On Application of Green Function Method to the Solution of 3D Incompressible Navier-Stokes equations

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September 5, 2014

Abstract

The fluid equations, named after Claude-Louis Navier and George Gabriel Stokes, describe the motion of fluid substances. These equations arise from applying Newton's second law to fluid motion, together with the assumption that the stress in the fluid is the sum of a diffusing viscous term (proportional to the gradient of velocity) and a pressure term - hence describing viscous flow. Due to specific of NS equations they could be transformed into full/partial inhomogeneous parabolic differential equations: differential equations in respect of space variables and the full differential equation in respect of time variable and time dependent inhomogeneous part. Velocity and outer forces densities components were expressed in form of curl for obtaining solutions satisfying continuity condition specifying divergence of velocities equality to zero. Finally, solution in 3D space for any shaped boundary was expressed in terms of 3D Green function and inverse Laplace transform accordantly.

1 Introduction

In physics, the fluid equations, named after Claude-Louis Navier and George Gabriel Stokes, describe fluid substances motion. These equations arise from applying Newton's second law to fluid motion, together with the assumption that the stress in the fluid is the sum of a diffusing viscous term (proportional to the gradient of velocity) and a pressure term - hence describing viscous flow. Equations were introduced in 1822 by the French engineer Claude Louis Marie Henri Navier [1] and successively re-obtained, by different arguments, by a several authors including Augustin-Louis Cauchy in 1823 [2], Simeon Denis Poisson in 1829, Adhemar Jean Claude Barre de Saint-Venant in 1837, and, finally, George Gabriel Stokes in 1845 [3]. Detailed and thorough analysis of the history of the fluid equations could be found in by Olivier Darrigol [4]. The invention of the digital computer led to many changes. John von Neumann, one of the CFD founding fathers, predicted already in 1946 that automatic computing machines' would replace the analytic solution of simplified flow equations by a numerical' solution of the full nonlinear flow equations for arbitrary geometries. Von Neumann suggested that this numerical approach would even make experimental fluid dynamics obsolete. Von Neumann's prediction did not fully come true, in the sense that both analytic theoretical and experimental research still coexist with CFD. Crucial properties of CFD methods such as consistency, stability and convergence need mathematical study [5].

Earlier author proposed solution of 3D Navier-Stokes equations by using of orthogonal function series [11]. Aims of this article are to propose new approach for solution of incompressible fluid equations based on Green function method in conjunction with Laplace transform.

2 Parabolic formulation of equations

Incompressible fluid equations are expressed as follow

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t}+\quad(\mathbf{v}\cdot\nabla)\mathbf{v}\right)-\mu\Delta\mathbf{v}+\nabla p = f \tag{1}$$

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\mathbf{v}) = 0 \tag{2}$$

where equation (2) for incompressible flow reduces to $\frac{d\rho}{dt} = 0$ or $\rho = const$ due to $\nabla \cdot \mathbf{v} = 0$. Equations of fluid motion (1) could be expressed in convective time derivative as follow

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \quad (\mathbf{v} \cdot \nabla) \tag{3}$$

So, we obtain

$$\frac{d\mathbf{v}}{dt} - a^2 \Delta \mathbf{v} = \frac{1}{\rho}(-\nabla p + f) \tag{4}$$

3 inhomogeneous parabolic like equation for full time derivative, where $a = \sqrt{\mu/\rho}$. Tensor of the inner pressure of fluid for existing solution of velocities could be found by using of equations

$$\mathbf{p}_{in}^i = -p\mathbf{e}_i + f_i\mathbf{e}_i + \mu\nabla_i(\sum_{i=1}^{3} v^i \mathbf{e}_i) \tag{5}$$

where $\mathbf{e}_i$ are eigenvectors of corresponding coordinate system.

3 Three dimensional inhomogeneous solution

Consider the initial-boundary value problem for $v = v(x, y, z, t)$

$$\begin{aligned}
\frac{dv^i}{dt} - a^2\Delta v^i &= \frac{1}{\rho}(-\nabla_i p + f_i) \text{ in } \Omega \times (0, \infty) && (6)\\
v^i(x, y, z, 0) &= v_0^i(x, y, z) \; x, y, z \in \Omega && (7)\\
\frac{\partial v^i}{\partial n} &= 0 \text{ on } \partial\Omega \times (0, \infty) && (8)
\end{aligned}$$

where $p = p(x, y, z, t)$ and $f = f(x, y, z, t)$, $\Omega \subset \mathbb{R}^{3n}$, n the exterior unit normal at the smooth parts of $\partial\Omega$, a^2 a positive constant and $v_0^x(x, y, z), v_0^y(x, y, z), v_0^z(x, y, z)$ a given function.

So, equation (4) could be rewritten as follow

$$\frac{1}{a^2}\frac{dv^i}{dt} = \frac{\partial^2 v^i}{\partial x^2} + \frac{\partial^2 v^i}{\partial y^2} + \frac{\partial^2 v^i}{\partial z^2} - Q^i(x, y, z, t), \; x, y, z \in \Omega, \; t > 0. \tag{9}$$

Continuity conditions will be satisfied, if our solution will be curl. So, we will apply inverse curl operator to the velocities v^i and Q^i as follow

$$\begin{aligned}
\mathbf{v}_o &= \nabla^{-1} \times \mathbf{v} && (10)\\
\mathbf{Q}_o &= \nabla^{-1} \times \mathbf{Q} && (11)
\end{aligned}$$

