \( a = 18 \). If \( n \) is any integer in the interval \( [e^{18.1}, e^{19}] \), then \( \pi(n) \nmid n \).

\[ \blacksquare \]

**Remark A.5.** If \( n \geq 60184 \) and \( n/\pi(n) \) is an integer, then \( n \) is located in the interval \( [e^k, e^{k+0.1}] \) for some positive integer \( k \).

\[ \blacksquare \]

**Question A.6.** Let \( b_0 \) be a sufficiently large positive integer and let \( b \) be any integer \( \geq b_0 \). In the interval \( [e^b, e^{b+0.1}] \), is there always an integer \( n \) that is divisible by \( \pi(n) \)?

\[ \blacksquare \]

**References**

