Co-prime Gap n-Tuples That Sum To A Number And Other Algebraic Forms

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Dedicated to my son Panha

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Abstract

We study the spacings of numbers co-prime to an even consecutive product of primes, $P_m\#$[4] and its structure exposed by the fundamental theorem of prime sieving (FTPS)[3]. We extend this to prove some parts of the Hardy-Littlewood general prime density conjecture[1] for all finite multiplicative groups modulo a primorial. We then use the FTPS to prove such groups have gap spacings which form arithmetic progressions as long as we wish. We also establish their densities and provide prescriptions to find them.

1 Co-prime Gap n-Tuples That Sum To A Number

The set of n-tuples which sum to 2 and 4 are the trivial single gaps and they will occur at counts[3]

$$T^m_{\{2\}} = T^m_{\{4\}} = P^{-2}_m\#$$

We also showed that the gap pairs $\{4, 2\}$ and $\{2, 4\}$ will occur infinitely often having counts in each set co-prime to $P_m\#$ of

$$T^m_{\{4,2\}} = T^m_{\{2,4\}} = P^{-3}_m\#$$

Now we notice these sets sum to 6. When we follow the steps with successive prime numbers we can precisely and simultaneously count all n-tuples that sum to any even number greater than zero. We do this with careful repeated application of the FTPS.

Proof. Choose any span $2n$. We first build the intermediate hybrid[3]

$$h^n_{\{i,j\}} = P^m_i + jP_m\#$$

in the expansion phase of building our next set. We only need to consider the sum in order to count how it will propagate in future generations. Since we only follow the bracketing co-prime numbers and ignore what happens internally, we can treat that span as if it were an individual gap. That is, as long as $2n$ is not divisible by the next prime, $P_{m+1} - 2 = P^{-2}_{m+1}$ copies will survive. In this case we mean both anchoring numbers remain co-prime to the next product, and there are exactly two cases where either anchor is removed.

However if the span is divisible by the next prime there will be exactly one occurrence where both anchoring primes are removed and in this case we have one extra copy. Or $P_{m+1} - 1 = P^{-1}_{m+1}$ instead of $P_{m+1} - 2 = P^{-2}_{m+1}$.

We start with the first prime number, $P_0 = 1$ which is not prime but it is co-prime to every number, and our initial condition[2]. This forms the trivial set $\{1, 2, 3, 4, 5, 6, 7, \dots\}$ from which we identify the number 2 as our first true prime number, $P_1 = 2$. That is, in this instance $P_{m+1} = 2$. It happens that every span $2n$ is divisible by the next prime. Now, we expect to write $(2 - 2)\# = 1$, but we must boost the count by $(2 - 1)\# = 1$! That is, we do nothing.

As soon as we apply 2 to our product we get a new set, also trivial $\{1, 3, 5, 7, 11, 13, \dots\}$ with $P_{m+1} = 3$. Since gaps of 2 and 4 not divisible by any future prime, their count is fixed. However, $3|j6$ so every span of any multiple of 6 has its count relative to 2 doubled. We write this as

$$\mathbf{T}^m_6 = 2P^m\#$$

Now we can count individual gaps of 6 because

$$\mathbf{T}^m_6 = T^m_{\{4,2\}} + T^m_{\{2,4\}} + T^m_{\{6\}}$$

so

$$T^m_6 = 2P^{-2}_m\# - 2P^{-3}_m\#$$

Now since our counting is multiplicative as we concider more complex numbers, we get the famous Hardy-Littlewood conjectured densities.

$$\mathbf{T}^m_{2n} = P^{-2}_m\# \prod_{p|n} \frac{p-1}{p-2}$$

where we start our primes from 2, remembering $(2-2)\# = 1$[3], so our products are twice as large. ☐

1.1 The Density Sophie Germain Co-primes

A Sophie Germain prime is one of the form P_a iff $P_b = 2P_a + 1$ is also a prime. A Sophie Germain co-prime is the same thing but using co-primeness instead.

Theorem 1. *There are $P^{-2}_m\# = T^m_2$ Sophie Germain co-primes in every set of numbers modulo $P_m\#$.*

Proof. We begin by looking at the set modulo $3\#$, these numbers can be generated by the equations $P^m_j = 1+6n$ and $P^m_j = 5+6n$. Now $2P^m_j+1 = 2(5+6n)+1 = 5+6*(2n+1)$. That is, every number of the form $5 + 6n$ is a Sophie Germain co-prime in the set $3\#$.

So we can ask how many are there in succeeding iterations. By construction there will be $P_m^{-2}\#$ such numbers in every future set because for every pair $2P_j^m + 1 = P_k^m$, if one member is divisible by the next prime, its partner can not be.

This can be generalized for the numbers $2(1+6n)-1 = 1+6*(2n)$, that is co-primes of the form $2P_j^m - 1 = P_k^m$ will also occur at the same densities. □

2 Arithmetic Progressions

Any set formed by multiplication modulo a primoral, is a factory of arithmetic progressions on the order of that primorial. And they are of infinite extent since, if P_j^m is coprime to $P_m\#$, so is every $P_j^m + nP_m\#$.

However we also have embedded subsequences as a result of the *"chopping"* process of evolving the set from previous primorials.

Proof. In moving from $P_{m_0-1}\#$ to $P_{m_0}\#$, we had $P_{m_0-1}^{-1}\#$ unique progressions of period $P_{m_0-1}\#$ (hybrid columns), each infinite in extent. In transforming that set to the set co-prime to $P_{m_0}\#$ the FTPS says every sequence must be cut (have an anchor removed). since each column (sequence) has exactly one solution per column per P_{m_0} rows. And if that solution occurs on row $b < P_{m_0}$, it occurs at every row $b + nP_{m_0}$.

This means this set has the same associated progressions but now their sequence is limited to $P_{m_0}-2 = P_{m_0}^{-2}$.

This leads to an important sub-conclusion. There can never be an arithmetic sequence of period $P_m\#$ whose length is greater than P_m^{-2}. For example, $3\# = 6$ so it is impossible to find progressions of $\{P_j^m, P_j^m+6, P_j^m+12, P_j^m+18, P_j^m+24\}$ all co-prime to any primorial $\geq 5\#$. Since these sets generate the primes, it is impossible to find such progressions among the prime numbers.

To see what happens next, we follow the co-prime anchors of a single progression labeled $\{a_1, a_2, \ldots, a_{P_{m_0}^{-1}}\}$. These are now all on a single row in a new hybrid with columns $a_j + nP_{m_0}\#$ since the sequence a_j is periodic $P_{m_0-1}\#$ we have another level of independence in it is not divisible by any higher prime. That is we have the condition that P_{m_0+1} exact copies are made of the sequence and exactly $P_{m_0}^{-2}$ have one member each knocked out leaving

$$P_{m_0-1}^{-1}\# P_{m_0+k}^{-P_{m_0}^{-2}}\#$$

in every future $k \geq 1$ Of course those that don't survive become shorter progressions. So every length must occur up to the limit of the particular prime index. □

References

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[4] Paulo Ribenboim. *The new Book of Prime Number Records 3rd ed.* Springer Press, Springer-Verlag, New York, Inc., 1995.