The method of obtaining inverse curl could be found in [9]. Now we apply Laplace transform to the equation (9) and obtain

$$\nabla^2\mathbf{v}(\mathbf{r}, s) + k^2 s\mathbf{v}(\mathbf{r}, s) = k^2\mathbf{v}(\mathbf{r}, 0) + \mathbf{Q}(\mathbf{r}, s) \tag{12}$$

where $k = \sqrt{-1}/a$. After inserting equations (10) and (11) into (12) equations transform as follow

$$\nabla^2\nabla \times \mathbf{v}_o(\mathbf{r}, s) + k^2 s\nabla \times \mathbf{v}_o(\mathbf{r}, s) = k^2\nabla \times \mathbf{v}_o(\mathbf{r}, 0) + \nabla \times \mathbf{Q}_o(\mathbf{r}, s) \tag{13}$$

$$\nabla \times (\nabla^2\mathbf{v}_o(\mathbf{r}, s) + k^2 s\mathbf{v}_o(\mathbf{r}, s)) = \nabla \times (k^2\mathbf{v}_o(\mathbf{r}, 0) + \mathbf{Q}_o(\mathbf{r}, s)) \tag{14}$$

$$\nabla^2\mathbf{v}_o(\mathbf{r}, s) + k^2 s\mathbf{v}_o(\mathbf{r}, s) = k^2\mathbf{v}_o(\mathbf{r}, 0) + \mathbf{Q}_o(\mathbf{r}, s) \tag{15}$$

This are three inhomogeneous Helmholtz equations. We will solve it using Green function method described in [10]. The Green function is then defined by

$$(\nabla^2 + k^2 s))G(\mathbf{r}_1, \mathbf{r}_2) = \delta^3(\mathbf{r}_1 - \mathbf{r}_2) \tag{16}$$

Define the basis functions ϕ_n as the solutions of homogeneous Helmholtz differential equation

$$\nabla^2\phi_n(\mathbf{r}) + k_n^2\phi_n(\mathbf{r}) = 0 \tag{17}$$

So, Green function is expressed as follow

$$G(\mathbf{r}_1, \mathbf{r}_2) = \sum_{n=0}^{\infty} \frac{\phi_n(\mathbf{r}_1)\phi_n(\mathbf{r}_2)}{k^2 s - k_n^2} \tag{18}$$

The general solution to (15) is therefore

$$\begin{aligned}
\mathbf{v}_o(\mathbf{r}_1, t) &= \int_\Omega G(\mathbf{r}_1, \mathbf{r}_2)(k^2 \mathbf{v}_o(\mathbf{r}, 0) + \mathbf{Q}_o(\mathbf{r}, s)) d^3\mathbf{r}_2 && (19)\\
&= \sum_{n=0}^{\infty} \int_\Omega \frac{\phi_n(\mathbf{r}_1)\phi_n(\mathbf{r}_2)(k^2 \mathbf{v}_o(\mathbf{r}_2, 0) + \mathbf{Q}_o(\mathbf{r}_2, s))}{k^2 s - k_n^2} d^3\mathbf{r}_2 && (20)
\end{aligned}$$

Now we apply curl transform for obtained velocities $\mathbf{v}_o$

$$\mathbf{v}(\mathbf{r}_1, t) = \nabla \times \sum_{n=0}^{\infty} \int_\Omega \frac{\phi_n(\mathbf{r}_1)\phi_n(\mathbf{r}_2)(k^2 \mathbf{v}_o(\mathbf{r}_2, 0) + \mathbf{Q}_o(\mathbf{r}_2, s))}{k^2 s - k_n^2} d^3\mathbf{r}_2 \tag{21}$$

Finally, we must apply the Laplace inverse transform to get resulting velocities $\mathbf{v}$

$$\mathbf{v}(\mathbf{r}_1, t) = \mathcal{L}^{-1}\{\nabla \times \sum_{n=0}^{\infty} \int_\Omega \frac{\phi_n(\mathbf{r}_1)\phi_n(\mathbf{r}_2)(k^2 \mathbf{v}_o(\mathbf{r}_2, 0) + \mathbf{Q}_o(\mathbf{r}_2, s))}{k^2 s - k_n^2} d^3\mathbf{r}_2\} \tag{22}$$

Now, if we want to know velocity in respect of some point $\mathbf{r}_0$ at each time moment, we must apply Galileo transform so that

$$\mathbf{v}(\mathbf{r}_1) = \mathbf{v}(\mathbf{r}_0 + \mathbf{v}(\mathbf{r}_1)t). \tag{23}$$

in case equalities

$$\frac{d\mathbf{v}(\mathbf{r}_1, t)}{dt} = \frac{\partial \mathbf{v}(\mathbf{r}_1, t)}{\partial t} + (\mathbf{v}(\mathbf{r}_1, t) \cdot \nabla)\mathbf{v}(\mathbf{r}_1, t) \tag{24}$$

are true.

4 Conclusions

Due to the form of fluid equations they could be transformed into the full/partial inhomogeneous parabolic differential equations: partial differential equations in respect to space variables and full differential equations in respect to the time variable and inhomogeneous time dependent part. Velocity and outer forces densities components were expressed in form of curl for obtaining solutions satisfying continuity condition specifying divergence of velocities equality to zero. Finally, solution in 3D space for any shaped boundary was expressed in terms of 3D Green function and inverse Laplace transform accordantly.

Acknowledgement

This work was partly supported by the Project of Scientific Groups (Lithuanian Council of Science), Nr. MIP-13204. The author is very grateful all participants of seminar on 31 January, 2014 and specially for prof. habil. dr. Konstantinas Pileckas for detailed discussions on completeness of obtained solutions.

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