Schrödinger’s cat paradox resolution using GRW collapse model.

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Abstract: Possible solution of the Schrödinger’s cat paradox is considered.We pointed out
that the collapsed state of the cat always shows definite and predictable outcomes even if
cat also consists of a superposition:

$$\left|\text{cat}\right\rangle = c_1\left|\text{live cat}\right\rangle + c_2\left|\text{death cat}\right\rangle.$$

Content

I. Introduction

As Weinberg recently reminded us [1], the measurement problem remains a fundamental conundrum. During measurement the state vector of the microscopic system collapses in a probabilistic way to one of a number of classical states, in a way that is unexplained, and cannot be described by the time-dependent Schrödinger equation [1].To review the essentials, it is sufficient to consider two-state systems. Suppose a nucleus $\mathbf{n}$, whose Hilbert space is spanned by orthonormal states $|s_i(t)\rangle,\ i = 1,2,$where $|s_1(t)\rangle = \left|\text{undecayed nucleus at instant } t\right\rangle$ and $|s_2(t)\rangle = \left|\text{decayed nucleus at instant } t\right\rangle$ is in the superposition state,

$$|\Psi_t\rangle_{\mathbf{n}} = c_1|s_1(t)\rangle + c_2|s_2(t)\rangle, |c_1|^2 + |c_2|^2 = 1. \tag{1.1}$$

A measurement apparatus A, which may be microscopic or macroscopic, is designed to distinguish between states $|s_i(t)\rangle$ by transitioning at each instant t into state $|a_i(t)\rangle$ if it finds $\mathbf{n}$ is in $|s_i(t)\rangle,\ i = 1,2.$ Assume the detector is reliable, implying the $|a_1(t)\rangle$ and $|a_2(t)\rangle$ are orthonormal at each instant t ,i.e., $\langle a_1(t)||a_2(t)\rangle = 0$ and that the measurement interaction does not disturb states $|s_i\rangle$ -i.e., the measurement is "ideal". When A measures $|\Psi_t\rangle_{\mathbf{n}}$, the Schrödinger equation's unitary time evolution then leads to the "measurement state" $|\Psi_t\rangle_{\mathbf{n}A}$:

$$|\Psi_t\rangle_{\mathbf{n}A} = c_1|a_1(t)\rangle + c_2|a_2(t)\rangle, |c_1|^2 + |c_2|^2 = 1. \tag{1.2}$$

of the composite system $\mathbf{n}A$ following the measurement.

Standard formalism of continuous quantum measurements [2],[3],[4],[5] leads to a definite but unpredictable measurement outcome, either $|a_1(t)\rangle$ or $|a_2(t)\rangle$ and that $|\Psi_t\rangle_{\mathbf{n}}$ suddenly "collapses" at instant t' into the corresponding state $|s_i(t')\rangle$. But unfortunately equation (1.2) does not appear to resemble such a collapsed state at instant t'?.

The measurement problem is as follows:

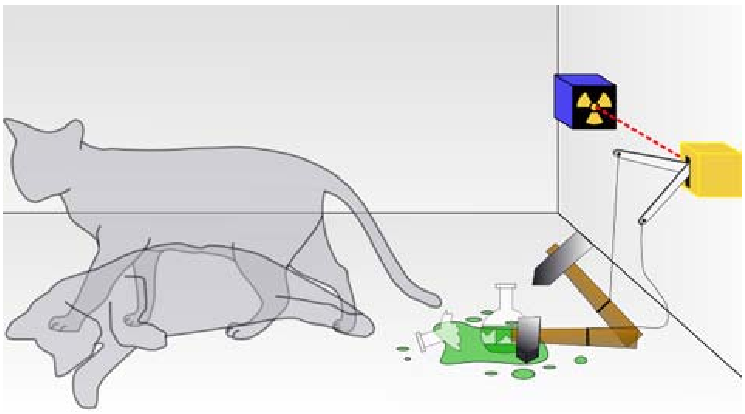
(**I**) How do we reconcile canonical collapse models postulate's

(**II**) How do we reconcile the measurement postulate's definite outcomes with the

"measurement state" $|\Psi_t\rangle_{\mathbf{n}A}$ at each instant t and

(**III**) how does the outcome become irreversibly recorded in light of the Schrödinger equation's unitary and, hence, reversible evolution?

This paper deals with only the special case of the measurement problem, known as Schrödinger's Cat paradox. For a good and complete explanation of this paradox see Leggett [6] and Hobson [7].



Pic.1.1.Schrödinger's cat.

Schrödinger's cat: a cat, a flask of poison, and a radioactive source are placed in a sealed box. If an internal monitor detects radioactivity (i.e. a single atom decaying), the flask is shattered, releasing the poison that kills the cat. The Copenhagen interpretation of quantum mechanics implies that after a while, the cat is simultaneously alive and dead. Yet, when one looks in the box, one sees the cat either alive or dead, not both alive and dead. This poses the question of when exactly quantum superposition ends and reality collapses into one possibility or the other.

The canonical collapse models.

In order to appreciate how canonical collapse models work, and what they are able to achieve, we briefly review the GRW model. Let us consider a system of n particles which, only for the sake of simplicity, we take to be scalar and spinless; the GRW model is defined by the following postulates: (**1**) The state of the system is represented by a wave function $\psi_t(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$ belonging to the Hilbert space $\mathcal{L}_2(\mathbb{R}^{3n})$. (**2**) At random times, the wave function experiences a sudden jump of the form:

$$\psi_t(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) \to \psi_t(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n; \widetilde{\mathbf{x}}_m) = \frac{\Re_m(\widetilde{\mathbf{x}}_m)\psi_t(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)}{\|\Re_m(\widetilde{\mathbf{x}}_m)\psi_t(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)\|_2}, \tag{1.3}$$

where $\psi_t(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$ is the state vector of the whole system at time t, immediately prior to the jump process and $\Re_n(\widetilde{\mathbf{x}}_m)$ is a linear operator which is conventionally chosen equal to:

$$\Re_m(\widetilde{\mathbf{x}}_m) = (\pi r_c^2)^{-3/4} \exp\left[-\frac{(\widehat{\mathbf{x}}_m - \widetilde{\mathbf{x}}_m)^2}{2r_c^2}\right], \tag{1.4}$$

where r_c is a new parameter of the model which sets the width of the localization process, and $\widehat{\mathbf{x}}_m$ is the position operator associated to the m-th particle of the system and the random variable $\widetilde{\mathbf{x}}_m$ corresponds to the place where the jump occurs. (**3**) It is assumed that the jumps are distributed in time like a Poissonian process with frequency $\lambda = \lambda_{GRW}$ this is the second new parameter of the model. (4) Between two consecutive jumps, the state vector evolves according to the standard Schrödinger equation.

The 1-particle master equation of the GRW model takes the form

$$\frac{d}{dt}\rho(t) = -\frac{i}{\hbar}\left[\widehat{\mathbf{H}}, \rho(t)\right] - T[\rho(t)]. \tag{1.5}$$

Here $\widehat{\mathbf{H}}$ is the standard quantum Hamiltonian of the particle, and $T[\cdot]$ represents the effect of the spontaneous collapses on the particle's wave function. In the position representation, this operator becomes:

$$\langle \mathbf{x}|T[\rho(t)]|\mathbf{y}\rangle = \lambda\left\{1 - \exp\left[-\frac{(\mathbf{x}-\mathbf{y})^2}{4r_c^2}\right]\right\}\langle \mathbf{x}|\rho(t)|\mathbf{y}\rangle. \tag{1.6}$$

Another modern approach to stochastic reduction is to describe it using a stochastic nonlinear Schrödinger equation, an elegant simplied example of which is the following one particle case known as Quantum Mechanics with Universal Position Localization [QMUPL]:

$$d|\psi_t(x)\rangle = \left[-\frac{i}{\hbar}\widehat{\mathbf{H}} - k(\widehat{q} - \langle q_t\rangle)^2 dt\right]|\psi_t(x)\rangle dt + \sqrt{2k}\,(\widehat{q} - \langle q_t\rangle)dW_t|\psi_t(x)\rangle. \tag{1.7}$$

Here $\widehat{q}$ is the position operator, $\langle q_t\rangle = \langle\psi_t|\widehat{q}|\psi_t\rangle$ it is its expectation value, and k is a constant, characteristic of the model, which sets the strength of the collapse mechanics, and it is chosen proportional to the mass m of the particle according to the formula: $k = (m/m_0)\lambda_0$, where m_0 is the nucleon's mass and λ_0 measures the collapse strength. It is easy to see that Eqn.(1.5) contains both non-linear and stochastic terms, which are necessary to induce the collapse of the wave function. For an examle let us consider a free particle ($\widehat{\mathbf{H}} = p^2/2m$), and a Gaussian state:

$$\psi_t(x) = \exp\left\{-a_t(x - \bar{x}_t)^2 + i\bar{k}_t x\right\}. \tag{1.8}$$

It is easy to see that $\psi_t(x)$ given by Eq.(1.6) is solution of Eq.(1.5), where

$$\frac{da_t}{dt} = k - \frac{2i\hbar}{m}a_t^2, \frac{d\bar{x}_t}{dt} = \frac{\hbar}{m}\bar{k}_t + \frac{\sqrt{k}}{2\operatorname{Re}(a_t)}\dot{W}_t, \frac{d\bar{k}_t}{dt} = -\sqrt{k}\,\frac{\operatorname{Im}(a_t)}{\operatorname{Re}(a_t)}\dot{W}_t. \tag{1.9}$$

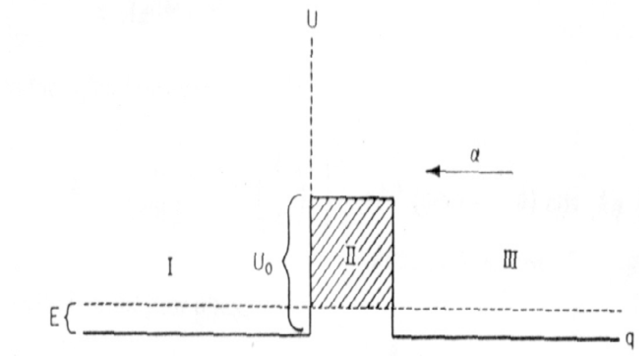
The CSL model is defined by the following stochastic differential equation in the Fock space:

$$\begin{aligned} d|\psi_t(\mathbf{x})\rangle = &\left[-\frac{i}{\hbar}\widehat{\mathbf{H}} - k\left(\widehat{M}(\mathbf{x}) - \langle M_t(\mathbf{x})\rangle\right)^2 dt\right]|\psi_t(\mathbf{x})\rangle dt + \\ &+ \sqrt{2k}\left(\widehat{M}(\mathbf{x}) - \langle M_t(\mathbf{x})\rangle\right)dW_t(\mathbf{x})|\psi_t(\mathbf{x})\rangle. \end{aligned} \tag{1.10}$$

II.Generalized Gamow theory of the alpha decay via tunneling using GRW collapse model.

