Kaluza-Cartan Theory With Completely Antisymmetric Torsion

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Abstract

Kaluza's 1921 theory of gravity and electromagnetism using a fifth wrapped-up spatial dimension is inspiration for many modern attempts to develop new physical theories. For a number of reasons the theory is incomplete and often considered untenable. An alternative approach is presented that includes torsion, unifying gravity and electromagnetism in a Kaluza-Cartan theory. Emphasis is placed on admitting important electromagnetic fields not present in Kaluza's original theory, and on a Lorentz force law. This is investigated via a non-Maxwellian kinetic definition of charge related to Maxwellian charge and 5D momentum. Two connections are used. General covariance and global properties are investigated via a reduced non-maximal atlas. Conserved super-energy is used in place of the energy conditions for 5D causality. Explanatory relationships between matter, charge and spin are present.

1 Introduction

Kaluza's 1921 theory of gravity and electromagnetism [1][2][3][4] using a fifth wrapped-up spatial dimension gives a taste of unification of electromagnetism with gravity in a way that has problems and is often believed to be untenable. However the underlying aim was particularly promising in terms of explanatory power. Modern works hold out hope for higher dimensional theories and non-abelian gauges [25], and the consequent hope for unification with quantum mechanics. Here an alternative approach is implemented that goes back to a simpler (and in the author's opinion more practical) root: fully unifying just gravity and electromagnetism. Such an approach, without finding use for lagrangians, or even Klein's famous contribution, is contrary to more common lines of thinking. The author makes no claims in this regard here except that the work be judged on its own terms and novelty. Certain requirements are evident: a Lorentz force law [6] must be explained, Maxwell's laws [6] must be present, the Lorentz transformation [6] must define inertial frames, general relativity [6] must be a limit for gravitational physics. The Lorentz force law is the most conceptually unsatisfying law within classical theory. It may not even be compatible with n-dimensional Noether theorems [26] - all the more reason to construct it, or an approximation as is the case here, from first principles. Yes, it does come from the Einstein-Maxwell stress-energy tensor [6], but where does *that* come from? The Lorentz force law is but the relativistic form of Coulomb's law. Surely it should be as fundamental geometrically as the inverse square law of gravity? It may equally be approximate. It is in this straight forward and relatively unambitious vein that search for a variant Kaluza theory is undertaken.

The Lorentz force law here requires a constant scalar field, this places constraints on admissible solutions. The emphasis is then on eliminating the constraint in Kaluza theory that prevents the so-called non-null electromagnetic solutions. Explicit existence proofs are not necessary. It is sufficient to show that the constraint that causes the problems on solutions has been weakened in the new theory. The constraint is the third field equation in [1], and equation (2.0.7) here. When the scalar field is constant this equation becomes one of two equations that characterize the null electromagnetic fields. This equation is as follows, and fields that satisfy this will be called 'nullish':

Definition 1.0.1: 'Nullish' electromagnetic fields satisfy: $F_{ab}F^{ab} = 0$. Null electromagnetic fields have the nullish property plus the following condition, where the star is the Hodge star operator: $F_{ab}(*F^{ab}) = 0$.

Kaluza's original theory [1] prohibits non-nullish solutions (or even near non-nullish solutions) for constant scalar field. Nullishness is too tight to admit important electromagnetic fields, in particular the essential electrostatic fields. That electrostatic or near-electrostatic fields are non-nullish and therefore a problem in any theory that omits them can be seen by comparing definition (1.0.1) with the following well-known fact from special relativity, that is by considering a special relativistic limit: $F_{ab}F^{ab} = 2(B \cdot B - E \cdot E)$.

At first the objective of the research undertaken here, that is before torsion was finally admitted, was actually to try to discount the need for torsion since its lack of presence is geometrically an obvious assumption in many physical theories. This is analogous to Euclid's fifth postulate in that its assumption is an addition and its removal actually enabled geometric theories like general relativity to be possible. Whilst few would consider it necessary to investigate such an assumption as torsion is essentially just a deformation of the connection [26], that was nevertheless the program here. The research was therefore based around showing that sufficient electromagnetic fields could be obtained (without torsion) from existing Kaluza theory without difficulties or arbitrary assumptions [10], and also whilst deriving the Lorentz Force law from first principles. That program evolved into explicitly allowing torsion in defining equations such as the Ricci curvature, noting that vanishing Ricci curvature plays a role in Kaluza-Klein theories - thus torsion contrary to some modern complaints can have physical significance. It is claimed that the theory presented here is an example of a Kaluza variant theory that better satisfies the requirements of observable classical physics in having a wide range of electromagnetic fields permitted whilst providing a Lorentz force law by construction. This paper therefore shows that such Kaluza variant theories exist and exemplifies a new route in the search for such theories. The theory presented here can thus be seen as at least an example of such a theory, and has value as such. In addition it can also be considered as a candidate empirical theory in itself, though this is not necessary for it to have the previously mentioned theoretical value.

A new kinetic charge will be defined as the 5th-dimensional component of momentum as in [8]. A Lorentz force law will follow. As momentum the kinetic charge has a divergence law via the (torsionless) Einstein tensor. Maxwellian charge also has a vector potential, see (4.4.1), and thus local conservation, but the kinetic charge and corresponding divergence law being covariant is in this sense more fundamental.

2 Conventions

The following conventions are adopted unless otherwise specified. Though unfamiliar in places these are necessary for following the multiple systems used.

Five dimensional metrics, tensors and pseudo-tensors and operators are given the hat symbol. Five dimensional indices, subscripts and superscripts are given capital Roman letters. Lower case indices can either be 4D or generic for definitions depending on context. Index raising is referred to a metric $\hat{g}_{AB}$ if 5-dimensional, and to g_{ab} if 4-dimensional. Terms that might repeat dummy variables or are otherwise in need of clarification use additional brackets. The domain of partial derivatives carries to the end of a term without need for brackets, so for example we have $\partial_a g_{db} A_c + g_{db} g_{ac} = (\partial_a (g_{db} A_c)) + (g_{db} g_{ac})$. Terms that might repeat dummy variables or are otherwise in need of clarification use additional brackets. Square brackets can be used to make dummy variables local in scope. Space-time is given signature (−, +, +, +), Kaluza space (−, +, +, +, +) in keeping with [6]. Under the Wheeler et al [6] nomenclature the sign conventions used here as a default are [+, +, +]. The first dimension (index 0) is time and the 5^{th} dimension (index 4) is the topologically closed Kaluza dimension. Time and distance are geometrized throughout such that c = 1. **G** is the gravitational constant. The scalar field component is labelled ϕ^2. The matrix of g_{cd} can be written as $|g_{cd}|$. The Einstein summation convention may be used without special mention. $\Box$ represents the 4D D'Alembertian [6].

