A series expansion for a real function

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Abstract

We show that a class $C^\infty(\mathbb{R})$ function can be written as an $n-$summation of terms involving its derivative. For many functions, under certain conditions, this summation can become a particular series expansion.

Theorem

Let be $f(x)$ a class $C^\infty(\mathbb{R})$ function with $\mathcal{D}$ domain. Given $a \in \mathcal{D}$ the function satisfies the following identity for all $n \in \mathbb{N}^+$

$$f(x) = f(a) - \sum_{k=1}^{n} \frac{(-1)^k}{k!} \left( x^k \frac{d^k f(x)}{dx^k} - a^k \frac{d^k f(x)}{dx^k}\bigg|_{x=a} \right) + R_n(x), \quad (1)$$

with

$$R_n(x) = \frac{(-1)^n}{n!} \int_a^x x^n \frac{d^{n+1} f(x)}{dx^{n+1}} \, dx.$$ 

In particular, if

$$\lim_{n \to +\infty} R_n(x) = 0$$

and $a = 0$ then

$$f(x) = f(0) - \sum_{k=1}^{\infty} \frac{(-x)^k}{k!} \frac{d^k f(x)}{dx^k}. \quad (2)$$

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Proof

We define the \( \hat{P} \) operator that, applied to a continuous function, it gives its primitive with zero constant. We define also the operator \( \hat{D} \) as

\[
\hat{D} = \frac{d}{dx}.
\]

Now consider the identity

\[
J := \int_a^x df(x) \frac{dx}{dx} = f(x) - f(a),
\]

we can integrate it for parties, considering the product

\[
1 \cdot \frac{df(x)}{dx},
\]

obtaining

\[
J = \left[ \hat{P}(1) \hat{D} f(x) \right]_a^x - \int_a^x \hat{P}(1) \hat{D}^2 f(x) \, dx.
\]

Iterating for many times we can write

\[
J = - \sum_{k=1}^{n} (-1)^k \left[ \hat{P}^k(1) \hat{D}^k f(x) \right]_a^x + (-1)^n \int_a^x \hat{P}^{n+1}(1) \hat{D}^{n+1} f(x) \, dx.
\]

Being

\[
\hat{P}^k(1) = \frac{x^k}{k!},
\]

the above expression, using (3), becomes

\[
f(x) = f(a) - \sum_{k=1}^{n} (-1)^k \left[ \frac{x^k}{k!} \hat{D}^k f(x) \right]_a^x + (-1)^n \int_a^x \frac{x^n}{n!} \hat{D}^{n+1} f(x) \, dx
\]

and this is an identity \( \forall n \in \mathbb{N}^+ \), equation (1). Taking the limit for \( n \to +\infty \) we have

\[
f(x) = f(a) - \sum_{k=1}^{\infty} (-1)^k \left[ \frac{x^k}{k!} \hat{D}^k f(x) \right]_a^x + \lim_{n \to +\infty} R_n(x),
\]

where

\[
R_n(x) := \frac{(-1)^n}{n!} \int_a^x x^n \hat{D}^{n+1} f(x) \, dx.
\]
In many cases we have
\[ \lim_{n \to +\infty} R_n(x) = 0 \] (4)
and, under this condition, we can write the series expansion
\[ f(x) = f(a) - \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \left( \frac{x^k}{k!} \right) \frac{d^k f(x)}{dx^k} \bigg|_{x=a}. \]

Putting \( a = 0 \), naturally if it is possible seen the domain \( D \), we obtain the series
\[ f(x) = f(0) - \sum_{k=1}^{\infty} \frac{(-x)^k}{k!} \frac{d^k f(x)}{dx^k}, \]
that is equation (2). In general for the condition (4) we can write
\[ \lim_{n \to +\infty} |R_n(x)| \leq \lim_{n \to +\infty} \frac{1}{n!} \left| \int_a^x \left| x^n \bar{f}^{n+1} f(x) \right| dx \right|. \] (5)

Suppose that \( x \in [-b, b] \), with \( b \in \mathbb{R}^+ \), so if
\[ \lim_{n \to +\infty} \frac{1}{\sqrt{n}} \left( \frac{b \cdot e}{n} \right)^n \left| \frac{d^{n+1} f(x)}{dx^{n+1}} \right| = 0 \]
uniformly, we have
\[ \lim_{n \to +\infty} R_n(x) = 0. \]

In fact from (5) we can write, using Stirling formula [1] and the uniform limit condition above,
\[ \lim_{n \to +\infty} |R_n(x)| \leq \lim_{n \to +\infty} \int_a^x \frac{e^n}{n^{n \sqrt{2\pi n}}} \left| x^n \bar{f}^{n+1} f(x) \right| dx \]
\[ \leq \left| \int_a^x \lim_{n \to +\infty} \frac{b^n e^n}{n^n \sqrt{2\pi n}} \left| \bar{f}^{n+1} f(x) \right| dx \right| = 0. \]

This could be a useful method to verify (4) for many functions, in particular we see that all functions that have limited derivative of all orders, like \( \sin(x) \), satisfy this condition, so for them it is possible to write (2).

**Example**

For example we can derive the well known series expansion
\[ e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + \sum_{k=1}^{\infty} \frac{x^k}{k!}. \] (6)
Let be \( f(x) = e^x \),

and \( a = 0 \), for all \( x \in [-b, b] \), with \( b \in \mathbb{R}^+ \) we can verify that

\[
\lim_{n \to +\infty} \mathcal{R}_n(x) = \lim_{n \to +\infty} \frac{(-1)^n}{n!} \int_0^x x^n e^x \, dx = 0,
\]

in fact

\[
|R_n(x)| \leq \frac{|x|}{n!} \max_{x \in [-b,b]} |x^n e^x| \leq \frac{b^{n+1}e^b}{n!}
\]

and, taking the limit for \( n \to \infty \),

\[
0 \leq \lim_{n \to +\infty} |R_n(x)| \leq \lim_{n \to +\infty} \frac{b^{n+1}e^b}{n!} = 0,
\]

independently by \( x \). So we apply the equation (2), hence

\[
e^x = 1 - e^x \sum_{k=1}^{\infty} \frac{(-x)^k}{k!}.
\]

from which

\[
\sum_{k=1}^{\infty} \frac{(-x)^k}{k!} = 1 - \frac{e^x}{e^x} = e^{-x} - 1,
\]

that is (6) with the change \( x \to -x \).

References