By 1928, George Gamow had solved the theory of the alpha decay via tunneling [7]. The alpha particle is trapped in a potential well by the nucleus. Classically, it is forbidden to escape, but according to the (then) newly discovered principles of quantum mechanics, it has a tiny (but non-zero) probability of "tunneling" through the barrier and appearing on the other side to escape the nucleus. Gamow solved a model potential for the nucleus and derived, from first principles, a relationship between the half-life of the decay, and the energy of the emission.

The α-particle has total energy E and is incident on the barrier from the right to left.



Pic. 2.1.The particle has total energy E and
is incident on the barrier $V(x)$ from right to left.

The Schrödinger equation in each of regions $\mathbf{I} = \{x|x < 0\}, \mathbf{II} = \{x|0 \leq x \leq l\}$ and $\mathbf{III} = \{x|x > l\}$ takes the folloving form

$$\frac{\partial^2 \Psi(x)}{\partial x^2} + \frac{2m}{\hbar^2}[E - U(x)]\Psi(x) = 0, \tag{2.1}$$

where

$$U(x) = \begin{cases} 0 \text{ for } x < 0 \\ U_0 \text{ for } 0 \leq x \leq l \\ 0 \text{ for } x > l \end{cases} \tag{2.2}$$

The solutions reads [8]:

$$\begin{aligned} \Psi_{\mathbf{III}}(x) &= C_+ \exp(ikx) + C_- \exp(-ikx), \\ \Psi_{\mathbf{II}}(x) &= B_+ \exp(k'x) + B_- \exp(-k'x), \\ \Psi_{\mathbf{I}}(x) &= A\cos(kx) = \frac{A}{2}[\exp(ikx) + \exp(-ikx)], \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} k &= \frac{2\pi}{\hbar}\sqrt{2mE}\,, \\ k' &= \frac{2\pi}{\hbar}\sqrt{2m(U_0 - E)}\,. \end{aligned} \tag{2.4}$$

At the boundary $x = 0$ we have the following boundary conditions:

$$\Psi_{\mathbf{I}}(0)|_{x=0} = \Psi_{\mathbf{II}}(0)|_{x=0}, \frac{\partial \Psi_{\mathbf{I}}(x)}{\partial x}\bigg|_{x=0} = \frac{\partial \Psi_{\mathbf{II}}(x)}{\partial x}\bigg|_{x=0}. \tag{2.5}$$

At the boundary $x = l$ we have the following boundary conditions

$$\Psi_{\mathbf{II}}(l)|_{x=l} = \Psi_{\mathbf{III}}(l)|_{x=l}, \frac{\partial \Psi_{\mathbf{II}}(x)}{\partial x}\bigg|_{x=l} = \frac{\partial \Psi_{\mathbf{III}}(x)}{\partial x}\bigg|_{x=l}. \tag{2.6}$$

From the boundary conditions (2.5)-(2.6) one obtains [8]:

$$\begin{gathered} B_+ = \frac{A}{2}\left(1 + i\frac{k}{k'}\right), B_- = \frac{A}{2}\left(1 - i\frac{k}{k'}\right), \\ C_+ = A[ch(k'l) + iDsh(k'l)], C_- = i(ASsh(k'l)\exp(ikl)), \\ D = \frac{1}{2}\left(\frac{k}{k'} - \frac{k'}{k}\right), S = \frac{1}{2}\left(\frac{k}{k'} + \frac{k'}{k}\right). \end{gathered} \tag{2.7}$$

From (2.7) one obtain the conservation law

$$|A|^2 = |C_+|^2 - |C_-|^2.$$

Let us introduce now a function $E_{\mathbf{II}}(x,l) = \theta_2(x,l)E_2(x,l)$ where

$$E_2(x,l) = \begin{cases} (\pi r_c^2)^{-1/4}\exp\left(-\frac{x^2}{2r_c^2}\right) \text{ for } -\infty < x < \frac{l}{2} \\ (\pi r_c^2)^{-1/4}\exp\left(-\frac{(x-l)^2}{2r_c^2}\right) \text{ for } \frac{l}{2} \leq x < \infty \end{cases} \tag{2.8}$$

$$\theta_2(x,l) = \begin{cases} 1 \text{ for } x \in [0,l] \\ 0 \text{ for } x \notin [0,l] \end{cases}$$

Assumption 2.1. We assume now that:

(i) at instant $t = 0$ the wave function $\Psi_{\mathbf{I}}(x)$ experiences a sudden jump of the form

$$\Psi_{\mathbf{I}}(x) \to \Psi_{\mathbf{I}}^{\#}(x) = \frac{\Re_{\mathbf{I}}(\widehat{x})\Psi_{\mathbf{I}}(x)}{\|\Re_{\mathbf{I}}(\widehat{x})\Psi_{\mathbf{I}}(x)\|_2}, \tag{2.9}$$

where $\Re_{\mathbf{I}}(\widehat{x})$ is a linear operator which is chosen equal to:

$$\Re_{\mathbf{I}}(\hat{x}) = (\pi r_c^2)^{-1/4}\theta_1(\hat{x},l)\exp\left[-\frac{\hat{x}^2}{2r_c^2}\right]; \tag{2.10}$$

where

$$\theta_1(x,l) = \begin{cases} 1 \text{ for } x \in [-l,0], \\ 0 \text{ for } x \notin [-l,0]. \end{cases}$$

Remark 2.1. Note that: **supp**$(\Psi_{\mathbf{I}}^{\#}(x)) \subseteq [-l,0]$

(ii) at instant $t = 0$ the wave function $\Psi_{\mathbf{II}}(x)$ experiences a sudden jump of the form

$$\Psi_{\mathbf{II}}(x) \to \Psi_{\mathbf{II}}^{\#}(x) = \frac{\Re_{\mathbf{II}}(\hat{x})\Psi_{\mathbf{II}}(x)}{\|\Re_{\mathbf{II}}(\hat{x})\Psi_{\mathbf{II}}(x)\|_2}, \tag{2.11}$$

where $\Re_{\mathbf{II}}(\hat{x})$ is a linear operator which is chosen equal to:

$$\Re_{\mathbf{II}}(\hat{x}) = E_{\mathbf{II}}(\hat{x},l); \tag{2.12}$$

Remark 2.2. Note that: **supp**$(\Psi_{\mathbf{II}}^{\#}(x)) \subseteq [0,l]$.

(iii) at instant $t = 0$ the wave function $\Psi_{\mathbf{III}}(x)$ experiences a sudden jump of the form

$$\Psi_{\mathbf{III}}(x) \to \Psi_{\mathbf{III}}^{\#}(x) = \frac{\Re_{\mathbf{III}}(\hat{x})\Psi_{\mathbf{III}}(x)}{\|\Re_{\mathbf{III}}(\hat{x})\Psi_{\mathbf{III}}(x)\|_2}, \tag{2.13}$$

where $\Re_{\mathbf{III}}(\hat{x})$ is a linear operator which is chosen equal to:

$$\Re_{\mathbf{III}}(\hat{x}) = (\pi r_c^2)^{-1/4}\exp\left[-\frac{(\hat{x}-l)^2}{2r_c^2}\right]. \tag{2.14}$$

Remark 2.3. Note that. We have choose operators (2.10),(2.12) and (2.14) such that the boundary conditions (2.5),(2.6) is satisfied.

Definition 2.1. Let $\Psi(x)$ be an solution of the Schrödinger equation (2.1). The stationary Schrödinger equation (2.1) is a weakly well preserved in region $\Gamma \subseteq \mathbb{R}$ by collapsed wave function $\Psi^{\#}(x)$ if there exist an wave function $\Psi(x)$ such that the estimate

$$\int_{\Gamma}\left\{\frac{\partial^2\Psi^{\#}(x)}{\partial x^2}+\frac{2m}{\hbar^2}[E-U(x)]\Psi^{\#}(x)\right\}dx=O(\hbar^{2+\alpha}), \tag{2.15}$$

where $\alpha \geq 1,$ is satisfied.

Proposition 2.1.The Schrödinger equation in each of regions $\mathbf{I},\mathbf{II},\mathbf{III}$ is a weakly well preserved by collapsed wave function $\Psi^{\#}_{\mathbf{I}}(x),\Psi^{\#}_{\mathbf{II}}(x)$ and $\Psi^{\#}_{\mathbf{III}}(x)$ correspondingly.

Proof. See Appendix B.

Definition 2.2.Let us consider the time-dependent Schrödinger equation:

$$i\hbar\frac{\partial\Psi(\mathbf{x},t)}{\partial t}=\widehat{\mathbf{H}}\Psi(\mathbf{x},t), \tag{2.16}$$
$$t\in[0,T],\mathbf{x}\in\mathbb{R}^{3n}.$$

The time-dependent Schrödinger equation (2.16) is a weakly well preserved by corresponding to $\Psi(\mathbf{x},t)$ collapsed wave function $\Psi^{\#}(\mathbf{x},t)$

$$\Psi^{\#}(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n,t)=$$

$$\Psi(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n,t;\widetilde{\mathbf{x}}_{m_1},\ldots,\widetilde{\mathbf{x}}_{m_k})=$$

$$=\frac{\Re_{m_1,\ldots,m_k}(\widetilde{\mathbf{x}}_{m_1},\ldots,\widetilde{\mathbf{x}}_{m_k})\Psi(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n,t)}{\|\Re_{m_1,\ldots,m_k}(\widetilde{\mathbf{x}}_{m_1},\ldots,\widetilde{\mathbf{x}}_{m_k})\Psi(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n,t)\|_2},$$
$$\Re_{m_1,\ldots,m_k}(\widetilde{\mathbf{x}}_{m_1},\ldots,\widetilde{\mathbf{x}}_{m_k})=\prod_{i=1}^{k}\Re_{m_i}(\widetilde{\mathbf{x}}_{m_i})$$

in region $\Gamma\subseteq\mathbb{R}^{3d}$ if there exist an wave function $\Psi(\mathbf{x},t)$ such that the estimate

$$\int_{\Gamma}\left\{i\hbar\frac{\partial\Psi^{\#}(\mathbf{x},t)}{\partial t}-\widehat{\mathbf{H}}\Psi^{\#}(\mathbf{x},t)\right\}d^{3d}x=O(\hbar^{\alpha}), \tag{2.17}$$
$$t\in[0,T],\mathbf{x}\in\mathbb{R}^{3d},$$

where $\alpha\geq 1,$ is satisfied.