Connection coefficients with torsion will take the form: Γ^c_{ab} or Γ^{abc}. The metric with an antisymmetric torsion tensor defines a unique connection ∇_a. The unique Levi-Civita connection (ie without torsion) is written as: $\mathit{\Gamma}^c_{ab}$, and the covariant Levi-Civita derivative operator (ie without torsion): $\triangle_a$. Define:

$$F_{ab} = \partial_a A_b - \partial_b A_a = \nabla_a A_b - \nabla_b A_a + \Gamma^c_{ab} A_c - \Gamma^c_{ba} A_c = \triangle_a A_b - \triangle_b A_a$$

$$F = dA \tag{2.0.1}$$

In order to distinguish tensors constructed using torsion G_{ab} and R_{ab} (i.e. where the Ricci tensor is defined in terms of Γ^c_{ab}) from those that do not use torsion (ie that are defined in terms of $\mathcal{F}^{c}_{ab}$), the torsionless case uses cursive: $\mathcal{G}_{ab}$ and $\mathcal{R}_{ab}$. On any given manifold with torsion both these parallel systems of connection coefficients and dependent tensors can be used. That is, the Ricci tensor (with torsion), R_{ab}, and the Ricci tensor, $\mathcal{R}_{ab}$, are both defined and are in general different. Further each of these can have hats on or hats off, giving: $\hat{R}_{AB}$ and $\hat{\mathcal{R}}_{AB}$. It is an extremely confusing part of this work that all four systems can be used at the same time in the same equations! This particularly occurs when the 4D components of a 5D tensor are being used, e.g. looking at $\hat{R}_{ab}$ and $\hat{\mathcal{R}}_{ab}$. Torsion introduces non-obvious conventions in otherwise established definitions. The order of the indices in the connection coefficients matters, and this includes in the Ricci tensor definition and the definition of the connection coefficient symbols themselves:

$$\nabla_a w_b = \partial_a w_b - \Gamma^c_{ab} w_c \tag{2.0.2}$$

Some familiar defining equations consistent with [1] define the Ricci tensor and Einstein tensors in terms of the connection coefficients along usual lines, noting that with torsion the order of indices can not be carelessly interchanged as they can with the symmetric Levi-Civita coefficients:

$$R_{ba} = \partial_c \Gamma^c_{ba} - \partial_b \Gamma^c_{ca} + \Gamma^c_{ba} \Gamma^d_{dc} - \Gamma^c_{da} \Gamma^d_{bc} \tag{2.0.3}$$

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab} = 8\pi \mathbf{G} T_{ab} \tag{2.0.4}$$

We will define $\alpha = \frac{1}{8\pi\mathbf{G}}$. Analogous definitions can also be used with the Levi-Civita connection to define $\mathcal{R}_{ab}$ and $\mathcal{G}_{ab}$ in the obvious way.

We also make reference to Kaluza's original field equations [1] in the text:

$$G_{ab} = \frac{k^2\phi^2}{2} \left\{ \frac{1}{4} g_{ab} F_{cd} F^{cd} - F^c_a F_{bc} \right\} - \frac{1}{\phi} \{ \nabla_a (\partial_b \phi) - g_{ab} \Box \phi \} \tag{2.0.5}$$

$$\nabla^a F_{ab} = -3 \frac{\partial^a \phi}{\phi} F_{ab} \tag{2.0.6}$$

$$\Box \phi = \frac{k^2 \phi^3}{4} F_{ab} F^{ab} \tag{2.0.7}$$

Note that there are convention differences above with Wald's [7] and Wheeler's Einstein-Maxwell equation [6], and thus also with this work.

In this paper much of the working has been omitted. More extensive detailing of the workings are given in [24], along with extra materials not core to this work. It should however be noted that the postulates are slightly different here. The Ricci curvature there is based on the conventions of [12], whereas here it's based on the conventions of Wald [7], but extended for torsion as defined in footnote 1 p31 of [7], and using the index position used in eqn(3.2.3) of [7] that defines the Riemannian curvature. Much basic working is detailed in [29]. The resulting difference to [24]'s notation is a reversal of the indices of the Ricci curvature. With torsion even the conventions of [7] and [6] cease to be identical. Indeed, the definition of Riemannian curvature in [6] would lead to a different sign in the definition of the torsion tensor and an inversion of the Ricci curvature indices with respect to [7].

In this paper orders of magnitude notation is used. In contrast to [24] a clearer notation will be used to indicate when terms are of a certain order of magnitude: $O(X)$. Versus when rounding has occured by ignoring terms of $O(X)$, ie "$O(X)$ terms discounted", which may be denoted $\backslash O(X)$.

3 Torsion

For both 5D and 4D manifolds torsion will be introduced into the connection coefficients as follows, using the notation of Hehl [11].

$$\frac{1}{2}(\Gamma^k_{ij} - \Gamma^k_{ji}) = S_{ij}{}^k \tag{3.0.1}$$

This relates to the notation of Kobayashi and Nomizu [12] and Wald [7] as follows:

$$T^i{}_{jk} = 2S_{jk}{}^i \equiv \Gamma^i_{jk} - \Gamma^i_{kj} \tag{3.0.2}$$

We have the contorsion tensor $K_{ij}{}^k$ [11] as follows, and a number of relations [11]:

$$\Gamma^k_{ij} = \frac{1}{2}g^{kd}(\partial_i g_{dj} + \partial_j g_{di} - \partial_d g_{ij}) - K_{ij}{}^k = \mathit{\Gamma}^k_{ij} - K_{ij}{}^k \tag{3.0.3}$$

$$K_{ij}{}^k = -S_{ij}{}^k + S_j{}^k{}_i - S^k{}_{ij} = -K_i{}^k{}_j \tag{3.0.4}$$

Notice how the contorsion is antisymmetric in the last two indices. With torsion included, the auto-parallel equation becomes [11]:

$$\frac{d^2x^k}{ds^2} + \Gamma^k_{(ij)}\frac{dx^i}{ds}\frac{dx^j}{ds} = 0 \tag{3.0.5}$$

$$\Gamma^k_{(ij)} = \mathit{\Gamma}^k_{ij} + S^k{}_{(ij)} - S_{(j}{}^k{}_{i)} = \mathit{\Gamma}^k_{ij} + 2S^k{}_{(ij)} \tag{3.0.6}$$

Only when torsion is completely antisymmetric is this the same as the extremals [11] which give the path of spinless particles and photons in Einstein-Cartan theory: this becomes none other than geodesics with respect to the Levi-Civita connection.

$$\frac{d^2x^k}{ds^2} + \mathit{\Gamma}^k_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = 0 \tag{3.0.7}$$

We can now apply this to some definitions in 5D that will be needed, invoking complete antisymmetry of torsion as required. Inspired by the Belinfante-Rosenfeld procedure [12][15], by defining the torsionless Einstein tensor in terms of torsion bearing components we can decompose it as follows:

$$\hat{\mathcal{G}}_{AB} = \hat{G}_{AB} + \hat{V}_{AB} + \hat{X}_{AB} \tag{3.0.8}$$

$$\hat{V}_{AB} = -\frac{1}{2}\hat{\nabla}^C(\hat{\sigma}_{ABC} + \hat{\sigma}_{BAC} + \hat{\sigma}_{CBA}) \tag{3.0.9}$$

Where σ is defined as the spin tensor in Einstein-Cartan theory simply extended to 5D.

$$\hat{\sigma}_{ABC} = 2\hat{S}_{ABC} + 2\hat{g}_{AC}\hat{S}_{BD}{}^{D} - 2\hat{g}_{BC}\hat{S}_{AD}{}^{D} \tag{3.0.10}$$

This simplifies definition (3.0.9):

$$\hat{V}_{AB} = -\frac{1}{2}\hat{\nabla}^C(\hat{\sigma}_{CBA}) = -\hat{\nabla}^C(\hat{S}_{CBA} + \hat{g}_{CA}\hat{S}_{BD}{}^{D} - \hat{g}_{BA}\hat{S}_{CD}{}^{D}) \tag{3.0.11}$$

