What Mathematics Is The Most Fundamental?

Felix M. Lev

Artwork Conversion Software Inc., 1201 Morningside Drive, Manhattan Beach, CA 90266, USA (Email: felixlev314@gmail.com)

Abstract

Standard mathematics involves such notions as infinitely small/large, continuity and standard division. However, some of these notions are treated differently in traditional and constructive versions. This mathematics is usually treated as fundamental while finite mathematics is treated as inferior. Standard mathematics has foundational problems (as follows, for example, from Gödel's incompleteness theorems) but people usually believe that this is less important than the fact that it describes many experimental data with high accuracy. We argue that the situation is the opposite: standard mathematics is only a special case of finite one in the formal limit when the characteristic of the ring or field used in finite mathematics goes to infinity. Therefore foundational problems in standard mathematics are not fundamental.

MSC: 03Axx

Keywords: standard mathematics, finite mathematics, infinity, Galois fields

Standard mathematics involves such notions as infinitely small/large, continuity and standard division. However, some of these notions are treated differently in traditional and constructive versions. These notions arose from a belief based on everyday experience that any macroscopic object can be divided into arbitrarily large number of arbitrary small parts. However, the very existence of elementary particles indicates that those notions have only a limited meaning. Indeed, consider the gram-molecule of water having the mass 18 grams. It contains the Avogadro number of molecules $6 \cdot 10^{23}$. We can divide this gram-molecule by ten, million, billion, but when we begin to divide by numbers greater than the Avogadro one, the division operation loses its meaning and we cannot obtain arbitrarily small parts. This example shows that mathematics involving the set of all rational numbers has only a limited applicability.

As a consequence, any description of macroscopic phenomena using continuity and differentiability can be only approximate. Water in the ocean can be described by differential equations of hydrodynamics but this is only an approximation since matter is discrete. The above remarks show that using standard mathematics in quantum physics is at least unnatural.

A problem arises whether standard mathematics can be substantiated as an abstract science. A detailed description of the problem of substantiating standard mathematics can be found in numerous textbooks and monographs (see e.g. Ref. [1]).

As it was shown by Russell and other mathematicians, the Cantor set theory contains several fundamental paradoxes. To avoid them several axiomatic set theories have been proposed and the most known of them is the ZFC theory developed by Zermelo and Fraenkel. However, the consistency of ZFC cannot be proven within ZFC itself and it was shown that the continuum hypothesis is independent of ZFC. Gödel's incompleteness theorems state that no system of axioms can ensure that all facts about natural numbers can be proven and the system of axioms in traditional mathematics cannot demonstrate its own consistency.

Brouwer proposed constructive mathematics in order to avoid foundational problems of traditional mathematics. Here there is no law of the excluded middle and it is required that any proof of existence be algorithmic. That is why constructive mathematics is treated such that, at least in principle, it can be implemented on a computer. Here "in principle" means that the number of steps might be not finite. With this meaning constructive mathematics, as well as traditional one, assumes that one can operate with any desired amount of resources and it is theoretically possible to consider an idealized case when a computer can operate with any desirable number of bits.

The absolute majority of mathematicians prefer the traditional version. Physics is also based only on traditional mathematics. Hilbert was a strong opponent of constructive mathematics. He said: "No one shall expel us from the paradise that Cantor has created for us" and "Taking the principle of excluded middle from the mathematician would be the same, say, as proscribing the telescope to the astronomer or to the boxer the use of his fists".

Let us look at mathematics from the point of view of philosophy of quantum theory. According to this philosophy, there should be no statements accepted without proof. The theory should contain only those statements that can be verified, where by "verified" physicists mean an experiment involving only a finite number of steps.

Let us pose a problem of whether 10+20 equals 30. Then we should describe an experiment which should solve this problem. Any computer can operate only with a finite number of bits and can perform calculations only modulo some number p. Say $p = 40$, then the experiment will confirm that 10+20=30 while if $p = 25$ then we will get that 10+20=5. So the statements that 10+20=30 and even that $2 \cdot 2 = 4$ are ambiguous because they do not contain information on how they should be verified. We believe the following observation is very important: although standard mathematics is a part of our everyday life, people typically do not realize that *standard mathematics is implicitly based on the assumption that one can have any desirable amount of resources*. So standard mathematics (including traditional and constructive versions) is based on the implicit assumption that we can consider a formal limit $p \to \infty$ and the correctness of the limit can be substantiated.

While Gödel's works on the incompleteness theorems are written in highly technical terms of mathematical logics, the fact that standard mathematics has foundational problems is clear from the philosophy of quantum theory. For instance, the first incompleteness theorem says that not all facts about natural numbers can be proven. However, from the philosophy of quantum theory this seems to be clear because if the number of numbers is not finite then we cannot verify that $a + b = b + a$ for any a and b.

The famous Kronecker's expression is: "*God made the natural numbers, all else is the work of man*". However only two operations are always possible in the set of natural numbers: addition and multiplication. In order to make addition reversible, we introduce negative integers and then we can work with the ring of integers Z. The next step is the transition to the field of rational numbers in which all four operations except division by zero are possible. However, if, instead of all natural numbers, we consider only a set F_p of p numbers 0, 1, 2, ... $p-1$ where p is prime and the operations are defined as usual but modulo p then we get a field without adding new elements.

One can also treat F_p as a set of elements $-(p-1)/2, -(p-3)/2, ...0, ...(p-3)/2, (p-1)/2$. The convenience of this representation is that for elements $x \in F_p$ such that $|x| \ll p$ addition, subtraction and multiplication are the same as in Z. In other words, for such elements we do not feel the existence of p. When p increases, the bigger and bigger part of F_p becomes similar to Z. This important observation implies that *standard mathematics can be treated as a special case of mathematics modulo p in the formal limit* $p \to \infty$.