Definition 2.3. Let $\Psi^{\#}(\mathbf{x},t)=\Psi^{\#}(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_d,t)$ be a function $\Psi(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_d,t;\widetilde{\mathbf{x}}_1,\ldots,\widetilde{\mathbf{x}}_d)$.Let us consider the Probability Current Law

$$\frac{\partial}{\partial t}P(\Gamma,t)+\int\limits_{\partial\Gamma}\mathbf{J}(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_d,t)\cdot\mathbf{n}d^{2d}x=O(\hbar^{\alpha}),$$
$$\mathbf{J}(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_d,t)=\Psi(\mathbf{x},t)\nabla\overline{\Psi(\mathbf{x},t)}-\overline{\Psi(\mathbf{x},t)}\nabla\Psi(\mathbf{x},t),$$
$$t\in[0,T],\mathbf{x}\in\mathbb{R}^{3d}, \tag{2.18}$$

corresponding to Schrödinger equation (2.16). Probability Current Law (2.18) is a

weakly well preserved by corresponding to $\Psi(\mathbf{x},t)$ collapsed wave function $\Psi^{\#}(\mathbf{x},t)$ in region $\Gamma\subseteq\mathbb{R}^{3d}$ if there exist an wave function $\Psi(\mathbf{x},t)$ such that the estimate

$$\frac{\partial}{\partial t}P(\Gamma,t)+\int\limits_{\partial\Gamma}\mathbf{J}^{\#}(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_d,t)\cdot\mathbf{n}d^{2d}x=O(\hbar^{\alpha}),$$
$$\mathbf{J}^{\#}(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_d,t)=\Psi^{\#}(\mathbf{x},t)\nabla\overline{\Psi^{\#}(\mathbf{x},t)}-\overline{\Psi^{\#}(\mathbf{x},t)}\nabla\Psi^{\#}(\mathbf{x},t)$$
$$=O(\hbar^{\alpha}),$$
$$t\in[0,T],\mathbf{x}\in\mathbb{R}^{3d}, \tag{2.19}$$

where $\alpha\geq 1,$ is satisfied.

Proposition 2.2. Assume that there exist an wave function $\Psi(\mathbf{x},t)$ such that the

estimate

(2.17) is satisfied. Then Probability Current Law (2.18) is a weakly well

preserved by corresponding to $\Psi(\mathbf{x},t)$ collapsed wave function $\Psi^{\#}(\mathbf{x},t)$ in region $\Gamma\subseteq\mathbb{R}^{3d}$, i.e. the estimate (2.19) is satisfied on the wave function $\Psi^{\#}(\mathbf{x},t)$.

III. Schrödinger's Cat paradox resolution

III.1. Resolution of the Schrödinger's cat paradox using canonical von Neumann postulate

Let $|s_1(t)\rangle$ and $|s_2(t)\rangle$ be

$$|s_1(t)\rangle=\left|\text{undecayed nucleus at instant } t\right\rangle,$$
$$|s_2(t)\rangle=\left|\text{decayed nucleus at instant } t\right\rangle. \tag{3.1}$$

In a good approximation we assume now that

$$|s_1(0)\rangle=\int_{-\infty}^{+\infty}\Psi^{\#}_{\mathbf{II}}(x)|x\rangle dx \tag{3.2}$$

and

$$|s_2(0)\rangle = \int_{-\infty}^{+\infty} \Psi_{\mathbf{I}}^{\#}(x)|x\rangle dx. \tag{3.3}$$

Remark 3.1. Note that: (i) $|s_2(0)\rangle = \Big|$decayed nucleus at instant $0\Big\rangle =$ $=\Big|$free α-particle at instant $0\Big\rangle$. (ii) Feynman propagator of a free α-particle are [9]:

$$K_2(x,t,x_0) = \left(\frac{m}{2\pi i\hbar t}\right)^{1/2} \exp\left\{\frac{i}{\hbar}\left[\frac{m(x-x_0)^2}{2t}\right]\right\}. \tag{3.4}$$

Therefore from Eq.(3.3),Eq.(2.9) and Eq.(3.4) we obtain

$$|s_2(t)\rangle = \int_{-\infty}^{+\infty} \Psi_{\mathbf{I}}^{\#}(x,t)|x\rangle dx,$$

$$\Psi_{\mathbf{I}}^{\#}(x,t) = \int\limits_{-\infty}^{0} \Psi_{\mathbf{I}}^{\#}(x_0)K_2(x,t,x_0)dx_0 =$$

$$(\pi r_c^2)^{-1/4} \times \left(\frac{m}{2\pi i\hbar t}\right)^{1/2} \times \int\limits_{-\infty}^{0} \theta_1(x_0,l)\exp\left(-\frac{x_0^2}{2r_c^2}\right)\exp\left(-i\frac{2\pi}{\hbar}\sqrt{2mE}\,x_0\right) \times$$

$$\times \exp\left\{\frac{i}{\hbar}\left[\frac{m(x-x_0)^2}{2t}\right]\right\}dx_0 =$$

$$(3.5)$$

$$(\pi r_c^2)^{-1/4} \times \left(\frac{m}{2\pi i\hbar_\varepsilon t}\right)^{1/2} \times \int\limits_{-l}^{0} \theta_1(x_0,l)\exp\left(-\frac{x_0^2}{2r_c^2}\right) \times$$

$$\times \exp\left\{\frac{i}{\hbar}\left[\frac{m(x-x_0)^2}{2t} - \pi\sqrt{4mE}\,x_0\right]\right\}dx_0 =$$

$$(\pi r_c^2)^{-1/4} \times \left(\frac{m}{2\pi i\hbar t}\right)^{1/2} \times \int\limits_{-l}^{0} \theta_1(x_0,l)\exp\left(-\frac{x_0^2}{2r_c^2}\right) \times \exp\left\{\frac{i}{\hbar}[S(t,x,x_0)]\right\}dx_0,$$

where

$$S(t,x,x_0) = \frac{m(x-x_0)^2}{2t} - \pi\sqrt{8mE}\,x_0. \tag{3.6}$$

We assume now that

$$\hbar \ll 2r_c^2 \ll l^2 < 1. \tag{3.7}$$

Oscillatory integral in RHS of Eq.(3.5) is calculated now directly using stationary phase approximation. The phase term $S(x,x_0)$ given by Eq.(3.6) is stationary when

$$\frac{\partial S(t,x,x_0)}{\partial x_0} = -\frac{m(x-x_0)}{t} - \pi\sqrt{8mE} = 0. \tag{3.8}$$

Therefore

$$\begin{gathered} -\frac{m(x-x_0)}{t} - \pi\sqrt{8mE} = 0, \\ -(x-x_0) = \pi t\sqrt{8E/m}, \end{gathered} \tag{3.9}$$

and thus stationary point $x_0(t,x)$ are

$$x_0(t,x) = \pi t\sqrt{8E/m} + x. \tag{3.10}$$

Thus from Eq.(3.5) and Eq.(3.10) using stationary phase approximation we obtain

$$\begin{gathered} |s_2(t)\rangle = |s_2(t)\rangle = \int_{-\infty}^{+\infty} \Psi_{\mathbf{I}}^{\#}(x,t)|x\rangle dx, \\ \Psi_{\mathbf{I}}^{\#}(x,t) = \\ (\pi r_c^2)^{-1/4} \times \theta_1(x_0(t,x),l)\exp\left[-\frac{x_0^2(t,x)}{2r_c^2}\right] \times \exp\left\{\frac{i}{\hbar}[S(t,x,x_0(t,x))]\right\} + O(\hbar), \end{gathered} \tag{3.11}$$

where

$$S(x,x_0(t,x)) = \frac{m(x-x_0(t,x))^2}{2t} - \pi\sqrt{8mE}\,x_0(t,x). \tag{3.12}$$

From Eq.(3.11) we obtain

$$\langle s_2(t)\|s_2(t)\rangle \simeq (\pi r_c^2)^{-1/2} \times \theta_1\left(x + \pi t\sqrt{8E/m}, l\right)\exp\left[-\frac{\left(x + \pi t\sqrt{8E/m}\right)^2}{r_c^2}\right]. \tag{3.13}$$

Remark 3.2. From the inequality (3.7) and Eq.(3.13) follows that α-particle at each instant $t \geq 0$ moves quasiclassically from right to left by the law

$$x(t) = -\pi t\sqrt{8E/m}, \tag{3.14}$$

i.e. i.e.,estimating the position $x(t,x_0,t_0;\hbar)$ at each instant $t \geq 0$ with final error r_c gives $|\langle x\rangle(t) - x(t)| \leq r_c, i = 1,\ldots,d$ with a probability

$$\mathbf{P}\{|\langle x\rangle(t,0,0;\hbar) - x(t)| \leq r_c\} = 1.$$

Remark 3.3. We assume now that a distance between radioactive source and internal monitor which detects a single atom decaying (see Pic.1) is equal to L.

Proposition 3.1. After α-decay at instant $t = 0$ the collaps: $\left|\text{live cat}\right\rangle \to \left|\text{death cat}\right\rangle$ arises at instant

$$T = \frac{L}{\pi\sqrt{8E/m}} \tag{3.15}$$

with a probability $\mathbf{P}_T\left(\left|\text{death cat}\right\rangle\right)$ to observe a state $\left|\text{death cat}\right\rangle$ at instant T is $\mathbf{P}_T\left(\left|\text{death cat}\right\rangle\right) = 1.$

Proof. Note that. In this case Schrödinger's cat in fact permorm the single measurement of α-particle position with accuracy of $\delta x = l$ at instant $t = T$ (given by Eq.(3.15)) by internal monitor (see Pic.1.1). The probability of getting a result L with accuracy of $\delta x = l$ given by

$$\int_{|L-x|\leq l/2} |\langle x \| s_2(T)\rangle|^2 dx = 1. \tag{3.16}$$

Therefore at instant T the α-particle kills Schrödinger's cat with a probability $\mathbf{P}_T\left(\left|\text{death cat}\right\rangle\right) = 1.$

Remark 3.4.Note that. When Schrödinger's cat has permormed this measurement the immediate post measurement state of α-particle (by von Neumann postulate **C.4**.) will end up in the state

$$|\Psi_T\rangle = \frac{\int_{|L-x|\leq l/2} |x\rangle\langle x \| s_2(T)\rangle dx}{\sqrt{\int_{|L-x|\leq l/2} |\langle x \| s_2(T)\rangle|^2 dx}} = \int_{|L-x|\leq l/2} |x\rangle\langle x \| s_2(T)\rangle dx. \tag{3.17}$$

From Eq.(3.17) one obtains

$$\langle x' \| \Psi_T\rangle = \int_{|L-x|\leq l/2} \langle x' \| x\rangle\langle x \| s_2(T)\rangle dx = \int_{|L-x|\leq l/2} \delta(x'-x)\langle x \| s_2(T)\rangle dx = \Psi^{\#}_{\mathbf{I}}(x',t). \tag{3.18}$$