By considering symmetries and antisymmetries we get a divergence law for 5D spin sources:

$$\hat{\nabla}^B\hat{V}_{AB} = 0 \tag{3.0.12}$$

Note that the mass-energy-charge divergence law for the torsionless Einstein tensor is in terms of the torsionless connection, but the spin source divergence law here is in terms of the torsion-bearing connection. However, for completely antisymmetric torsion we have:

$$\hat{\nabla}^A\hat{V}_{AB} = 0 \tag{3.0.13}$$

$$\hat{\nabla}_C\hat{\mathcal{G}}_{AB} = \hat{\triangle}_C\hat{\mathcal{G}}_{AB} + \hat{K}_{CA}{}^{D}\hat{\mathcal{G}}_{DB} + \hat{K}_{CB}{}^{D}\hat{\mathcal{G}}_{AD}$$

$$\hat{\nabla}^A\hat{\mathcal{G}}_{AB} = -\hat{K}_{B}{}^{AD}\hat{\mathcal{G}}_{AD} = -\hat{K}_{B}{}^{AD}\hat{\mathcal{G}}_{DA} = +\hat{K}_{B}{}^{AD}\hat{\mathcal{G}}_{AD} = 0 \tag{3.0.14}$$

$$\hat{\nabla}^A\hat{\mathcal{G}}_{AB} = 0 \tag{3.0.15}$$

$$\hat{\nabla}^A(\hat{G}_{AB} + \hat{X}_{AB}) = 0 \tag{3.0.16}$$

And so there is a stress-energy divergence law with respect to the torsion connection also, at least in the completely antisymmetric case. Further, still assuming complete antisymmetry of torsion, by definition of the Ricci tensor:

$$\begin{aligned}\hat{R}_{AB} &= \hat{\mathcal{R}}_{AB} + \hat{K}_{DA}{}^{C}\hat{K}_{BC}{}^{D} - \partial_C\hat{K}_{BA}{}^{C} - \hat{K}_{BA}{}^{C}\hat{\digamma}^{D}_{DC} + \hat{K}_{DA}{}^{C}\hat{\digamma}^{D}_{DC} - \hat{K}_{DB}{}^{C}\hat{\digamma}^{D}_{AC} \\ &= \hat{\mathcal{R}}_{AB} - \hat{K}_{AD}{}^{C}\hat{K}_{BC}{}^{D} - \hat{\nabla}^C\hat{S}_{ABC} \end{aligned} \tag{3.0.17}$$

$$\hat{G}_{[AB]} = \hat{R}_{[AB]} = -\hat{\nabla}^C\hat{S}_{ABC} = -\hat{V}_{AB} \tag{3.0.18}$$

$-\hat{V}_{AB}$ is the antisymmetric part of $\hat{G}_{AB}$ at this limit. And $\hat{X}_{AB}$ becomes a symmetric spin-torsion coupling adjustment. Complete antisymmetry of torsion simplifies the issue of divergence laws that otherwise appears more complex in [24].

4 Overview Of Kaluza-Cartan Theory

4.1 Postulates

The following K1-K4 are the core Postulates of the present Kaluza-Cartan theory.

POSTULATE (K1): **Geometry**. A *Kaluza-Cartan manifold* is a 5D smooth Lorentzian manifold with completely antisymmetric (and necessarily metric) torsion connection.

POSTULATE (K2): **Well-behaved**. Kaluza-Cartan space is assumed globally hyperbolic in the sense that there exists a 4D spatial cauchy surface plus time, such that the 4D hypersurface is a simply connected 3D space extended around a 1D loop. And Kaluza-Cartan space is oriented and time-oriented.

POSTULATE (K3) version1: We start with effectively the old and original version: **Cylinder condition (original)**. One spatial dimension is topologically closed and 'small', the Kaluza dimension. This is taken to mean that there are global unit vectors that define this direction, the Kaluza direction. The partial derivatives of all tensors in this Kaluza direction are taken to be zero in some coordinate system. The other spatial dimensions and time dimension are 'large'. 'Large' here simply means that the considerations given to 'small' do not apply. We further add an additional constraint, **Cylinder condition (additional)**. The covariant derivative $\hat{\nabla}_\kappa$ (with torsion) of all tensors in the Kaluza direction differ from the partial derivative terms (i.e zero) by order $O(l) \approx O(h^2)$ terms.

POSTULATE (K4): **Geodesic Assumption**. That any model of a charged particle approximately follows 5D auto-parallels.

Definitions 4.1.1: The *Kaluza-Cartan vacuum* is a Ricci flat region of a Kaluza-Cartan manifold with respect to the torsion connection definition of the Ricci tensor. Similarly the *Kaluza vacuum* is a Ricci flat region with respect to the Levi-Civita connection. They are different: $\hat{R}_{AB} = 0$ and $\hat{\mathcal{R}}_{AB} = 0$ respectively. Here they are both defined in terms of the geometry implied by the cylinder condition. Kaluza vacuum will be associated with nullish electromagnetic solutions when there is no torsion, Kaluza-Cartan vacuum will encompass all electromagnetic fields. *Kaluza-Cartan matter* and *Kaluza mass-energy* follow as complements to their vanishing respective Ricci tensors.

Observe that Kaluza-Cartan matter, unlike Kaluza mass-energy, but like common matter, does not have its own divergence law. This is not a problem, it just means that the Kaluza mass-energy more closely resembling the original Einsteinian mass-energy is the correct definition with respect to conservation, and the Kaluza-Cartan matter potentially more similar to tangible matter. Less exploratory than [24], here the electromagnetic field will be a priori identified with the Kaluza-Cartan vacuum:

Definition 4.1.2: The *classical fields limit* is defined to be an area of Kaluza-Cartan space which is Kaluza-Cartan vacuum, or at least approximately so to at least \$O(h^2)$, and is to be identified with extended fields such as gravitational and electromagnetic, and characterized by the absence of what is normally called matter.

LIMIT POSTULATE (B1): There is a Kaluza atlas, see definition (4.2.1), possibly only over a region, such that $\phi^2 = 1$ at every point. The scalar field results from the the decomposition of the Kaluza metric into 4D metric, potential vector and scalar field. It is contained within the metric explicitly in (4.4.1). Thus B1 is a constraint on the 5D metric.

Additional postulates that can be interpreted as forming conditions necessary for a classical limit now follow. L1-L2 constitute a *weak field limit* that will be applied by way of approximation for the classical limit of behaviour. The deviation from the 5D-Minkowski metric is given by a tensor $\hat{h}_{AB}$. This tensor belongs to a set of small tensors that we might label $O(h)$. Whilst this uses a notation similar to orders of magnitude, and is indeed analogous, the meaning here is a little different. This is the weak field approximation of general relativity using a more flexible notation. Partial derivatives, to whatever order, of metric terms in a particular set $O(x)$ will be in that same set at the weak field limit. In principle we are doing nothing more than following the weak field limit procedure [6] of general relativity. In the weak field approximation of general relativity, terms that consist of two $O(h)$ terms multiplied together get discounted and are treated as vanishing at the limit. We might use the notation $O(h^2)$ to signify such terms. There is the weak field approximation given by discounting $O(h^2)$ terms. But we might also have a less aggressive limit given by, say, discounting $O(h^3)$ terms, and so on. We can talk about weak field limits (plural) that discount $O(h^n)$ terms for $n > 1$ based on the same underlying construction. The use of orders of magnitude in these axioms can be interpeted, in addition to imposing geometric constraints, as essentially a choice of scale. That choice of scale, even if mathematically arbitrary, is physically meaningful: the classical scale.

LIMIT POSTULATE (L1): The metric can be written as follows in terms of the 5D Minkowski tensor and $\hat{h} \in O(h)$: $\hat{g}_{AB} = \hat{\mu}_{AB} + \hat{h}_{AB}$.

Torsion will also be considered a weak field under normal observational conditions, similarly to L1. Torsion is defined in terms of the Christoffel symbols. Christoffel symbols are in part constructed from the partial derivatives of the metric and that part is constrained by L1 to be $O(h)$. The contorsion term being the difference. See [11]. The contorsion (and therefore the torsion) will be treated as $O(h)$ accordingly.

LIMIT POSTULATE (L2): The contorsion (and therefore the torsion) will be an $O(h)$ term at the weak field limits.

SUPER-ENERGY POSTULATE (SE1): That *The conserved superenergy with torsion hypothesis* holds. That is, that the divergence (Levi-Civita) of the generalized torsion Bel superenergy tensor [17][18] is vanishing everywhere using the superenergy tensor of the torsion-bearing Riemannian curvature tensor as defined in [29], where [29] gives the necessary and straight forward extension of Wald's definitions [7] to include torsion.

