The Contradiction within Equations of Motion with Constant Acceleration

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This paper is prepared to demonstrate the violation of rules of mathematics in the algebraic derivation of the equations of motion with constant acceleration in Newtonian mechanics and shows the inconsistency and contradiction within these equations even if we have used the correct calculus derivation and also demonstrates the impact of these contradictions on Newtonian mechanics.

THEORETICAL BACKGROUND

When the acceleration is constant (Straight-Line Motion), four equations relate the position \( x \) and the velocity \( v \) at any time \( t \) to the initial position \( x_0 \), the initial velocity \( v_0 \) (both measured at time \( t = 0 \)), and the acceleration \( a \):

\[
v = v_0 + at
\]

\[
x = x_0 + v_0 t + \frac{1}{2}at^2
\]

\[
v^2 = v_0^2 + 2a(x - x_0)
\]

\[
x - x_0 = \left( \frac{v_0 + v}{2} \right) t
\]

where:

\[
\frac{v_0 + v}{2} = v_{av} \quad \text{(average velocity)}
\]

And for rotation with constant angular acceleration we have

\[
\omega = \omega_0 + \alpha t
\]

\[
\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2
\]

\[
\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)
\]

\[
\theta - \theta_0 = \left( \frac{\omega_0 + \omega}{2} \right) t
\]

where:

\[
\frac{\omega_0 + \omega}{2} = \omega_{av} \quad \text{(average angular velocity)}
\]

The Derivation of The Equations of the Motion with Constant Acceleration

The simplest kind of accelerated motion is straight-line motion with constant acceleration and when the x-acceleration \( a \) is constant, the average x-acceleration \( a_{av} \) for any time interval is the same as \( a \).

\[
a = a_{av} = \frac{v_2 - v_1}{t_2 - t_1}
\]

Now we let \( t_1 = 0 \) and let \( t \) be any later time. We use the symbol \( v_0 \) for the x-velocity at the initial time \( t = 0 \); the x-velocity at the later time \( t \) is \( v \). Then Eq.(6) becomes

\[
a = \frac{v - v_0}{t - 0}
\]

\[
v = v_0 + at
\]

Next well derive an equation for the position \( x \) as a function of time when the x-acceleration is constant. To do this, we use two different expressions for the average x-velocity \( v_{av} \) during the interval from \( t = 0 \) to any later time \( t \). The first expression comes from the definition

\[
v_{av} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{\Delta x}{\Delta t}
\]

which is true whether or not the acceleration is constant. We call the position at time \( t = 0 \) the initial position, denoted by \( x_0 \). The position at the later time \( t \) is simply \( x \). Thus for the time interval \( \Delta t = t - 0 \) the displacement is \( \Delta x = x - x_0 \) and Eq.(8) gives

\[
v_{av} = \frac{x - x_0}{t - (t_0 = 0)} = \frac{x - x_0}{t}
\]

We can also get a second expression for \( v_{av} \) that is valid only when the x-acceleration is constant, so that the x-velocity changes at a constant rate. In this case the average x-velocity for the time interval from \( 0 \) to \( t \) is simply

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1 Young, Hugh D.
Sears and Zemansky’s university physics : with modern physics.
the average of the x-velocities at the beginning and end of the interval:

\[ v_{av} = \frac{v_0 + v}{2} \]  \hspace{1cm} (10)

(This equation is not true if the x-acceleration varies during the time interval.) We also know that with constant x-acceleration, the x-velocity \( v \) at any time \( t \) is given by Eq.(7). Substituting that expression for \( v \) into Eq.(12), we find

\[ v_{av} = \frac{1}{2}(v_0 + v_o + at) \]

\[ = v_0 + \frac{1}{2}at \]  \hspace{1cm} (11)

Finally, we set Eqs.(9) and (11) equal to each other and simplify:

\[ v_o + \frac{1}{2}at = \frac{x - x_o}{t} \]  \hspace{1cm} or

\[ x = x_o + v_o t + \frac{1}{2}at^2 \]  \hspace{1cm} (12)