Therefore the state $|\Psi_T\rangle$ again kills Schrödinger's cat with a probability $\mathbf{P}_T\left(\left|\text{death cat}\right\rangle\right) = 1.$

Suppose now that a nucleus $\mathbf{n}$, whose Hilbert space is spanned by orthonormal states $|s_i(t)\rangle,\ i = 1,2,$ where $|s_1(t)\rangle = \left|\text{undecayed nucleus at instant } t\right\rangle$ and $|s_2(t)\rangle = \left|\text{decayed nucleus at instant } t\right\rangle$ is in the superposition state

$$|\Psi_t\rangle_{\mathbf{n}} = c_1|s_1(t)\rangle + c_2|s_2(t)\rangle, |c_1|^2 + |c_2|^2 = 1. \tag{3.19}$$

Remark 3.5. Note that: (i) $|s_1(0)\rangle = \left|\text{undecayed nucleus at instant } t = 0\right\rangle =$
$= \left|\ \alpha\text{-particle iside region } (0,l] \text{ at instant } t = 0\right\rangle$. (ii) Feynman propagator of α-particle inside region $(0,l]$ are [9]:

$$K_2(x,t,x_0) = \left(\frac{m}{2\pi i\hbar t}\right)^{1/2} \exp\left\{\frac{i}{\hbar}[S(t,x,x_0)]\right\}, \tag{3.20}$$

where

$$S(t,x,x_0) = \frac{m(x-x_0)^2}{2t} + mt(U_0 - E). \tag{3.21}$$

Therefore from Eq.(2.11)-Eq.(2.12) and Eq.(3.20)-Eq.(3.21) we obtain

$$\begin{gathered}|s_1(t)\rangle = \int_{-\infty}^{+\infty} \Psi^{\#}_{\mathbf{II}}(x,t)|x\rangle dx,\\ \Psi^{\#}_{\mathbf{II}}(x,t) = \int_0^l \Psi^{\#}_{\mathbf{II}}(x_0) K_2(x,t,x_0)dx_0 = \\ \left(\frac{m}{2\pi i\hbar t}\right)^{1/2} \int_0^l E(x_0,l)\Psi_{\mathbf{II}}(x_0)\theta_l(x_0)\exp\left\{\frac{i}{\hbar}[S(t,x,x_0)]\right\}dx_0,\end{gathered} \tag{3.22}$$

where

$$\theta_l(x) = \begin{cases} 1 \text{ for } x \in [0,l] \\ 0 \text{ for } x \notin [0,l] \end{cases}$$

Remark 3.6.We assume for simplification now that

$$k' \leq 1. \tag{3.23}$$

Therefore oscillatory integral in RHS of Eq.(3.22) is calculated now directly using stationary phase approximation. The phase term $S(x,x_0)$ given by Eq.(3.21) is stationary when

$$\frac{\partial S(t,x,x_0)}{\partial x_0} = -\frac{m(x-x_0)}{t} = 0. \tag{3.24}$$

and thus stationary point $x_0(t,x)$ are

$$\begin{gathered} -x + x_0 = 0 \\ x_0(t,x) = x. \end{gathered} \tag{3.25}$$

Thus from Eq.(3.22) and Eq.(3.25) using stationary phase approximation we obtain

$$\begin{gathered}\Psi^{\#}_{\mathbf{II}}(x,t) = \\ E(x_0(t,x),l)\Psi_{\mathbf{II}}(x_0(t,x))\theta_l(x_0(t,x))\exp\left\{\frac{i}{\hbar}[S(t,x,x_0(t,x))]\right\} + O(\hbar) = \\ = E(x,l)\Psi_{\mathbf{II}}(x)\theta_l(x)\exp\left\{\frac{i}{\hbar}[mt(U_0-E)]\right\} + O(\hbar) = \\ E(x,l)\theta_l(x)O(1)\exp\left\{\frac{i}{\hbar}[mt(U_0-E)]\right\} + O(\hbar).\end{gathered} \tag{3.26}$$

Therefore from Eq.(3.22) and Eq.(3.26) we obtain

$$\langle s_1(t)|s_1(t)\rangle = |\Psi^{\#}_{\mathbf{II}}(x,t)|^2 = E^2(x,l)\theta_l(x)O(1) + O(\hbar). \tag{3.27}$$

Remark 3.7. Note that for each instant $t > 0$:

supp$(\Psi^{\#}_{\mathbf{II}}(x,t)) \cap$**supp**$(\Psi^{\#}_{\mathbf{I}}(x,t)) = \varnothing$.

Remark 3.8. Note that. From Eq.(3.11),Eq.(3.13), Eq.(3.19), Eq.(3.22)-Eq.(3.27) and Eq.(A.13) by Remark 3.7 we obtain

$$\begin{gathered}
{}_{\mathbf{n}}\langle\Psi_t|\widehat{x}|\Psi_t\rangle_{\mathbf{n}} = |c_1|^2\langle s_1(t)|\widehat{x}|s_1(t)\rangle + |c_2|^2\langle s_2(t)|\widehat{x}|s_2(t)\rangle + \\
c_1c_2^*\langle s_2(t)|\widehat{x}|s_1(t)\rangle + c_1^*c_2\langle s_2(t)|\widehat{x}|s_1(t)\rangle^* = \\
|c_1|^2\langle s_1(t)|\widehat{x}|s_1(t)\rangle + |c_2|^2\langle s_2(t)|\widehat{x}|s_2(t)\rangle = |c_1|^2 l + |c_2|^2 T\pi\sqrt{8E/m}\,.
\end{gathered} \tag{3.28}$$

Proposition 3.2. Suppose that a nucleus $\mathbf{n}$ is in the superposition state $|\Psi_t\rangle_{\mathbf{n}}$ ($|\Psi_t\rangle_{\mathbf{n}}$-particle) given by Eq.(3.19). Then the collaps: $|\text{live cat}\rangle \to |\text{death cat}\rangle$ arises at instant

$$T = \frac{L}{\sqrt{8\pi^2 E/m}}. \tag{3.29}$$

with a probability $\mathbf{P}_T(|\text{death cat}\rangle)$ to observe a state $|\text{death cat}\rangle$ at instant T is $\mathbf{P}_T(|\text{death cat}\rangle) = 1$.

Proof. Note that. In this case Schrödinger's cat in fact permorm the single measurement of $|\Psi_t\rangle_{\mathbf{n}}$-particle position with accuracy of $\delta x = l$ at instant $t = T$ (given by Eq.(3.15)) by internal monitor (see Pic.1.1). The probability of getting a

result L at instant $t = T$ with accuracy of $\delta x = l$ given by

$$\begin{gathered}
\int_{|L-x|\leq l/2} |\langle x||\Psi_T\rangle_{\mathbf{n}}|^2 dx = \int_{|L-x|\leq l/2} |\langle x|c_1|s_1(T)\rangle + c_2|s_2(T)\rangle|^2 dx = \\
\int_{|L-x|\leq l/2} |c_1\langle x||s_1(T)\rangle + c_2\langle x||s_2(T)\rangle|^2 dx = \\
\int_{|L-x|\leq l/2} |c_1^2\Psi^{\#2}_{\mathbf{II}}(x,T) + c_2^2\Psi^{\#2}_{\mathbf{I}}(x,T) + 2c_1c_2\Psi^{\#}_{\mathbf{I}}(x,T)\Psi^{\#}_{\mathbf{II}}(x,T)|dx.
\end{gathered} \tag{3.30}$$

From Eq.(3.30) by Remark 3.7 one obtains

$$\int_{|L-x|\leq l/2} |\langle x||\Psi_T\rangle_{\mathbf{n}}|^2 dx = \int_{|L-x|\leq l/2} |c_2^2\Psi^{\#2}_{\mathbf{I}}(x,T)|dx = |c_2|^2\int_{|L-x|\leq l/2} |\Psi^{\#}_{\mathbf{I}}(x,T)|^2 dx = |c_2|^2. \tag{3.31}$$

Note that. When Schrödinger's cat has permormed this measurement the immediate post measurement state of α-particle (by von Neumann postulate **C.4**.) will end up in the state

$$|\Psi_T\rangle_{\mathbf{n}} = \frac{\int_{|L-x|\leq l/2} |x\rangle\langle x||\Psi_T\rangle_{\mathbf{n}} dx}{\sqrt{\int_{|L-x|\leq l/2} |\langle x||\Psi_T\rangle_{\mathbf{n}}|^2 dx}} = \frac{\int_{|L-x|\leq l/2} |x\rangle\langle x|(c_1|s_1(t)\rangle + c_2|s_2(t)\rangle) dx}{\sqrt{\int_{|L-x|\leq l/2} |\langle x||\Psi_T\rangle_{\mathbf{n}}|^2 dx}} =$$
$$\frac{c_1 \int_{|L-x|\leq l/2} |x\rangle\langle x||s_1(t)\rangle + c_2 \int_{|L-x|\leq l/2} |x\rangle\langle x||s_2(t)\rangle dx}{\sqrt{\int_{|L-x|\leq l/2} |\langle x||\Psi_T\rangle_{\mathbf{n}}|^2 dx}}. \tag{3.32}$$

From Eq.(3.32) by Eq.(3.31) and by Remark 3.7 one obtains

$$|\Psi_T\rangle_{\mathbf{n}} = \frac{\int_{|L-x|\leq l/2} |x\rangle\langle x||\Psi_T\rangle_{\mathbf{n}} dx}{\sqrt{\int_{|L-x|\leq l/2} |\langle x||\Psi_T\rangle_{\mathbf{n}}|^2 dx}} = \frac{\int_{|L-x|\leq l/2} |x\rangle\langle x|(c_1|s_1(t)\rangle + c_2|s_2(t)\rangle) dx}{\sqrt{\int_{|L-x|\leq l/2} |\langle x||\Psi_T\rangle_{\mathbf{n}}|^2 dx}} =$$
$$= \frac{c_2}{|c_2|} \int_{|L-x|\leq l/2} |x\rangle\langle x||s_2(t)\rangle dx. \tag{3.32}$$

Obviously by Remark 3.4 the staite $|\Psi_T\rangle_{\mathbf{n}}$ kills Schrödinger's cat with a probability $\mathbf{P}_T\big(|\text{death cat}\rangle\big) = 1.$

III.2. **Resolution of the Schrödinger's cat paradox using generalized von Neumann postulate**.