This both ensures local causality of the torsion-bearing Riemannian curvature tensor as proven in [16] and provides a well-defined conservation law, by definition, for 5D Kaluza-Cartan space, that can potentially be used in place of the energy conditions. This also overcomes the need for a lagrangian-based approach. This is analogous to the famous Bel-Robinson superenergy tensor [28][17][18] which is completely symmetric, curiously, in exactly the 4D and 5D cases. The use of torsion was included here in contrast to [24] as the SE1 of [24] imposes constraints on the electromagnetic field that are here avoided. The superenergy thus defined remains a doubly symmetric 4-tensor, but need not be symmetric in the sense of (31) in [17]. The local causality (concept detailed in [16]) imposed by SE1 here also ensures the local causality of the tensor $\hat{R}_{AB}$. This is therefore a more natural superenergy tensor for Kaluza-Cartan theory than the version without torsion [24]. There is no claim here that this is the unique or correct way to impose causality, it simply shows that alternatives to more common approaches such as lagrangian methods exist. As mentioned in [24] it has not been proven that the Levi-Civita divergence operator can be replaced by the torsion-bearing covariant derivative and for the results of [16] to still hold.

Completely antisymmetric torsion ensures that normal coordinates are defined. Though, unlike [24], the Kaluza direction is not here identical with such. We here use index 4 for the 5D normal coordinates approximating the Kaluza direction, and index κ to indicate the Kaluza direction proper. We will however assume that the normal neighbourhoods are small enough to minimize any such differences. For the sake of rigour and future analysis this can be included, for the purposes of this work, in the postulate list:

APPROXIMATION LIMIT (L3): The approximation that index $\kappa \approx$ index 4 in normal coordinates will be made.

All the so-called limit postulates are potentially disposable if the theory is to be interpreted in the widest possible context. In this preliminary paper however they are necessary for the classical limit.

4.2 The Cylinder Condition And Charts

The cylinder condition by construction allows for an atlas of charts wherein the Kaluza dimension is approximately presented by the fourth index. The atlases that are compliant are restricted. This means that the cylinder condition can be represented by a subatlas of the maximal atlas. The set of local coordinate transformations that are compliant with this atlas (called a Kaluza atlas) is non-maximal by construction. A further reduction in how the atlas might be interpreted is also implied by setting c=1, and constant **G**. The existence of a single unit for space and time can be assumed, and this must be scaled in unison for all dimensions. Consistently with cgs units we can choose either centimetres or seconds. This would leave velocities (and other geometrically unitless quantities) unchanged in absolute magnitude. This doesn't prevent reflection of an axis however, and indeed reflection of the Kaluza dimension is here equivalent to a (kinetic) charge inversion. However, given orientability and an orientation we can remove even this ambiguity. We can further reduce a Kaluza atlas by removing boosts in the Kaluza dimension. Space-time is taken to be a subframe within a 5D frame within a Kaluza subatlas of a region wherein uncharged matter can be given a rest frame via a 4D Lorentz transformation. Boosting uncharged matter along the Kaluza axis will give it kinetic charge. The Kaluza atlas represents the 4D view that kinetic charge is 4D covariant. The justification for this assertion will be given later. Rotations into the Kaluza axis can likewise be omitted. This results in additional constraints on the connection coefficients associated with charts of this subatlas, and enables certain geometrical objects to be more easily interpreted in space-time. The use of this subatlas does not prevent the theory being generally covariant, but simplifies the way in which we look at the Kaluza space through a 4D physical limit.

Definition 4.2.1: A*Kaluza atlas* is:

(i) A subatlas (possibly just over a region) of the maximal atlas of Kaluza-Cartan space where boosts and rotations into the Kaluza dimension (as defined by the cylinder condition K3) are explicitly omitted.

(ii) All partial derivatives in the Kaluza direction are vanishing.

(iii) Inversion in the Kaluza direction and rescalings can also be omitted so as to establish units and orientation.

(iv) For each point on the Kaluza atlas a chart exists with 'torsion-normal' coordinates where index 4 is the Kaluza dimension.

4.3 Kinetic Charge

Kinetic charge is defined as the 5D momentum component in terms of the 5D Kaluza rest mass of a hypothesised particle: ie (i) its rest mass in the 5D Lorentz manifold (m_{k0}) and (ii) its proper Kaluza velocity ($dx_4/d\tau^*$) with respect to a frame in the maximal atlas that follows the particle. And equally it can be defined in terms of (i) the relativistic rest mass (m_0), relative to a projected frame where the particle is stationary in space-time, but where non-charged particles are stationary in the Kaluza dimension, and in terms of (ii) coordinate Kaluza velocity (dx_4/dt_0):

Prov. Definition 4.3.1: kinetic charge: $Q^* = m_{k0}dx_4/d\tau^* = m_0 dx_4/dt_0$

This makes sense because mass can be written in fundamental units (i.e. in distance and time). And the velocities in question defined relative to particular frames. It is not a generally covariant definition but it is nevertheless mathematically meaningful. This kinetic charge can be treated in 4D space-time, and the Kaluza atlas, as a scalar: the first equation above is covariant with respect to the Kaluza atlas. It can be generalized to a 4-vector, and it is also conserved as shown. In general relativity at the special relativistic Minkowski limit the conservation of momenergy can be given in terms of the stress-energy tensor as follows [9], $j \neq 0$. This is approximately true at a weak field limit and can be applied equally to Kaluza theory, via the (torsionless) connection. We have a description of conservation of (torsionless) momentum in the 5th dimension.

$$\frac{\partial \hat{T}^{00}}{\partial t} + \frac{\partial \hat{T}^{i0}}{\partial x_i} = 0 \ , \ \frac{\partial \hat{T}^{0j}}{\partial t} + \frac{\partial \hat{T}^{ij}}{\partial x_i} = 0 \text{ and } \frac{\partial \hat{T}^{04}}{\partial t} + \frac{\partial \hat{T}^{i4}}{\partial x_i} = 0 \tag{4.3.2}$$

We also have i=4 vanishing by the cylinder condition. Thus the conservation of kinetic charge becomes (when generalized to different space-time frames) the property of a 4-vector current, which we know to be locally conserved: $\partial_0 \hat{T}^{04} + \partial_1 \hat{T}^{14} + \partial_2 \hat{T}^{24} + \partial_3 \hat{T}^{34} = 0$.

To make sense of this in 5D we need to change the provisional definition above and make it density-based as follows (imagine a ring rather than a particle). The alternative definition can be made in terms of the mass density ρ_0, coupled with the Kaluza dimension's size or Kaluza length λ. In this way we do not presuppose that the rest mass we observe in space-time is necessarily the m_0 above: what happens for example to the apparent rest mass in 4D if the Kaluza distance changes and the density compressed or rarefacted? m_0 makes most sense as a definition of rest mass in 4D when this does not happen. Generalization demands the following definition, replacing m_0 with a density:

Definition 4.3.3: 5D kinetic charge: $Q^* = \lambda \rho_{k0} dx_4/d\tau^* = \lambda \rho_0 dx_4/dt_0$

This leads to a density-slice definition of 4D density-based kinetic charge as follows (noting that it is not 4D-divergence free in the event that λ changes):

Definition 4.3.4: 4D kinetic charge density: $Q^{**} = \rho_{k0} dx_4/d\tau^* = \rho_0 dx_4/dt_0$

Kinetic charge current density is the 4-vector, induced from 5D Kaluza-Cartan space as follows (using the Kaluza atlas to ensure it is well-defined as a 4-vector):

$$J^{**a} = -\alpha \hat{\mathcal{G}}^{a4} \tag{4.3.5}$$

And a measure of the total current can be give as:

$$J^{*a} = -\alpha \lambda \hat{\mathcal{G}}^{a4} \tag{4.3.6}$$

Using Wheeler et al [6] p.131, and the appropriate space-time (or Kaluza atlas) frame, we have:

$$Q^* = J^*_a (1, 0, 0, 0)^a \tag{4.3.7}$$

So we have a scalar, then a vector representation of relativisitic invariant charge current, and finally a 2-tensor unification with conserved (torsionless) mass-energy via the (torsionless) Einstein tensor. It follows that the vanishing of the divergence of kinetic charge in 4D is only approximate, in 5D not.