It’s often useful to have a relationship for position, x-velocity, and (constant) x-acceleration that does not involve the time. To obtain this, we first solve Eq.(7) for \( t \) and then substitute the resulting expression into Eq.(12):

\[ t = \frac{v - v_o}{a} \]

\[ x = x_o + v_o \left( \frac{v - v_o}{a} \right) + \frac{1}{2}a \left( \frac{v - v_o}{a} \right)^2 \]

We transfer the term \( x_o \) to the left side and multiply through by \( 2a \) then simplify, we get:

\[ v^2 = v_o^2 + 2a(x - x_o) \]  \hspace{1cm} (13)

After Evaluating we obtain

\[ v = at \]  \hspace{1cm} (15)

\[ x = \frac{1}{2}at^2 \]  \hspace{1cm} (16)

We have two different definitions for average velocity \( v_{av} \) as we have been mentioned in the basic derivation of the equations: Eqs.(9) and (10). Evaluate Eq.(9) at \( x = 0 \) the final position, and \( t = 0 \) the final time which implies that the final velocity \( v \) is equal to 0, the body accelerates uniformly to rest, then we obtain

\[ v_{av} = \frac{0 - x_o}{0 - t_o} = v_o \]  \hspace{1cm} (17)

Consequently Eq.(10) will becomes

\[ v_{av} = \frac{v_o + 0}{2} = \frac{1}{2}v_o \]  \hspace{1cm} (18)

The initial velocity \( v_o \) in Eq.(17) is the same initial velocity \( v_o \) in Eq.(18) and in the basic derivation they have been assumed that both averages are equal to each other and they have been equated as been done on the steps to Eq.(12). Then let us also equate Eq.(17) and (18) as they did, we find

\[ v_o = \frac{1}{2}v_o \]  \hspace{1cm} (19)

Because both initial velocities are one-thing that implies that the initial velocity \( v_o \) is equal to the half-itself which leads to inescapable mathematical conclusion that \( 1 = 2 \) which is not true. The same mathematical conclusion can be found if we evaluate \( x_o = 0 \) and \( t_o = 0 \), the body accelerates uniformly from rest, we will find that the final velocity \( v \) is equal to the half-itslf:

\[ v = \frac{1}{2}v \]  \hspace{1cm} (20)

In order for us to validly equate Eq.(9) and (10), it is mathematically necessary to balance the equation and that be done by multiply Eq.(9) by \( \frac{1}{2} \), or multiply Eq.(10) by 2 then we equate them and continue with the derivation of the Eq.(12). Then we find

\[ v_o + v = \frac{x - x_o}{t} \]  \hspace{1cm} (21)

Substitute Eq.(7) we get

\[ v_o + v_o + at = \frac{x - x_o}{t} \]  \hspace{1cm} or

\[ \frac{1}{2}x = \frac{1}{2}x_o + v_o t + \frac{1}{2}at^2 \]  \hspace{1cm} (22)

Because \( x_o \) is the arbitrary constant of integration, we can rewrite Eq.(22) as:

\[ \frac{1}{2}x = x_o + v_o t + \frac{1}{2}at^2 \]  \hspace{1cm} (23)

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**ANALYSIS**

I. The Violation of Mathematical Rules in Algebraic Derivation of Equations of Motion with Constant Acceleration

At the beginning let us mention the integration of Eq.(7) and evaluate it at \( x_o = 0 \) and \( t_o = 0 \) which implies that \( v_o = 0 \) also. Then we have

\[ v = v_o + at \]

then

\[ \int v \, dt = \int v_o \, dt + \int at \, dt \]  \hspace{1cm} and

\[ x = x_o + v_o t + \frac{1}{2}at^2 \]  \hspace{1cm} (14)
Taking the $\frac{d}{dt}$ we obtain
\[ \frac{1}{2}v = v_o + at \]  

(24)

Evaluate Eq.(23) and (24) at $x_o = 0$ and $t_o = 0$ which implies that $v_o = 0$, we get
\[ x = at^2 \quad \text{and} \quad v = 2at \]  

(25)

(26)

Equation (23) is not equal to Eq.(14), and Eq.(24) is not equal to Eq.(7), and if we have equated them we will end up with the same inescapable mathematical conclusion that $1 = 2$ which emphasizes that they are not equal.