Proposition 3.3. Suppose that a nucleus $\mathbf{n}$ is in the superposition state given by Eq.(3.19). the collaps: $|\text{live cat}\rangle \to |\text{death cat}\rangle$ arises at instant

$$T = \frac{L}{|c_2|^2 \sqrt{8\pi^2 E/m}}. \tag{3.33}$$

with a probability $\mathbf{P}_T\big(|\text{death cat}\rangle\big)$ to observe a state $|\text{death cat}\rangle$ at instant T is $\mathbf{P}_T\big(|\text{death cat}\rangle\big) = 1.$

Proof. Let us consider now a state $|\Psi_t\rangle_{\mathbf{n}}$ given by Eq.(3.19). This state consists of a summ of two wave packets $c_1\Psi^{\#}_{\mathbf{II}}(x,t)$ and $c_2\Psi^{\#}_{\mathbf{I}}(x,t)$. Wave packet $c_1\Psi^{\#}_{\mathbf{II}}(x,t)$ present an $\alpha_{\mathbf{II}}$-particle which lives inside region $\mathbf{II}$ with a probability $|c_1|^2$ (see Pic. 2.1). Wave packet $c_2\Psi^{\#}_{\mathbf{I}}(x,t)$ present an $\alpha_{\mathbf{I}}$-particle which lives inside region $\mathbf{I}$ with a probability $|c_2|^2$ (see Pic. 2.1) and moves from the right to the left. Note that $\mathbf{I} \cap \mathbf{II} = \varnothing$. From Eq.(3.28) follows that $\alpha_{\mathbf{I}}$-particle at each instant $t \geq 0$ moves quasiclassically from right to left by the law

$$x(t) = -|c_2|^2 \pi t \sqrt{8E/m}, \tag{3.34}$$

From Eq.(3.34) one obtains

$$T = T_{\text{col}} \simeq \frac{L}{|c_2| \sqrt{8\pi^2 E/m}}. \tag{3.35}$$

Note that. In this case Schrödinger's cat in fact permorm a single measurement of $|\Psi_t\rangle_{\mathbf{n}}$-particle position with accuracy of $\delta x = l$ at instant $t = T = T_{\text{col}}$ (given by Eq.(3.35)) by internal monitor (see Pic.1.1). The probability of getting the result L at

instant $t = T_{\text{col}}$ with accuracy of $\delta x = l$ by Remark 3.7 and by generalized von Neumann postulate. C.5. III (see apendix C) given by

$$\begin{aligned}&\int_{|L-x|\leq l/2} |\langle x||\Psi_{T\text{col}}\rangle_{\mathbf{n}}|^2 dx = \int_{|L-x|\leq l/2} |\langle x|c_1|s_1(T_{\text{col}})\rangle + c_2|s_2(T_{\text{col}})\rangle|^2 dx = \\ &\int_{|L-x|\leq l/2} |c_2|^{-2}\langle x|c_2|^{-2}||s_2(T_{\text{col}})\rangle^2 dx = \int_{|L-x|\leq l/2} |c_2|^{-2}|\Psi^{\#}_{\mathbf{I}}(x|c_2|^{-2}, T_{\text{col}})|^2 dx = 1.\end{aligned} \quad (3.36)$$

Therefore the staite $|\Psi_{T_{\text{col}}}\rangle_{\mathbf{n}}$ kills Schrödinger's cat with a probability $\mathbf{P}_{T\text{col}}\big(\big|\text{death cat}\big\rangle\big) = 1.$

Thus is the collapsed state of the cat always shows definite and predictable
outcomes even if cat also consists of a superposition:

$$\big|\text{cat}\big\rangle = c_1\big|\text{live cat}\big\rangle + c_2\big|\text{death cat}\big\rangle.$$

Contrary to van Kampen's [10] and some others' opinions, "looking" at the outcome changes nothing, beyond informing the observer of what has already happened.

We remain: there are widespread claims that Schrödinger's cat is not in a definite alive or dead state but is, instead, in a superposition of the two. van Kampen, for example, writes "The whole system is in a superposition of two states: one in which no decay has occurred and one in which it has occurred. Hence, the state of the cat also consists of a superposition: $\big|\text{cat}\big\rangle = c_1\big|\text{live cat}\big\rangle + c_2\big|\text{death cat}\big\rangle$. The state remains a superposition until an observer looks at the cat" [10].

Appendix. **A**.

The time-dependent Schrodinger equation governs the time evolution of a quantum mechanical system:

$$i\hbar\frac{\partial\Psi(\mathbf{x},t)}{\partial t} = \widehat{\mathbf{H}}\Psi(\mathbf{x},t). \quad (A.1)$$

The average, or expectation, value $\langle x_i\rangle$ of an observable x_i corresponding to a quantum mechanical operator $\widehat{x}_i$ is given by:

$$\langle x_i \rangle(t, \mathbf{x}_0, t_0; \hbar) = \frac{\int\limits_{\mathbb{R}^d} x_i |\Psi(\mathbf{x}, t, \mathbf{x}_0, t_0; \hbar)|^2 d^d x}{\int\limits_{\mathbb{R}^d} |\Psi(\mathbf{x}, t, \mathbf{x}_0, t_0; \hbar)|^2 d^d x}. \quad (A.2)$$

$$i = 1, \dots, d.$$

Remark A.1. We assume now that: the solution $\Psi(\mathbf{x}, t, \mathbf{x}_0, t_0; \hbar)$ of the time-dependent Schrödinger equation (A.1) has a good approximation by a delta function such that

$$|\Psi(\mathbf{x}, t, \mathbf{x}_0, t_0; \hbar)|^2 \simeq \prod_{i=1}^{d} \delta(x_i - x_i(t, \mathbf{x}_0, t_0)), \quad (A.3)$$

$$x_i(t, \mathbf{x}_0, t_0) = x_{i,0},$$

$$i = 1, \dots, d.$$

Remark A.2. Note that under conditions given by Eq.(A.3) QM-system which governed by Schrödinger equation Eq.(A.1) completely evolve quasiclassically i.e. estimating the position $\{x_i(t, \mathbf{x}_0, t_0; \hbar)\}_{i=1}^{d}$ at each instant t with final error δ gives $|\langle x_i \rangle(t, \mathbf{x}_0, t_0; \hbar) - x_i(t, \mathbf{x}_0, t_0)| \leq \delta, i = 1, \dots, d$ with a probability

$$\mathbf{P}\{|\langle x_i \rangle(t, \mathbf{x}_0, t_0; \hbar) - x_i(t, \mathbf{x}_0, t_0)| \leq \delta\} \simeq 1.$$

Thus from Eq.(A.2) and Eq.(A.3) we obtain

$$\langle x_i \rangle(t, \mathbf{x}_0, t_0; \hbar) \simeq$$

$$\simeq \frac{\int\limits_{\mathbb{R}^d} x_i \prod_{i=1}^{d=1} \delta(x_i - x_i(t, \mathbf{x}_0, t_0)) d^d x}{\int\limits_{\mathbb{R}^d} \prod_{i=1}^{d=1} \delta(x_i - x_i(t, \mathbf{x}_0, t_0)) d^d x} = x_i(t, \mathbf{x}_0, t_0). \quad (A.4)$$

$$i = 1, \dots, d.$$

Thus under condition given by Eq.(A.3) one obtain

$$\langle x_{i,t} \rangle(t, \mathbf{x}_0, t_0; \hbar) \simeq x_i(t, \mathbf{x}_0, t_0), \quad (A.5)$$

$$i = 1, \dots, d.$$

Remark A.3.Let $\Psi_i(\mathbf{x}, t, \mathbf{x}_0, t_0), i = 1, 2$ be the solutions of the time-dependent Schrödinger equation (A.1). We assume now that $\Phi(\mathbf{x}, t, \mathbf{x}_0, \mathbf{y}_0, t_0)$ is a linear superposition such that

$$\begin{gathered}\Phi(\mathbf{x},t,\mathbf{x}_0,\mathbf{y}_0,t_0) = c_1\Psi_1(\mathbf{x},t,\mathbf{x}_0,t_0) + c_2\Psi_2(\mathbf{x},t,\mathbf{y}_0,t_0). \\ |c_1|^2 + |c_2|^2 = 1.\end{gathered} \tag{A.6}$$

Then we obtain

$$\begin{gathered}|\Phi(\mathbf{x},t,\mathbf{x}_0,\mathbf{y}_0,t_0)|^2 = (\Phi(\mathbf{x},t,\mathbf{x}_0,\mathbf{y}_0,t_0)\Phi^*(\mathbf{x},t,\mathbf{x}_0,\mathbf{y}_0,t_0)) = \\ = ([c_1\Psi_1(\mathbf{x},t,\mathbf{x}_0,t_0) + c_2\Psi_2(\mathbf{x},t,\mathbf{y}_0,t_0)]) \times \\ \times([c_1^*\Psi_1^*(\mathbf{x},t,\mathbf{x}_0,t_0) + c_2^*\Psi_2^*(\mathbf{x},t,\mathbf{x}_0,\mathbf{y}_0,t_0)]) = \\ = |c_1|^2\left(|\Psi_1(\mathbf{x},t,\mathbf{x}_0,t_0)|^2\right) + c_1^*c_2(\Psi_1^*(\mathbf{x},t,\mathbf{x}_0)\Psi_2(\mathbf{x},t,\mathbf{y}_0,t_0)) + \\ |c_2|^2\left(|\Psi_2(\mathbf{x},t,\mathbf{y}_0,t_0)|^2\right) + c_1c_2^*(\Psi_1(\mathbf{x},t,\mathbf{x}_0)\Psi_2^*(\mathbf{x},t,\mathbf{y}_0,t_0)).\end{gathered} \tag{A.7}$$

Definition A.1. Let $\langle\mathbf{x}\rangle(t,\mathbf{x}_0,\mathbf{y}_0,t_0)$ be a vector-function

$$\langle\mathbf{x}\rangle(t,\mathbf{x}_0,\mathbf{y}_0,t_0) : [0,T]\times\mathbb{R}^d\times\mathbb{R}^d\times[0,T]\to\mathbb{R}^d$$

$$\langle\mathbf{x}\rangle(t,\mathbf{x}_0,\mathbf{y}_0,t_0) = \{\langle x_1\rangle(t,\mathbf{x}_0,\mathbf{y}_0,t_0),\ldots,\langle x_d\rangle(t,\mathbf{x}_0,\mathbf{y}_0,t_0)\}, \tag{A.8}$$

where

$$\langle x_i\rangle(t,\mathbf{x}_0,\mathbf{y}_0,t_0) = \int\limits_{\mathbb{R}^d} x_i|\Phi(\mathbf{x},t,\mathbf{x}_0,\mathbf{y}_0,t_0)|^2 d^d x =$$

$$= |c_1|^2 \int\limits_{\mathbb{R}^d} x_i|\Psi_1(\mathbf{x},t,\mathbf{x}_0,t_0)|^2 d^d x +$$

$$+|c_2|^2 \int\limits_{\mathbb{R}^d} x_i|\Psi_2(\mathbf{x},t,\mathbf{y}_0,t_0)|^2 d^d x + \qquad (A.9)$$

$$+c_1^* c_2 \int\limits_{\mathbb{R}^d} x_i \Psi_1^*(\mathbf{x},t,\mathbf{x}_0,t_0)\Psi_2(\mathbf{x},t,\mathbf{y}_0,t_0) d^d x +$$

$$+c_1 c_2^* \int\limits_{\mathbb{R}^d} x_i \Psi_1(\mathbf{x},t,\mathbf{x}_0,t_0)\Psi_2^*(\mathbf{x},t,\mathbf{y}_0,t_0) d^d x.$$