Definition 4.3.8: *Kinetic charge current* is defined to be the 4-vector $J^{*a} = -\alpha\lambda\hat{\mathcal{G}}^{a4}$, with respect to the Kaluza atlas that represents this total charge current in 4D. Note the divergence of the (torsionless) Einstein tensor:

$$\hat{\triangle}_A \hat{\mathcal{G}}^{AB} = 0 \text{ and } \hat{\triangle}_A \hat{\mathcal{G}}^{A4} = 0 \approx \hat{\triangle}_a \hat{\mathcal{G}}^{a4}$$

Due to complete antisymmetry, and unlike in [24], the above also holds with respect to the torsion connection covariant derivative.

4.4 Two Types Of Geometrized Charge

Components used in [1] will be used here as the Kaluza-Cartan metric. The vector potential and electromagnetic fields formed via the metric are sourced in Maxwell charge Q_M. Maxwell's law are automatically satisfied, using (2.0.1) to define F with respect to the potential: dF=0 follows from dd = 0. d*F= 4π*J can be set by construction. d*J=0. A_a is to be identified with the electromagnetic potential, ϕ^2 is to be a scalar field, and g_{ab} the metric of 4D space-time:

Definition 4.4.1: The 5D Kaluza-Cartan metric.

$$\hat{g}_{AB} = \begin{bmatrix} g_{ab} + k^2\phi^2 A_a A_b & k\phi^2 A_a \\ k\phi^2 A_b & \phi^2 \end{bmatrix} \text{ and } \hat{g}^{AB} = \begin{bmatrix} g^{ab} & -kA^a \\ -kA^b & \frac{1}{\phi^2} + k^2 A_i A^i \end{bmatrix} \tag{4.4.1}$$

This gives (without torsion [1]) nullish solutions under the original Kaluza cylinder condition and constant scalar field, such that $G_{ab} = -\frac{k^2}{2}\ F_{ac}F^c_b$. Compare this with [7] where we have $G_{ab} = 2F_{ac}F^c_b$ in geometrized units for ostensibly the same fields. The units need to be agreed between the two schemes. We would need to set either $k = 2$ or $k = -2$ for compatibility of results and formulas. And this is particularly important as we wish to derive the Lorentz force law with the same units as [7]. N.B. the sign change introduced by [1] - where it appears that the Einstein tensor was defined relative to $(+, -, -, -)$, despite the 5D metric tensor being given in a form that can only be $(-, +, +, +, +)$, which is confusing. This makes no fundamental difference, but must be noted. It is a confusion seemingly introduced by accident in [1]. The use of conventions in this type of work are excruciatingly tricky.

The geometrized units, Wald [7] p470-471, define units of mass in terms of fundamental units. This leads to an expression for kinetic charge in terms of Kaluza momentum when $k = 2$ and $\mathbf{G} = 1$. $\mathbf{G}$ and k are not independent however. If we fix one, the other is fixed too: A consequence of requiring the Lorentz force law written in familiar form and compatibility with the units used in [7]. The relation between $\mathbf{G}$ and k is given in equation (6.3.8) via the derivation of the Lorentz force law. Simple compatibility with Wald [7] results where $k = 2$ and $\mathbf{G} = 1$. The sign of k is also fixed by (6.1.4). The result of dimensional analysis gives kinetic charge, Q^*, in terms of a total 5D momentum component P_4 and its corresponding density P_4^*:

$$Q^* = \frac{c}{\sqrt{\mathbf{G}}}\lambda P_4^* = \frac{c}{\sqrt{\mathbf{G}}}P_4 \tag{4.4.2}$$

5 The Field Equations

5.1 The New Cylinder Condition And Scalar Field, $k = 1$

Here we look at how the new Kaluza-Cartan cylinder condition affects the connection coefficients of any coordinate system within the Kaluza atlas. The following requires the selection of coordinates (the Kaluza atlas) that set the partial derivatives in the Kaluza dimension to zero and from the relationship between these two and the Christoffel symbols given in Wald [7] p33 eqn (3.1.14) as applied to a number of test vectors. Note that there is no symmetry of the (with torsion) connection coefficients suggested here. That is, these terms are forced zero by the fact that both the partial derivatives and the covariant derivatives in the Kaluza direction are zero. Cf equation (2.0.2), where the consequences of setting both the partial derivatives and the covariant derivative to zero can be seen on the connection coefficients.

$$2\hat{\Gamma}^A_{4c} = \sum_d \hat{g}^{Ad}(\partial_4 g_{cd} + \partial_4\phi^2 A_c A_d + \partial_c\phi^2 A_d - \partial_d\phi^2 A_c) + \hat{g}^{A4}\partial_c\phi^2 - 2\hat{K}_{4c}{}^A \text{ is } O(h^2) \tag{5.1.1}$$

$$2\hat{\Gamma}^A_{44} = 2\sum_d \hat{g}^{Ad}\partial_4\phi^2 A_d - \sum_d \hat{g}^{Ad}\partial_d\phi^2 + \hat{g}^{A4}\partial_4\phi^2 - 2\hat{K}_{44}^{\ A} \text{ is } O(h^2) \tag{5.1.2}$$

$$2\hat{K}_{4c}{}^A = \hat{g}^{Ad}(\partial_c\phi^2 A_d - \partial_d\phi^2 A_c) + \hat{g}^{A4}\partial_c\phi^2 + O(h^2) \tag{5.1.3}$$

$$2\hat{K}_{44}^{\ A} = -\hat{g}^{Ad}\partial_d\phi^2 + O(h^2) = 0 \tag{5.1.4}$$

This puts limits on the scalar field notwithstanding postulate B1. These equations, now applying B1, give the contorsion a clear interpretation in terms of the electromagnetic field:

$$2\hat{K}_{4c}{}^A = \hat{g}^{Ad}(\partial_c A_d - \partial_d A_c) + O(h^2) \tag{5.1.5}$$

$$\hat{K}_{4c}{}^a = \frac{1}{2}F^a_c + O(h^2) \tag{5.1.6}$$

$$\hat{K}_{4c}{}^4 = -\frac{1}{2}A^d F_{cd} + O(h^2) = 0 \tag{5.1.7}$$

Here the postulate that $O(l) \approx O(h^2)$ is seen not to be arbitrary. We also have from (3.0.3) the following:

$$\hat{\Gamma}^4_{4c} = \hat{\Gamma}^4_{c4} = \hat{\digamma}^{\,4}_{4c} = -\frac{1}{2}A^d F_{cd} \tag{5.1.8}$$

Certain analogous equations to those in [24] cease to apply. And thus the postulate in [24] labelled L3 is nolonger necessary here.