In the general case, division in F_p is not the same as in standard mathematics. For example, $1/2$ in F_p equals $(p+1)/2$, i.e. a very large number if p is large. However, this does not mean that mathematics modulo p cannot describe physics. It is important to realize that spaces in quantum theory are projective. In Refs. [2, 3] we have proposed an approach called GFQT where quantum theory is based on a Galois field with characteristic p. It has been shown that in the formal limit $p \to \infty$ GFQT recovers the predictions of standard continuous theory. Then the fact that standard mathematics describes many experiments with a high accuracy can be explained as a consequence of the fact that in real life the number p is very large.

In addition, GFQT gives a new look at many fundamental problems in physics (see the discussion in Refs. [4, 5]). In particular, GFQT gives a natural explanation of the existence of antiparticles and of the fact that a particle and its antiparticle have equal masses and opposite charges. Another striking example is that gravity can be treated not as an interaction but simply as a manifestation of the fact that nature is finite and is described by a Galois field of characteristic p. In this approach the gravitational constant is not a parameter taken from the outside (e.g. from the condition that theory should describe experiment) but a quantity which should be calculated. The actual calculation is problematic because it requires the knowledge of details of wave functions for macroscopic bodies. However, reasonable qualitative arguments show [5] that the gravitational constant is proportional to $1/lnp$. Therefore, gravity is a consequence of the finiteness of nature and disappears in the limit $p \to \infty$. A qualitative estimation based on additional assumptions gives that lnp is of the order of 10^{80} and therefore p is a huge number of the order of $exp(10^{80})$.

Also as noted above, in quantum theory standard division has a limited applicability. This might be an indication that (as Metod Saniga pointed out), in the spirit of Ref. [6], the ultimate quantum theory will be based even on a finite ring and not a field.

The above discussion has a well-known historical analogy. For many years people believed that our Earth was flat and infinite, and only after a long period of time they realized that it was finite and had a curvature. It is difficult to notice the curvature when we deal only with distances much less than the radius of the curvature R. Analogously one might think that the set of numbers describing physics has a "curvature" defined by a very large number p but we do not notice it when we deal only with numbers much less than p.

One might argue that introducing a new fundamental constant is not justified. However, the history of physics tells us that new theories arise when a parameter, which in the old theory was treated as infinitely small or infinitely large, becomes finite. For example, from the point of view of nonrelativistic physics, the velocity of light c is infinitely large but in relativistic physics it is finite. Similarly, from the point of view of classical theory, the Planck constant $\hbar$ is infinitely small but in quantum theory it is finite. Therefore, it is natural to think that in the future quantum physics the quantity p will be not infinitely large but finite. A problem arises whether p is a constant or is different in various periods of time. Moreover, in view of the problem of time in quantum theory, an extremely interesting scenario is that the world time is defined by p.

Regardless of whether or not we accept that the ultimate quantum theory will be based on finite mathematics with the characteristic p, a problem of what mathematics should be treated as fundamental still remains. Many physicists and mathematicians think that standard mathematics is fundamental while finite one is inferior. Typical reasons are that standard mathematics contains more numbers than finite one and that the whole history of mankind has proven that standard mathematics describes reality with an unprecedented accuracy. For those reasons, the fact that standard mathematics has foundational problems might be treated as less important.

However, any realistic calculations can involve only a finite number of numbers and any experiment has a finite accuracy. In standard mathematics there are no operations modulo p. This fact can be treated such that in the formal limit $p \to \infty$ such operations disappear. As noted above, the ring of integers Z can be treated as a special case of a finite ring with the characteristic p in the formal limit

$p \to \infty$. Therefore, the situation is the opposite: standard mathematics is a special case of finite one in the formal limit $p \to \infty$, and an illusion of continuity arises because p is very large.

Hence standard mathematics might be treated only as a technique which in many cases describes reality with a high accuracy while the fact that this mathematics has foundational problems indeed does not have a fundamental role. The philosophy of Brouwer, Cantor, Fraenkel, Gödel, Hilbert, Kronecker, Russell, Zermelo and other great mathematicians working on foundation of standard mathematics was based on macroscopic experience in which the notions of infinitely small, infinitely large and continuity are natural. However, as noted above, those notions contradict the existence of elementary particles and are not natural in quantum theory.

Acknowledgements: I am very grateful to Sergei Dolgobrodov, Volodya Netchitailo, Mikhail Aronovich Olshanetsky, Misha Partenskii and Teodor Shtilkind for numerous stimulating discussions.

References

[1] J. Ferreiros. Labyrinth of Thought: A History of Set Theory and Its Role in Modern Mathematics. Birkhäuser Verlag AG, Basel-Boston-Berlin (2007).

[2] F. Lev. Modular Representations as a Possible Basis of Finite Physics. J. Math. Phys., **30**, 1985-1998 (1989).

[3] F. Lev. Finiteness of Physics and its Possible Consequences. J. Math. Phys., **34**, 490-527 (1993).

[4] F. Lev. de Sitter Symmetry and Quantum Theory. Phys. Rev., **D85**, 065003 (2012).

[5] F. Lev. Quantum Theory over a Galois Field and Applications to Gravity and Particle Theory. arxiv:1104.4647, vixra:1402.0116.

[6] M. Planat and M. Saniga. Finite Geometries in Quantum Theory: From Galois (fields) to Hjelmslev (rings). J. Mod. Phys., **B20**, 1885-1892 (2006).