Definition A.2. Let $\Delta(t,\mathbf{x}_0,\mathbf{y}_0,t_0)$ be a vector-function

$$\Delta(t,\mathbf{x}_0,\mathbf{y}_0,t_0) : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$$

$$(\Delta(t,\mathbf{x}_0,\mathbf{y}_0,t_0)) = \{\delta_1(t,\mathbf{x}_0,\mathbf{y}_0,t_0),\ldots,\delta_d(t,\mathbf{x}_0,\mathbf{y}_0,t_0)\}, \qquad (A.10)$$

where

$$\delta_i(t,\mathbf{x}_0,\mathbf{y}_0,t_0) = \delta[x_i(t,\mathbf{x}_0,\mathbf{y}_0,t_0)] =$$

$$= c_1^* c_2 \int\limits_{\mathbb{R}^d} x_i \Psi_1^*(\mathbf{x},t,\mathbf{x}_0,t_0)\Psi_2(\mathbf{x},t,\mathbf{y}_0,t_0) d^d x + \qquad (A.11)$$

$$+c_1 c_2^* \int\limits_{\mathbb{R}^d} x_i \Psi_1(\mathbf{x},t,\mathbf{x}_0,t_0)\Psi_2^*(\mathbf{x},t,\mathbf{y}_0,t_0) d^d x.$$

Substituting Eqs.(A.11) into Eqs.(A.9) gives

$$
\begin{aligned}
&\langle x_i\rangle(t,\mathbf{x}_0,\mathbf{y}_0,t_0) = \int\limits_{\mathbb{R}^d} x_i|\Phi(\mathbf{x},t,\mathbf{x}_0,\mathbf{y}_0,t_0)|^2 d^d x = \\
&= |c_1|^2 \int\limits_{\mathbb{R}^d} x_i|\Psi_1(\mathbf{x},t,\mathbf{x}_0,t_0)|^2 d^d x + \\
&+|c_2|^2 \int\limits_{\mathbb{R}^d} x_i|\Psi_2(\mathbf{x},t,\mathbf{y}_0,t_0)|^2 d^d x + \delta_i(t,\mathbf{x}_0,\mathbf{y}_0,t_0) = \\
&= |c_1|^2\langle x_i\rangle(t,\mathbf{x}_0,t_0) + |c_2|^2\langle x_i\rangle(t,\mathbf{y}_0,t_0) + \delta_i(t,\mathbf{x}_0,\mathbf{y}_0,t_0).
\end{aligned} \tag{A.12}
$$

Substitution equations (A.5) into equations (A.12) gives

$$
\begin{aligned}
&\langle x_i\rangle(t,\mathbf{x}_0,\mathbf{y}_0,t_0) = \int\limits_{\mathbb{R}^d} x_i|\Phi(\mathbf{x},t,\mathbf{x}_0,\mathbf{y}_0,t_0)|^2 d^d x = \\
&= |c_1|^2\langle x_i\rangle(t,\mathbf{x}_0,t_0) + |c_2|^2\langle x_i\rangle(t,\mathbf{y}_0,t_0) + \delta_i(t,\mathbf{x}_0,\mathbf{y}_0,t_0) \\
&\simeq |c_1|^2 x_i(t,\mathbf{x}_0,t_0) + |c_2|^2 x_i(t,\mathbf{y}_0,t_0) + \delta_i(t,\mathbf{x}_0,\mathbf{y}_0,t_0).
\end{aligned} \tag{A.13}
$$

Appendix. **B**.

The Schrödinger equation (2.1) in region $\mathbf{I} = \{x|x < 0\}$ has the folloving form

$$
\hbar^2 \frac{\partial^2\Psi_{\mathbf{I}}(x)}{\partial x^2} + 2mE\Psi_{\mathbf{I}}(x) = 0. \tag{B.1}
$$

From Schrödinger equation (B.1) follows

$$\hbar^2 \int_{-\infty}^{0} \frac{\partial^2 \Psi_{\mathbf{I}}(x)}{\partial x^2} dx + 2mE \int_{-\infty}^{0} \Psi_{\mathbf{I}}(x) dx = 0. \qquad (B.2)$$

Let $\Psi_{\mathbf{I}}^{\#}(x)$ be a function

$$\Psi_{\mathbf{I}}^{\#}(x) = \phi(x)\Psi_{\mathbf{I}}(x), \qquad (B.3)$$

where

$$\phi(x) = (\pi r_c^2)^{-1/4} \exp\left(\frac{x^2}{2r_c^2}\right) \qquad (B.4)$$

see Eq.(2.9). Note that

$$\begin{aligned} \frac{\partial^2[\phi(x)\Psi_{\mathbf{I}}(x)]}{\partial x^2} &= \frac{\partial}{\partial x}\left[\Psi_{\mathbf{I}}(x)\frac{\partial \phi(x)}{\partial x} + \phi(x)\frac{\partial \Psi_{\mathbf{I}}(x)}{\partial x}\right] = \\ & 2\frac{\partial \Psi_{\mathbf{I}}(x)}{\partial x}\frac{\partial \phi(x)}{\partial x} + \Psi_{\mathbf{I}}(x)\frac{\partial^2 \phi(x)}{\partial x^2} + \phi(x)\frac{\partial^2 \Psi_{\mathbf{I}}(x)}{\partial x^2}. \end{aligned} \qquad (B.5)$$

Therefore substitution (B.2) into LHS of the Schrödinger equation (B.1) gives

$$\begin{aligned} & \hbar^2 \int_{-\infty}^{0} \frac{\partial^2 \Psi_{\mathbf{I}}^{\#}(x)}{\partial x^2} dx + 2mE \int_{-\infty}^{0} \Psi_{\mathbf{I}}^{\#}(x) dx = \\ & \hbar^2 \int_{-\infty}^{0} \frac{\partial^2 \phi(x)\Psi_{\mathbf{I}}(x)}{\partial x^2} dx + 2Em \int_{-\infty}^{0} \phi(x)\Psi_{\mathbf{I}}(x) dx = \\ & 2\hbar^2 \int_{-\infty}^{0} \frac{\partial \Psi_{\mathbf{I}}(x)}{\partial x}\frac{\partial \phi(x)}{\partial x} dx + \hbar^2 \int_{-\infty}^{0} \Psi_{\mathbf{I}}(x)\frac{\partial^2 \phi(x)}{\partial x^2} dx + \\ & + \int_{-\infty}^{0} \phi(x)\left\{\hbar^2 \frac{\partial^2 \Psi_{\mathbf{I}}(x)}{\partial x^2} + 2Em \int_{-\infty}^{0} \Psi_{\mathbf{I}}(x)\right\} dx. \end{aligned} \qquad (B.6)$$

Note that

$$\int_{-\infty}^{0} \phi(x)\left\{\hbar^2 \frac{\partial^2 \Psi_{\mathbf{I}}(x)}{\partial x^2} + 2Em \int_{-\infty}^{0} \Psi_{\mathbf{I}}(x)\right\}dx = 0. \qquad (B.7)$$

Therefore from Eq.(B.6) and Eq.(2.3)-Eq.(2.4) one obtain

$$\hbar^2 \int_{-\infty}^{0} \frac{\partial^2 \Psi_{\mathbf{I}}^{\#}(x)}{\partial x^2} dx + 2mE \int_{-\infty}^{0} \Psi_{\mathbf{I}}^{\#}(x) dx =$$

$$\hbar^2 \int_{-\infty}^{0} \frac{\partial^2 \phi(x)\Psi_{\mathbf{I}}(x)}{\partial x^2} dx + 2Em \int_{-\infty}^{0} \phi(x)\Psi_{\mathbf{I}}(x) dx = \qquad (B.8)$$

$$= 2\hbar^2 \int_{l}^{\infty} \frac{\partial \Psi_{\mathbf{I}}(x)}{\partial x} \frac{\partial \phi(x)}{\partial x} dx + \hbar^2 \int_{l}^{\infty} \Psi_{\mathbf{I}}(x) \frac{\partial^2 \phi(x)}{\partial x^2} dx.$$

From Eq.(B.6) one obtain

$$\frac{\partial \phi(x)}{\partial x} = (\pi r_c^2)^{-1/4} \frac{\partial}{\partial x} \exp\left[-\frac{x^2}{2r_c^2}\right] = -(\pi r_c^2)^{-1/4} r_c^{-2} x \exp\left[-\frac{x^2}{2r_c^2}\right],$$

$$\frac{\partial^2 \phi(x)}{\partial x^2} = -(\pi r_c^2)^{-1/4} r_c^{-2} \exp\left[-\frac{x^2}{2r_c^2}\right] + \qquad (B.9)$$

$$+(\pi r_c^2)^{-1/4} r_c^{-4} x^2 \exp\left[-\frac{x^2}{2r_c^2}\right].$$

From Eq.(B.9) and Eq.(2.3)-Eq.(2.4) one obtain

$$\hbar^2 \int\limits_{-\infty}^{0} \frac{\partial \Psi_{\mathbf{I}}(x)}{\partial x} \frac{\partial \phi(x)}{\partial x} dx =$$

$$-\frac{\hbar^2}{(\pi r_c^2)^{1/4} r_c^2} \int\limits_{-\infty}^{0} \frac{\partial \exp(ikx)}{\partial x} x \exp\left[-\frac{x^2}{2r_c^2}\right] dx = \tag{B.10}$$

$$-\frac{2\pi\sqrt{2mE}\,\hbar}{(\pi r_c^2)^{1/4} r_c^2} \int\limits_{-\infty}^{0} x \exp\left(i\frac{2\pi\sqrt{2mE}}{\hbar} x\right) \exp\left[-\frac{x^2}{2r_c^2}\right] dx,$$

$$k = \frac{2\pi}{\hbar}\sqrt{2mE}.$$

and

$$\hbar^2 \int\limits_{-\infty}^{0} \Psi_{\mathbf{I}}(x) \frac{\partial^2 \phi(x)}{\partial x^2} dx = -\frac{\hbar^2}{(\pi r_c^2)^{3/4} r_c^2} \int\limits_{-\infty}^{0} \exp(ikx) \exp\left[-\frac{x^2}{2r_c^2}\right] dx + \tag{B.11}$$

$$+\frac{\hbar^2}{(\pi r_c^2)^{1/4} r_c^2} \int\limits_{-\infty}^{0} x^2 \exp(ikx) \exp\left[-\frac{x^2}{2r_c^2}\right] dx.$$

Appendix C. Generalized Postulates for Continuous Valued Observables.