5.2 The First Field Equation With Torsion, $k = 1$

Looking at the Ricci tensors gives:

$$\hat{R}_{ba} = \partial_c \hat{\Gamma}^c_{ba} - \partial_b \hat{\Gamma}^c_{ca} + \hat{\Gamma}^C_{ba} \hat{\Gamma}^D_{DC} - \hat{\Gamma}^C_{Da} \hat{\Gamma}^D_{bC} \tag{5.2.1}$$

$$\hat{\mathcal{R}}_{ab} = \partial_c \hat{\digamma}^{\ c}_{ba} - \partial_b \hat{\digamma}^{\ c}_{ca} + \frac{1}{2} \partial_b (A^d F_{ad}) + \hat{\digamma}^{\ c}_{ba} \hat{\digamma}^{\ D}_{Dc} - \hat{\digamma}^{\ C}_{Da} \hat{\digamma}^{\ D}_{bC} \tag{5.2.2}$$

In the Kaluza theory without torsion, where the electromagnetic fields are identified with a Ricci flat Kaluza vacuum (ie $\hat{\mathcal{R}}_{ab} = 0$), the Ricci flatness leads to a constraint helping to impose nullish solutions when there is no scalar field. This is the analagous equation to (2.0.5). Without sources the remaining significant term is a nullish solution:

$$\begin{aligned} \mathcal{R}_{ab} &= \mathcal{R}_{ab} - \hat{\mathcal{R}}_{ab} \\ &= -\frac{1}{2} A_b \partial_c F_a^{\ c} - \frac{1}{2} A_a \partial_c F_b^{\ c} + \frac{1}{2} F_{ac} F_b^{\ c} \\ &\quad -\frac{1}{2} (A_b F_a^{\ c} + A_a F_b^{\ c}) \digamma^{\ d}_{dc} + \frac{1}{2} \digamma^{\ c}_{da} A_b F_c^{\ d} + \frac{1}{2} A_a F_b^{\ c} \digamma^{\ d}_{bc} \\ &\quad +\frac{1}{4} (A_d F_a^{\ c} + A_a F_d^{\ c})(A_b F_c^{\ d} + A_c F_b^{\ d}) + \frac{1}{4} A^d F_{ad} A^c F_{bc} \end{aligned} \tag{5.2.3}$$

Similarly [24] identifying electromagnetism with a sourceless Kaluza vacuum in Kaluza-Cartan theory is exactly the same thing, as torsion is just a deformation of the connection. However, and this is a crucial point in this theory, by identifying electromagnetism with the Kaluza-Cartan vacuum instead of the Kaluza vacuum (ie $\hat{R}_{ab} = 0$) simply eliminates this constraint:

$$\begin{aligned} \mathcal{R}_{ab} = \mathcal{R}_{ab} - \hat{R}_{ba} &= \partial_c \digamma^{\ c}_{ba} - \partial_b \digamma^{\ c}_{ca} + \digamma^{\ c}_{ba} \digamma^{\ d}_{dc} - \digamma^{\ c}_{da} \digamma^{\ d}_{bc} \\ &\quad -\partial_c \hat{\Gamma}^c_{ba} + \partial_b \hat{\Gamma}^c_{ca} - \hat{\Gamma}^C_{ba} \hat{\Gamma}^D_{DC} + \hat{\Gamma}^C_{Da} \hat{\Gamma}^D_{bC} \qquad (5.2.4) \\ &\approx \partial_c \hat{K}_{ba}^{\ \ c} \text{ to } \backslash O(h^2) \qquad (5.2.5) \end{aligned}$$

At first this looks like an unlikely field equation linking local contorsion sources to electromagentic fields: because of symmetry versus anti-symmetry both sides must be of order smaller than $O(h)$, so this equation simply says that the left and right side here are smaller than $O(h)$. We have to add in less significant terms to order $\backslash O(h^3)$. We also separate the symmetric and antisymmetric parts:

$$\mathcal{R}_{ab} = \mathcal{R}_{ab} - \hat{R}_{(ba)} = -\frac{1}{2} \partial_c (A_b F^c_a) - \frac{1}{2} \partial_c (A_a F^c_b) + \frac{1}{4} \partial_b (A_c F^c_a) + \frac{1}{4} \partial_a (A_c F^c_b)$$

$$= \frac{1}{2}F_{bc}F^c_a + \frac{1}{4}(A_c\partial_b F^c_a + A_c\partial_a F^c_b - \partial_c(A_b F^c_a) - \partial_c(A_a F^c_b)) - \frac{1}{2}(A_b\partial_c F^c_a + A_a\partial_c F^c_b) \tag{5.2.6}$$

$$\hat{R}_{[ba]} = 0 = -\partial_c \hat{K}_{ba}{}^c - \frac{1}{4}\partial_b(A_c F^c_a) + \frac{1}{4}\partial_a(A_c F^c_b) - \digamma^{\,d}_{dc}\hat{K}_{ba}{}^c + \digamma^{\,c}_{ad}\hat{K}_{bc}{}^d - \digamma^{\,c}_{bd}\hat{K}_{ac}{}^d \tag{5.2.7}$$

These are more reasonable field equation.

5.3 The Second Field Equation With Torsion

Rederivation of the second field equation under the present cylinder condition and postulates gives:

$$\hat{\mathcal{R}}_{a4} = \frac{1}{2}\partial_c F^c_a + \frac{1}{2}F^c_a \digamma^{\,d}_{dc} + \frac{1}{4}F^c_a A^d F_{cd} - \frac{1}{2}(\digamma^{\,c}_{da} + \frac{1}{2}(A_d F^c_a + A_a F^c_d))F^d_c$$

Looking at this at an \O$(h^2)$ L1-L2 weak field limit (re-inserting general k):

$$\hat{\mathcal{R}}_{a4} \to \frac{k}{2}\partial_c F^c_a \tag{5.3.1}$$

This couldn't be a clearer conception of Maxwell charge. This coincides with the Einstein (without torsion) tensor at the same limit, thus providing an alternative conception of the conservation of Maxwell charge locally (cf 6.1.1 and 6.1.2):

$$\hat{\mathcal{G}}_{a4} \to \hat{\mathcal{R}}_{a4} \to \frac{k}{2}\partial_c F^c_a \tag{5.3.2}$$

On the other hand, by definition (and the cylinder condition), we can deduce to \O$(h^3)$ by using (5.1.6):

$$\hat{R}_{4a} \to \partial_c \Gamma^c_{4a} \text{ and is } O(h^2) \text{ small} \tag{5.3.3}$$

Whereas $\hat{R}_{b4}$ simplifies at the \O$(h^2)$ limit to:

$$\hat{R}_{b4} \to \partial_c \Gamma^c_{b4} \to \partial_c F^c_b \tag{5.3.4}$$

This is also approximately conserved Maxwell charge (re-inserting general k) given at the \O$(h^2)$ L1-L2 weak field limit. Using equation (5.1.6):

$$\hat{R}_{b4} \to k\partial_c F^c_b \tag{5.3.5}$$

This means that the Kaluza-Cartan vacuum may not have stray charges in it of any significance, which is a required quality of a sourceless electromagnetic field. Any low significance charge source, further, necessarily implies antisymmetric components of the Kaluza-Cartan Ricci tensor: $\frac{1}{2}(\hat{R}_{4a} - \hat{R}_{a4})$, which at the classical field limit also implies no spin sources by (3.0.18). The Kaluza-Cartan vacuum can not contain significant spin or charge sources. The first field equations along with (5.3.3) suggests that the definition for the classical field limit might be more general if relaxed to order \O$(h^3)$. This does not effect the main results, and is not dealt with further here.

5.4 The Third Field Equation With Torsion, $k = 1$

This section shows how torsion releases the constraint of the third torsionless field equation (2.0.7), thus allowing non-nullish solutions. The constraint that the Ricci tensor be zero leads to no non-nullish solutions in the original Kaluza theory. This is caused by setting $\hat{\mathcal{R}}_{44} = 0$ in that theory and observing the terms. The result is that (when the scalar field is constant) $0 = F_{cd}F^{cd}$ in the original Kaluza theory. The same issue arises here:

$$\hat{\mathcal{R}}_{44} = \partial_C \hat{\digamma}^{C}_{\ 44} - \partial_4 \hat{\digamma}^{C}_{\ C4} + \hat{\digamma}^{C}_{\ 44}\hat{\digamma}^{D}_{\ DC} - \hat{\digamma}^{C}_{\ D4}\hat{\digamma}^{D}_{\ 4C} = -\hat{\digamma}^{c}_{\ d4}\hat{\digamma}^{d}_{\ 4c} = -\frac{1}{4}F^{c}_{d}F^{d}_{c} \tag{5.4.1}$$

Whilst we can have non-nullish solutions, we can only have them outside of a Kaluza vacuum. By definition and further calculation we get:

$$\hat{R}_{44} = -\frac{1}{4}F_{d}^{\ c}F^{d}_{c} + \hat{K}_{4d}^{\ \ c}\hat{K}_{4c}^{\ \ d} \text{ which is } O(h^3) \text{ small} \tag{5.4.2}$$

There is no reason in general for equation (5.4.1) to be 0, and so non-nullish solutions are generally available in the presence of torsion, providing we are not constrained to the Kaluza vacuum as with Kaluza's original theory.