Although we have used the same equation (Eq.(7)) in our derivation we arrived at different results (compare Eq.(23) with Eq.(14)). In the first method we have used the calculus (integration) and in the second method we have used the algebra, and even when we tried to verify the algebraic result by differentiating Eq.(23) to arrive to Eq.(7) we failed (compare Eq.(24) with Eq.(7)).

The only way to derive algebraically Eq.(14) from Eq.(7) using the averages (Eqs.(9) and (10)) is by violating the sound mathematical rules for set up a correct equation (see Eq.(19) and Eq.(20)).

Due to the previous argument we will consider this algebraic derivation as invalid, because the assumption that the average rate of change is equal to the arithmetic mean is false assumption, and because after we correct this assumption (balancing the equation (see Eq.(21)) we couldn’t derive Eq.(14) from Eq.(7), but we arrived at a different result (see Eq.(23)).

II. The Contradiction within Equations of Motion with Constant Acceleration

In the following discussion we are going to depend upon the calculus derivation of the equations of motion with constant acceleration (Eqs.(7) and (14)), and we are going to put aside the algebraic derivation because we have been considered it invalid. Also we are going to assume that $x_o = 0$, $t_o = 0$ and $v_o = 0$, simply we are going to use Eqs.(15) and (16). Then we have
\[ v = at \quad \text{and} \quad x = \frac{1}{2}at^2 \]

Solving the equations for the acceleration $a$, we get
\[ a = \frac{v}{t} \]  

(27)

\[ a = \frac{2x}{t^2} \]  

(28)

Because the motion is under constant acceleration which implies that the velocity is not constant, we are going to use the average velocity definitions (Eqs.(9) and (10)) one at time and that because they are not equal to each other as we have been proofed earlier so we couldn’t substitute one definition in Eq.(27) and the other in Eq.(28). They are not exchangeable unless under certain condition (see Eq.(21)).

Let us first rewrite Eqs.(27) and (28) in term of average velocity, then we have
\[ a = \frac{v_{av}}{t} \]  

(29)

\[ a = 2 \frac{v_{av}}{t} \]  

(30)

Beginning with the average rate of change (Eq.(9)), we obtain
\[ v_{av} = \frac{x - (x_o = 0)}{t - (t_o = 0)} = \frac{x}{t} \]  

(31)

Substitute Eq.(31) into Eqs.(29) and (30) then equating them, we find
\[ \frac{x}{t} = 2 \frac{x}{t^2} \quad \text{and} \quad \frac{x}{t^2} = 2 \frac{x}{t^2} \]  

(32)

Equation (32) leads to an inescapable mathematical conclusion that $1 = 2$ which is a contradiction because $1 \neq 2$.

Now substitute Eq.(31) into Eq.(13) and equating it to Eqs.(29) and (30) consecutively, we find
\[ \frac{x}{t} = \frac{(x/t)^2}{2x} \quad \text{and} \quad \frac{x}{t^2} = \frac{x}{2t^2} \]  

(33)

Equation (33) leads to an inescapable mathematical conclusion that $1 = 2$ which is a contradiction because $1 \neq 2$.

\[ 2 \frac{x}{t} = \frac{(x/t)^2}{2x} \quad \text{and} \quad 2 \frac{x}{t^2} = \frac{x}{2t^2} \]  

(34)

Equation (34) leads to an inescapable mathematical conclusion that $1 = 4$ which is a contradiction because $1 \neq 4$. 
Now we switch to the arithmetic average (Eq. (10)), then we obtain

\[ v_{av} = \frac{(v_0 = 0) + v}{2} = \frac{v}{2} \]  

(35)

Substitute Eq. (35) into Eqs. (29) and (30) then equating them, we find

\[ \frac{\left( \frac{v}{2} \right)}{t} = 2 \left( \frac{\left( \frac{v}{2} \right)}{t} \right) \]

\[ \frac{v}{2t} = \frac{v}{t} \]  

(36)

Again, Eq (36) leads to an inescapable mathematical conclusion that \( 1 = 2 \) which is a contradiction because \( 1 \neq 2 \).