Suppose we have an observable Q of a system that is found, for instance through an exhaustive series of measurements, to have a continuous range of values $\theta_1 < q < \theta_2$. **Then we claim the following**:

C.1. Any given quantum system is identified with some infinite-dimensional Hilbert space $\mathbf{H}$.

Definition C.1. The *pure states* correspond to vectors of norm 1. Thus the set of all pure states corresponds to the unit sphere $\mathbf{S}^{\infty} \subset \mathbf{H}$ in the Hilbert space $\mathbf{H}$.

Definition C.2. The projective Hilbert space $P(\mathbf{H})$ of a complex Hilbert space $\mathbf{H}$ is the set of equivalence classes $[\mathbf{v}]$ of vectors $\mathbf{v}$ in $\mathbf{H}$, with $\mathbf{v} \neq \mathbf{0}$, for the equivalence relation given by $\mathbf{v} \sim_P \mathbf{w} \Leftrightarrow \mathbf{v} = \lambda \mathbf{w}$ for some non-zero complex number $\lambda \in \mathbb{C}$. The equivalence classes for the relation $\sim_P$ are also called rays or projective rays.

Remark C.1.The physical significance of the projective Hilbert space $P(\mathbf{H})$ is that in canonical quantum theory, the states $|\psi\rangle$ and $\lambda|\psi\rangle$ represent the same physical state of the quantum system, for any $\lambda \neq 0$. It is conventional to choose a state $|\psi\rangle$ from the ray $[|\psi\rangle]$ so that it has unit norm $\langle\psi|\psi\rangle = 1$.

Remark C.2. In contrast with canonical quantum theory we have used instead contrary to $\sim_P$ equivalence relation $\sim_Q$, see Def.C.3.

C.2.The states $\{|q\rangle : \theta_1 < q < \theta_2\}$ form a complete set of δ-function normalized basis states for the state space $\mathbf{H}$ of the system.

That the states $\{|q\rangle : \theta_1 < q < \theta_2\}$ form a complete set of basis states means that any state $|\psi\rangle \in \mathbf{H}$ of the system can be expressed as:$|\psi\rangle = \int_{\theta_1}^{\theta_2} c_\psi(q)dq$ while δ-function normalized means that $\langle q|q'\rangle = \delta(q-q')$ from which follows $c_\psi(q) = \langle q|\psi\rangle$ so that $|\psi\rangle = \int_{\theta_1}^{\theta_2}|q\rangle\langle q|\psi\rangle dq$. The completeness condition can then be written as $\int_{\theta_1}^{\theta_2}|q\rangle\langle q|dq = \widehat{\mathbf{1}}$.

Definition C.3. A connected set in $\mathbb{R}$ is a set $X \subset \mathbb{R}$ that cannot be partitioned into two nonempty subsets which are open in the relative topology induced on the set. Equivalently, it is a set which cannot be partitioned into two nonempty subsets such that each subset has no points in common with the set closure of the other.

C.3.For the system in state $|\psi\rangle$ such that (i) $|\psi\rangle \in \mathbf{S}^\infty$ and (ii) supp$(c_\psi(q)) \triangleq \{q|c_\psi(q) \neq 0\}$ is

a connected set in $\mathbb{R}$, the probability $P(q;|\psi\rangle)dq$ of obtaining the result q lying in the range $(q, q+dq)$ on measuring Q is $|\langle q|\psi\rangle|^2 dq = |c_\psi(q)|^2 dq$.

Completeness means that for any state $|\psi\rangle \in \mathbf{S}^\infty$ it must be the case that $\int_{\theta_1}^{\theta_2}|\langle q|\psi\rangle|^2 dq \neq 0$, i.e. there must be a non-zero probability to get some result on measuring Q.

C.4.(**von Neumann measurement postulate**) Assume that (i) $|\psi\rangle \in \mathbf{S}^\infty$ and (ii) supp$(c_\psi(q))$ is a connected set in $\mathbb{R}$. Then if on performing a measurement of Q with an accuracy δq, the result is obtained in the range $(q - \frac{1}{2}\delta q, q + \frac{1}{2}\delta q)$, then the system will end up in the state

$$\frac{\widehat{P}(q,\delta q)|\psi\rangle}{\sqrt{\langle\psi|\widehat{P}(q,\delta q)|\psi\rangle}} = \frac{\int_{|q-q'|\leq\delta q/2}|q'\rangle\langle q'||\psi\rangle dq'}{\sqrt{\int_{|q-q'|\leq\delta q/2}|\langle q'||\psi\rangle|^2 dq'}}. \tag{C.1}$$

C.5.(**Generalized measurement postulates**)

C.5. I. For the system in state $|\psi^a\rangle = a|\psi\rangle \in \mathbf{H}$, where (i) $|\psi\rangle \in \mathbf{S}^\infty, |a| \neq 1$, (ii) **supp**$(c_\psi(q))$ is a connected set in $\mathbb{R}$ and (iii) $|\psi\rangle = \int_{\theta_1}^{\theta_2} c_\psi(q)|q\rangle dq$ the probability $P(q;|\psi^a\rangle)dq$ of obtaining the result q lying in the range $(q - \frac{1}{2}dq, q + \frac{1}{2}dq)$ on measuring Q is

$$P(q;|\psi^a\rangle)dq = |a|^{-2}|c_\psi(q|a|^{-2})|^2 dq. \tag{C.2}$$

Then if on performing a measurement of Q with an accuracy δq, the result is obtained in the range $(q - \frac{1}{2}\delta q, q + \frac{1}{2}\delta q)$, then the system immediately after measurement will end up in the state

$$\frac{\widehat{P}(q,\delta q)|\psi^a\rangle}{\sqrt{\langle\psi|\widehat{P}(q,\delta q)|\psi^a\rangle}} = \frac{\int_{|q-q'|\leq\delta q/2}|q'\rangle\langle q'||\psi^a\rangle dq'}{\sqrt{\int_{|q-q'|\leq\delta q/2}|\langle q'||\psi^a\rangle|^2 dq'}}. \tag{C.3}$$

C.5. II. Let $|\Psi^{a_1,a_2}\rangle = |\psi_1^{a_1}\rangle + |\psi_2^{a_2}\rangle \in \mathbf{H}$, where (i) $|\psi_i^{a_i}\rangle = a_i|\psi_i\rangle \in \mathbf{H}, |\psi_i\rangle \in \mathbf{S}^\infty, |a_i| \neq 1, i = 1,2$ (ii) $\mathbf{supp}(c_{\psi_i}(q)), i = 1,2$ is a connected set in $\mathbb{R}$ (iii) $(\mathbf{supp}(c_{\psi_1}(q))) \cap (\mathbf{supp}(c_{\psi_2}(q))) = \varnothing$

and (iv) $|\psi_i\rangle = \int_{\theta_1}^{\theta_2} c_{\psi_i}(q)|q\rangle dq, i = 1,2.$ Then on performing a measurement of Q with an accuracy $\delta q \ll 1$ the wave function $\langle q||\Psi^{a_1,a_2}\rangle$ experiences a sudden jump of the form

(1) $|\Psi^{a_1,a_2}\rangle \to |\psi_1^{a_1}\rangle$ or (2) $|\Psi^{a_1,a_2}\rangle \to |\psi_2^{a_2}\rangle$ or (3) $|\Psi^{a_1,a_2}\rangle \to \{|\psi_1^{a_1}\rangle, |\psi_2^{a_2}\rangle\}$.

C.5. III. The probability of getting a result q with an accuracy δq such that $(q - \frac{1}{2}\delta q, q + \frac{1}{2}\delta q) \in \mathbf{supp}(c_{\psi_1}(q))$ or $(q - \frac{1}{2}\delta q, q + \frac{1}{2}\delta q) \in \mathbf{supp}(c_{\psi_2}(q))$ given by

$$\int_{|q-q'|\leq\delta q/2}\left[|\langle q'||\psi_1^{a_1}\rangle|^2 + |\langle q'||\psi_2^{a_2}\rangle|^2\right]dq'.$$

C.5. IV. If the system is initially in the state $|\Psi^{a_1,a_2}\rangle$, then the state of the system immediately after measurement given by

$$\frac{\widehat{P}(q,\delta q)|\psi^a\rangle}{\sqrt{\langle\psi|\widehat{P}(q,\delta q)|\psi^a\rangle}} = \frac{\int_{|q-q'|\leq\delta q/2}(|q'\rangle\langle q'||\psi_1^{a_1}\rangle + |q'\rangle\langle q'||\psi_2^{a_2}\rangle)dq'}{\sqrt{\int_{|q-q'|\leq\delta q/2}\left[|\langle q'||\psi_1^{a_1}\rangle|^2 + |\langle q'||\psi_2^{a_2}\rangle|^2\right]dq'}}.$$

Definition C.4. Let $|\psi^a\rangle$ be a state $|\psi^a\rangle = a|\psi\rangle$, where $|\psi\rangle \in \mathbf{S}^\infty, |a| \neq 1$ and $|\psi\rangle = \int_{\theta_1}^{\theta_2} c_\psi(q)|q\rangle dq.$ Let $|\psi_a\rangle$ be an state such that $|\psi_a\rangle \in \mathbf{S}^\infty.$ States $|\psi^a\rangle$ and $|\psi_a\rangle$ is a Q-equivalent: $|\psi^a\rangle \sim_Q |\psi_a\rangle$ iff

$$P(q;|\psi^a\rangle)dq = |a|^{-2}|c_\psi(q|a|^{-2})|^2 dq = P(q;|\psi_a\rangle)dq \tag{C.4}$$

C.6. For any state $|\psi^a\rangle = a|\psi\rangle$, where $|\psi\rangle \in \mathbf{S}^\infty, |a| \neq 1$ and $|\psi\rangle = \int_{\theta_1}^{\theta_2} c_\psi(q)|q\rangle dq$ there exist an state $|\psi_a\rangle \in \mathbf{S}^\infty$ such that: $|\psi^a\rangle \sim_Q |\psi_a\rangle$.

Remark C.3. Formal motivetion of the postulate symple and clear. Let

$|\psi_t^a\rangle, t \in [0,+\infty)$ be a state $|\psi_t^a\rangle = a|\psi_t\rangle$, where $|\psi_t\rangle \in \mathbf{S}^\infty, |a| \neq 1$ and $|\psi_t\rangle = \int_{-\infty}^{+\infty} c_\psi(q)|q\rangle dq$.