6 The Lorentz Force Law

6.1 Kinetic And Maxwell Charge

Toth [8] derives a Lorentz-like force for static scalar field in the original Kaluza theory for a charge that is the momentum term in the fifth dimension. Here we make use of K4 to investigate this further. To investigate the relationship between kinetic charge and Maxwell charge we need the \$O(h^2)$ weak field limit defined by L1 (cf equation 5.3.2) and discounting $O(h^2)$ terms:

$$\hat{\mathcal{G}}^{a4} = \hat{\mathcal{R}}^{a4} - \frac{1}{2}\hat{g}^{a4}\hat{\mathcal{R}} = \hat{\mathcal{R}}^{a4} - \frac{1}{2}(-A^a)\hat{\mathcal{R}} \to \hat{\mathcal{R}}^{a4}$$

$$\hat{\mathcal{R}}^{a4} = \partial_C \hat{\digamma}^{C4a} - \partial^4 \hat{\digamma}^{C\ a}_{\ C} + \hat{\digamma}^{Cba}\hat{\digamma}^{D}_{\ DC} - \hat{\digamma}^{C\ a}_{\ D}\hat{\digamma}^{Db}_{\ \ C}$$

$$\hat{\mathcal{G}}^{a4} \to \hat{\mathcal{R}}^{a4} = \partial_c \hat{\digamma}^{c4a} \tag{6.1.1}$$

Putting k back in, and then using (4.3.8), we get:

$$\hat{\mathcal{R}}^{a4} \to \frac{1}{2}\partial_c k F^{ac} \tag{6.1.2}$$

$$J^*_a \to -\frac{\alpha k}{2}\lambda \partial_c F_a^{\ c} \tag{6.1.3}$$

So kinetic and Maxwell charges are related by a simple formula. The right hand side being Maxwell's charge current (see p.81 of [6]), and has the correct sign to identify a positive kinetic charge Q^* with a positive Maxwell charge source $4\pi Q_M$, whenever $\alpha k > 0$. In the appropriate space-time frame, and

Kaluza atlas frame, using (4.3.7), and approaching the \O$(h^2)$ limit given by L1:

$$4\pi Q_M \to +\frac{2}{\alpha k\lambda}Q^* \tag{6.1.4}$$

This correlates the two definitions of charge at the required limit and differs from [24] only due to the use of densities in the definition - allowing for the possibility of varying Kaluza length. Nevertheless we use throughout the same notation as [24], noting that $m_X \equiv p_X\lambda$.

6.2 A Lorentz-Like Force Law

Christoffel symbols will now be used to investigate the geodesic equation. We will here initially use $k = 1$, a general k can be added in later.

$$\hat{\Gamma}^c_{(4b)} = \tfrac{1}{2}\phi^2 F^c_b \quad -\tfrac{1}{2}g^{cd}A_b\delta_d\phi^2 \tag{6.2.1}$$

$$\hat{\Gamma}^c_{44} = \tfrac{1}{2}\hat{g}^{\mathrm{cD}}(\delta_4\hat{g}_{4D} + \delta_4\hat{g}_{4D} - \delta_D\hat{g}_{44}) = \text{-}\ \tfrac{1}{2}g^{\mathrm{cd}}\delta_d\phi^2 \tag{6.2.2}$$

$$\hat{\Gamma}^c_{(ab)} = \Gamma^c_{(ab)} + \tfrac{1}{2}g^{cd}(\delta_a(\phi^2 A_d A_b) + \delta_b(\phi^2 A_a A_d) - \delta_d(\phi^2 A_a A_b)) - A^c(\delta_a\phi^2 A_b + \delta_b\phi^2 A_a) \tag{6.2.3}$$

So, for any coordinate system within the maximal atlas:

$$\begin{aligned} 0 &= \tfrac{d^2x^a}{d\tau^2} + \hat{\Gamma}^a_{(BC)}\tfrac{dx^B}{d\tau}\tfrac{dx^C}{d\tau} \\ &= \tfrac{d^2x^a}{d\tau^2} + \hat{\Gamma}^a_{(bc)}\tfrac{dx^b}{d\tau}\tfrac{dx^c}{d\tau} + (\phi^2 F^a_b - g^{ad}A_b\delta_d\phi^2)\tfrac{dx^b}{d\tau}\tfrac{dx^4}{d\tau} - \tfrac{1}{2}g^{\mathrm{ad}}\delta_d\phi^2\tfrac{dx^4}{d\tau}\tfrac{dx^4}{d\tau} \end{aligned} \tag{6.2.4}$$

Taking $\phi^2 = 1$ and the charge-to-mass ratio to be:

$$Q'/m_{k0} = \frac{dx^4}{d\tau} \tag{6.2.5}$$

We derive a Lorentz-like force law. Then put k and variable ϕ back in:

$$\frac{d^2x^a}{d\tau^2} + \hat{\Gamma}^a_{(bc)}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = -(Q'/m_{k0})F^a_b\frac{dx^b}{d\tau} \tag{6.2.6}$$

$$= -k(Q'/m_{k0})(\phi^2 F^a_b - g^{ad}A_b\delta_d\phi^2)\frac{dx^b}{d\tau} - \frac{1}{2}g^{\mathrm{ad}}\delta_d\phi^2\frac{dx^4}{d\tau}\frac{dx^4}{d\tau} \tag{6.2.7}$$

6.3 Constant Kinetic Charge And The Lorentz Force Law

Having derived a Lorentz-like force law we look also at the momentum of the charge in the Kaluza dimension. We look at this acceleration as with the Lorentz force law. We have, with torsion (and $k = 1$):

$$\begin{aligned} 0 &= \frac{d^2x^4}{d\tau^2} + \hat{\Gamma}^4_{(BC)}\frac{dx^B}{d\tau}\frac{dx^C}{d\tau} \\ &= \frac{d^2x^4}{d\tau^2} + \hat{\Gamma}^4_{(bc)}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} + 2\hat{\Gamma}^4_{(4c)}\frac{dx^4}{d\tau}\frac{dx^c}{d\tau} + \frac{1}{2}A^d\delta_d\phi^2\frac{dx^4}{d\tau}\frac{dx^4}{d\tau} \end{aligned} \tag{6.3.1}$$

The two equations (6.3.1),(6.2.7) under B1 become (for all k):

$$\frac{d^2x^a}{d\tau^2} + \hat{\Gamma}^a_{(bc)}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = -k(Q'/m_{k0})F^a_b\frac{dx^b}{d\tau} \tag{6.3.2}$$

$$\frac{d^2x^4}{d\tau^2} + \hat{\Gamma}^4_{(bc)}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = -k^2(Q'/m_{k0})A_cF^c_b\frac{dx^b}{d\tau} \tag{6.3.3}$$

Multiplying both sides of (6.3.2) by $\frac{d\tau}{d\tau'}\frac{d\tau}{d\tau'}$, where τ' is an alternative affine coordinate frame, gives:

$$\frac{d^2x^a}{d\tau'^2} + \hat{\Gamma}^a_{(bc)}\frac{dx^b}{d\tau'}\frac{dx^c}{d\tau'} = -k\frac{d\tau}{d\tau'}(Q'/m_{k0})F^a_b\frac{dx^b}{d\tau'} \tag{6.3.4}$$