Note that:

(**i**) any process of continuous measurements on measuring Q for the system in state $|\psi_t\rangle$ and the system in state $|\psi_t^a\rangle$ one can to describe by an $\mathbb{R}$-valued stochastic processes $X_t(\omega) = X_t(\omega; |\psi_t\rangle)$ and $Y_t^a(\omega) = Y_t^a(\omega; |\psi_t^a\rangle)$ given on an probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and a measurable space $(\mathbb{R}, \Sigma)$.

(**ii**) We assume now that: $\forall \Theta \subseteqq \Omega$

$$\mathbf{E}_\Theta[X_t(\omega)] = \int_{\Theta\subset\Omega} X_t(\omega) d\mathbf{P}(\omega) = \mathbf{E}_\Theta[X_t(\omega; |\psi_t\rangle)] = \langle\psi_t|\widehat{Q}_{\Delta(\Theta)}|\psi_t\rangle,$$

$$\mathbf{E}[Y_t^a(\omega)] = \int_\Omega Y_t^a(\omega) d\mathbf{P}(\omega) = \mathbf{E}[Y_t^a(\omega; |\psi_t^a\rangle)] = \langle\psi_t^a|\widehat{Q}|\psi_t^a\rangle = |a|^2\langle\psi_t|\widehat{Q}_{\Delta(\Theta)}|\psi_t\rangle, \quad (C.5)$$

where $\Delta(\Theta) \subseteqq [\theta_1, \theta_2]$ and

$$\widehat{Q}_{\Delta(\Theta)} = \int_{\Delta(\Theta)} q|q\rangle\langle q| dq. \quad (C.6)$$

From Eq.(C.5) one obtain

$$\mathbf{E}_\Theta[(Y_t^a(\omega))] = |a|^2 \mathbf{E}_\Theta[X_t(\omega)] \quad (C.7)$$

From Eq.(C.7) one obtains

$$Y_t^a(\omega) = |a|^2 X_t(\omega). \quad (C.8)$$

(**iii**) Let $\rho_t(x)$ be a probability density of the stochastic process $X_t(\omega)$ and let $\rho_t^a(y)$ be a probability density of the stochastic process $Y_t^a(\omega)$. From Eq.(C.8) one obtains

$$\rho_t^a(y) = a^{-2}\rho_t\left(\frac{y}{a^2}\right). \quad (C.9)$$

C.**7**.The observable Q is represented by a Hermitean operator $\widehat{Q}$ whose eigenvalues are the possible results $\{q : \theta_1 < q < \theta_2\}$, of a measurement of Q, and the associated eigenstates are the states $\{|q\rangle : \theta_1 < q < \theta_2\}$, i.e. $\widehat{Q}\,|q\rangle = q|q\rangle$.

The name 'observable' is often applied to the operator $\widehat{Q}$ itself. The spectral decomposition of the observable $\widehat{Q}$ is then $\widehat{Q} = \int_{\theta_1}^{\theta_2} q|q\rangle\langle q|dq$.

Definition C.**5**. Let $|\psi^a\rangle$ be a state $|\psi^a\rangle = a|\psi\rangle$, where $|\psi\rangle \in \mathbf{S}^\infty, |a| \neq 1$ and $|\psi\rangle = \int_{\theta_1}^{\theta_2} c_\psi(q)|q\rangle dq$. Let $|\psi_a\rangle$ be an state such that $|\psi_a\rangle \in \mathbf{S}^\infty$. States $|\psi^a\rangle$ and $|\psi_a\rangle$ is a $\widehat{Q}$-equivalent ($|\psi^a\rangle \sim_{\widehat{Q}} |\psi_a\rangle$) iff: $\langle\psi^a|\,\widehat{Q}|\psi^a\rangle = \langle\psi_a|\,\widehat{Q}|\psi_a\rangle$.

C.**8**.For any state $|\psi^a\rangle = a|\psi\rangle$,where $|\psi\rangle \in \mathbf{S}^\infty, |a| \neq 1$ and $|\psi\rangle = \int_{\theta_1}^{\theta_2} c_\psi(q)|q\rangle dq$ there exist an state $|\psi_a\rangle \in \mathbf{S}^\infty$ such that: $|\psi^a\rangle \sim_{\widehat{Q}} |\psi_a\rangle$.

Appendix D. The Position Representation.Position observable of a particle in one dimension.

The position representation is used in quantum mechanical problems where it is the position of the particle in space that is of primary interest. For this reason, the position representation, or the wave function, is the preferred choice of representation.

D.**1**. In one dimension, the position x of a particle can range over the values $-\infty < x < +\infty$. Thus the Hermitean operator $\widehat{x}$ corresponding to this observable will have eigenstates $|x\rangle$ and associated eigenvalues x such that:
$\widehat{x}|x\rangle = x|x\rangle, -\infty < x < +\infty$.

D.**2**. As the eigenvalues cover a continuous range of values, the completeness relation will be expressed as an integral: $|\psi_t\rangle = \int_{-\infty}^{+\infty}|x\rangle\langle x|\psi_t\rangle dx$, where $\langle x|\psi_t\rangle = \psi(x,t)$ is the wave function associated with the particle at each instant t. Since there is a continuously infinite number of basis states $|x\rangle$, these states are δ-function normalized: $\langle x|x'\rangle = \delta(x - x')$.

D.**3**. The operator $\widehat{x}$ itself can be expressed as: $\widehat{x} = \int_{-\infty}^{+\infty} x|x\rangle\langle x|dx$.

Definition D.**1**. A connected set is a set $X \subset \mathbb{R}$ that cannot be partitioned into two nonempty subsets which are open in the relative topology induced on the set. Equivalently, it is a set which cannot be partitioned into two nonempty subsets such that each subset has no points in common with the set closure of the other.

D.**4**.The wave function is, of course, just the components of the state vector $|\psi_t\rangle \in \mathbf{S}^\infty$ with respect to the position eigenstates as basis vectors. Hence, the wave function is often referred to as being the state of the system in the position representation. The probability amplitude $\langle x|\psi_t\rangle$ is just the wave function, written $\langle x|\psi_t\rangle \triangleq \psi(x,t)$ and is such that $|\psi(x,t)|^2 dx$ is the probability $P(x,t;|\psi_t\rangle)$ of the particle being observed to have a coordinate in the range x to $x + dx$

Definition D.**2**. Let $|\psi_t^a\rangle, t \in [0,+\infty)$ be a state $|\psi_t^a\rangle = a|\psi_t\rangle$, where

$|\psi_t\rangle \in \mathbf{S}^\infty, |a| \neq 1$ and $|\psi_t\rangle = \int_{-\infty}^{+\infty} \psi(x,t)|x\rangle dx.$ Let $|\psi_{t,a}\rangle, t \in [0,+\infty)$ be an state such that $|\psi_{t,a}\rangle \in \mathbf{S}^\infty,\ t \in [0,+\infty).$ States $|\psi_t^a\rangle$ and $|\psi_{t,a}\rangle$ is x-equivalent ($|\psi_t^a\rangle \sim_x |\psi_{t,a}\rangle$) iff

$$P(x,t;|\psi_t^a\rangle)dx = |a|^{-2}|\psi(x|a|^{-2},t)|^2 dx = P(x,t;|\psi_{t,a}\rangle)dx \qquad (D.1)$$

D.5.From postulate C.5 (see Appendix C) follows: for any state $|\psi_t^a\rangle = a|\psi_t\rangle,$where $|\psi_t\rangle \in \mathbf{S}^\infty, |a| \neq 1, t \in [0,+\infty)$ and $|\psi_t\rangle = \int_{-\infty}^{+\infty} \psi(x,t)|x\rangle dx$ there exist an state $|\psi_{t,a}\rangle \in \mathbf{S}^\infty,\ t \in [0,+\infty)$ such that: $|\psi_t^a\rangle \sim_x |\psi_{t,a}\rangle.$

Definition D.2. Let $|\psi_t^a\rangle, t \in [0,+\infty)$ be a state $|\psi_t^a\rangle = a|\psi_t\rangle,$ where $|\psi_t\rangle \in \mathbf{S}^\infty, |a| \neq 1$ and $|\psi_t\rangle = \int_{-\infty}^{+\infty} \psi(x,t)|x\rangle dx.$ Let $|\psi_{t,a}\rangle, t \in [0,+\infty)$ be an state such that $|\psi_{t,a}\rangle \in \mathbf{S}^\infty,\ t \in [0,+\infty).$ States $|\psi_t^a\rangle$ and $|\psi_{t,a}\rangle$ is $\hat{x}$-equivalent ($|\psi_t^a\rangle \sim_{\hat{x}} |\psi_{t,a}\rangle$) iff: $\langle \psi_t^a | \hat{x} | \psi_t^a \rangle = \langle \psi_{t,a} | \hat{x} | \psi_{t,a} \rangle.$

D.6.From postulate C.7 (see Appendix C) follows: for any state $|\psi_t^a\rangle = a|\psi_t\rangle,$where $|\psi_t\rangle \in \mathbf{S}^\infty, |a| \neq 1, t \in [0,+\infty)$ and $|\psi_t\rangle = \int_{-\infty}^{+\infty} \psi(x,t)|x\rangle dx$ there exist an state $|\psi_{t,a}\rangle \in \mathbf{S}^\infty,\ t \in [0,+\infty)$ such that: $|\psi_t^a\rangle \sim_{\hat{x}} |\psi_{t,a}\rangle.$

Definition D.3. The pure state $|\psi_t\rangle \in \mathbf{S}^\infty, t \in [0,+\infty), |\psi_t\rangle = \int_{-\infty}^{+\infty} \psi(x,t)|x\rangle dx$ is a weakly Gaussian in the position representation iff

$$|\psi(x,t)|^2 dx = \frac{1}{\sigma_t\sqrt{2\pi}} \exp\left[-\frac{(x-\bar{x}_t)^2}{\sigma_t^2}\right] dx, \qquad (D.2)$$

where $\bar{x}_t$ and σ_t an given functions which depend only on variable t.

D.7.From statement D.5 follows: for any state $|\psi_t^a\rangle = a|\psi_t\rangle,$where $|\psi_t\rangle \in \mathbf{S}^\infty, |a| \neq 1,\ t \in [0,+\infty)$ and $|\psi_t\rangle = \int_{-\infty}^{+\infty} \psi(x,t)|x\rangle dx$ is a weakly Gaussian state there exist an weakly Gaussian state $|\psi_{t,a}\rangle \in \mathbf{S}^\infty,\ t \in [0,+\infty)$ such that:

$$P(x,t;|\psi_t^a\rangle)dx = |a|^{-1}|\psi(x|a|^{-1},t)|^2 dx =$$
$$= \frac{1}{|a|\sigma_t\sqrt{2\pi}} \exp\left[-\frac{(x-|a|\bar{x}_t)^2}{|a|^2\sigma_t^2}\right] dx. \qquad (D.3)$$

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