Given $Q^* = Q'\frac{d\tau}{d\tau^*}$ and therefore $\frac{m_{k0}}{m_0}Q^* = Q'\frac{d\tau}{dt_0}$ by definition, we can set the frame such that $\tau' = t_0$ via the projected 4D space-time frame of the charge. And the Lorentz force is derived:

$$\frac{d^2x^a}{d\tau'^2} + \hat{\Gamma}^a_{(bc)}\frac{dx^b}{d\tau'}\frac{dx^c}{d\tau'} = -k(Q^*/m_0)F^a_b\frac{dx^b}{d\tau'} \tag{6.3.5}$$

In order to ensure the correct Lorentz force law using the conventions of Wald [7] p69, this can be rewritten as follows, using the antisymmetry of $F^a_b = -F^a{}_b$:

$$= k(Q^*/m_0)F^a{}_b\frac{dx^b}{d\tau'} \tag{6.3.6}$$

Using (6.1.4) - only here does the calculation vary from [24] - as its L1 weak field limit is approached, this can be rewritten again in terms of the Maxwell charge:

$$\to k(\frac{\alpha k}{2}(4\pi Q_M\lambda)/m_0)F^a{}_b\frac{dx^b}{d\tau'} \tag{6.3.7}$$

The result is that we must relate $\mathbf{G}$ and k to obtain the Lorentz force law in acceptable terms:

$$\frac{d^2x^a}{d\tau'^2} + \hat{\Gamma}^a_{(bc)}\frac{dx^b}{d\tau'}\frac{dx^c}{d\tau'} \to (Q_M/\rho_0)F^a{}_b\frac{dx^b}{d\tau'} \quad \text{and} \quad k = 2\sqrt{\mathbf{G}} \tag{6.3.8}$$

This shows that the Lorentz force law proper can be derived given (6.1.4) and the required limit. This of course suggests that in this variant of the theory the universality of Lorentz force law is dependent on the constancy, or approximate constancy, or local constancy, of the Kaluza length. This is in contrast to the analysis in [24] which did not make this apparent.

7 Discussion And Interpretation

The sourceless electromagnetic field was here identified with Kaluza-Cartan vacuum. This is the Kaluza-Cartan vacuum as opposed to the Kaluza vacuum. It has tight restrictions on the presence of charge and spin sources: it approximately follows the sourceless Maxwells laws and is deterministic, though it has not been proven to be stable beyond certain limits, limits defined by the smallness of the tensor X_{AB} in the discussion on conservation laws. The theory as a whole reproduces classical theory as a limit. However the derivations of classical behaviour did not necessitate this identification as such as the derivations, such as that of the Lorentz force law, are quite general. Thus variants are possible depending on exactly which solutions are empirically required, and to what level of limit accuracy required. Indeed this work is a second variant formulation, a sequel to [24].

The fundamental divergence-free law for mass-energy belongs to the (torsionless) Einstein tensor. Kinetic charge is similarly fundamental in its zero divergence under definition (4.3.8). The correlation with Maxwell charge follows as a limit from (6.1.4), and the Lorentz force law provided the Kaluza length remains more or less constant. With respect to spin, spin current obeys the divergence law (3.0.12). This is then also a fundamental quantity in Kaluza-Cartan theory, and complementary to (torsionless) mass-energy, though divergence-free only relative to the 5D torsion connection. Matter and fields are able to transfer Kaluza matter (the complement of the Kaluza vacuum) to and from each other. Unlike (torsionless) mass-energy, the divergence law for Kaluza-Cartan matter depends on the torsion tensor as seen by combining (3.0.12) and (3.0.16). It is only vanishing because the torsion is completely antisymmetric.

Maxwell charge requires spin, at least at a local $\backslash O(h^2)$ L1-L2 weak field limit. This follows from (3.0.18) and (5.3.5). By definition of kinetic charge, components of 5D (torsionless) mass-energy are also required. A matter model defined by Kaluza-Cartan matter can have charge, but stray charges in a Kaluza-Cartan vacuum region are limited in significance by the weak field assumptions. Further a minimum component of Kaluza-Cartan matter and (torsionless) conserved mass-energy is required to form a charge model in addition to the (with torsion) divergence-free spin. The weak field assumptions therefore keep a certain amount of matter and spin assigned to any charge model. Spin then can be interpreted as the fundamental quantity (3.0.12) that gives matter-charge models their stable character.

Divergence laws arise due to anti-symmetry: (3.0.16) and (3.0.18). The result is the appearance of Maxwell charge as a significant term in (3.0.18), via (5.3.5) and (5.3.3). Certain components of the spin current also get identified at this limit with the Maxwell current.

Similar conclusions were present for [24] at the antisymmetric limit, using slightly different postulates.

8 Conclusion

Kaluza's 1921 theory of gravity and electromagnetism using a fifth wrapped-up spatial dimension is inspiration for many modern attempts to develop new physical theories. However for a number of reasons it is often considered untenable. Here a Kaluza-Cartan theory is presented.

The Kaluza dimension (in the direction of which partial derivatives are treated as vanishing) is identified with a cylinder condition but use is made in addition of a connection deformed by torsion, and this is used also in defining the geometry. This resolves a number of foundational problems with Kaluza's original theory. When the scalar field is set constant (and well-behaved assumptions are made about the paths of charged particles), and a weak field limit defined, then an improved unification of gravity and electromagnetism results. Improved because the Lorentz force law is derived from first principles, and because a more complete range of electromagnetic fields (i.e. the nullish solutions) become possible without making arbitrary assumptions or making too many constraints on a variable scalar field. The current theory was in effect derived from the need for these missing solutions (including electrostatic fields) and to derive the Lorentz force law simultaneously.

The concept of super-energy is applied here axiomatically to make sure that the resulting theory is causal. Other ways to do this may be equally valid, or experimentally correct. But the construction here was the simplest way to deal with such issues - the use of super-energy is presented as a hypothesis. The theory as presented here no doubt has foundational problems of its own: Is a realistic charge model possible? What might limit transformation of energy between matter sources and fields? It is not proposed that this theory is a fait accompli, but a work in progress needing development and testing. Indeed this work is a sequel to another variant in [24]. Here completely antisymmetric torsion is used.

Why go to all this effort to unify electromagnetism and gravitation and to make electromagnetism fully geometric? Because experimental differences could be detectable given sufficient technology on the one hand, and, on the other, simply because such an attempt at unification might be right or lead in the right direction. It may widen the search. This theory differs from both general relativity and Einstein-Cartan theory, it differs from other Kaluza theories. It also differs significantly from scalar-tensor theories as here the scalar is postulated to be without consequence, though this may equally be taken as presenting just a non-scalar limit. It should, in principle, be empirically testable. The expected ω-consistency of Einstein-Cartan theory together with the derivation of a Lorentz force law via the Kaluza part of the theory gives a theoretical motivation, as does the fact that the other approaches beyond general relativity have not fulfilled their full promise in terms of approaching unification. Attempting to extend and unify classical theory prior to a unification with quantum mechanics may even be a necessary step in a future unification whether Kaluza-Cartan theory turns out to be the right way or not. It may be that current attempts at unification are more difficult than necessary as the problem may not yet have been framed correctly. It is often asserted that the true explanation for gravitational theory and space-time curvatures will most likely, by reductionist logic, emerge out of its constituent quantum phenomena. Such an approach has merit, but is overly optimistic, and does not optimize the search [22][23]. Taking a global, more 'synthetic', 'post-reductionist' perspective, as attempted here, can often be more difficult, but may also be more insightful.

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9 Appendix

10 References

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