Now substitute Eq. (35) into Eq. (13) and equating it to Eqs. (29) and (30) consecutively, we find

\[ \left( \frac{v}{2} \right) \left( \frac{v}{2} \right) = \frac{1}{2} \left( \frac{v}{2} \right)^2 \]

\[ (v_{av}) \left( \frac{v}{2} \right) = \frac{1}{2} \left( \frac{v}{2} \right)^2 \]

\[ \left( \frac{v}{2} \right)^2 = \frac{1}{2} \left( \frac{v}{2} \right)^2 \]  

and

\[ \left( \frac{v}{2} \right)^2 = \frac{1}{2} \left( \frac{v}{2} \right)^2 \]  

(37)

Again, Eq (37) leads to an inescapable mathematical conclusion that \( 1 = 2 \) which is a contradiction because \( 1 \neq 2 \).

Equation (38) leads to an inescapable mathematical conclusion that \( 1 = 4 \) which is a contradiction because \( 1 \neq 4 \).

Similarly the equations of Rotation with Constant Angular Acceleration have exactly the same contradictions and the whole previous analysis also hold true for it.

II. The Effect of The Contradiction on Newtonian Mechanics

Assuming that \( x_o = 0 \), \( t_o = 0 \) which implies \( v_o = 0 \). We are going to limit the discussion to the average rate of change which simply mean that \( v_{av} = v = x/t \).

1. Kinetic Energy and The Work-Energy Theorem

From Eq. (13) we have

\[ a = \frac{v^2 - v_0^2}{2(x - x_o)} \]

When we multiply this equation by \( m \) and equate to the net force \( F \), we find

\[ F = ma = m \frac{v^2 - v_0^2}{2(x - x_o)} \]  

and

\[ F(x - x_o) = \frac{1}{2} m v^2 - \frac{1}{2} m v_0^2 \]  

(39)

Which is the work-energy theorem, where \( K = \frac{1}{2} m v^2 \) is the definition of kinetic energy.

Evaluate Eq. (39) at \( x_o = 0 \) and \( v_o = 0 \), we obtain

\[ Fx = \frac{1}{2} m v^2 \]  

(40)

Now when we multiply Eq. (27) by \( m v \), we find

\[ (ma)(vt) = m v^2 \]  

and

\[ Fx = m v^2 \]  

(41)

Equating Eq. (40) and (41), we get

\[ m v^2 = \frac{1}{2} m v^2 \]  

and

\[ m v^2 = \frac{1}{2} m v^2 \]  

Again, that leads to the inescapable mathematical conclusion that \( 1 = 2 \) which is a contradiction because \( 1 \neq 2 \).

And when we multiply Eq. (28) by \( m x \), we find

\[ (ma)x = 2 m \left( \frac{x}{t} \right)^2 \]  

and

\[ Fx = 2 m v^2 \]  

(42)
Now equating Eq.(40) and (42), we get

\[ 2mv^2 = \frac{1}{2} mv^2 \quad \text{and} \]

Again, that leads to the inescapable mathematical conclusion that \( 1 = 4 \) which is a contradiction because \( 1 \neq 4 \).

2. Impulse-Momentum Theorem

The definition of the impulse-momentum theorem is:

\[ \vec{J} = \vec{p} - \vec{p}_o \quad \text{and} \]

\[ J = mv - mv_o \]  \hspace{1cm} (43)

Evaluate Eq.(43) at \( v_o = 0 \), then we obtain

\[ J = mv \]  \hspace{1cm} (44)

Now when we multiply Eq.(27) by \( m \), we find

\[ (ma)t = mv \quad \text{and} \]

\[ Ft = mv \quad \text{and} \]

\[ J = mv \]  \hspace{1cm} (45)

Equating Eq.(44) and (45), we get

\[ mv = mv \]  \hspace{1cm} (46)

Which is free from contradiction and set as example of contradiction-free equation.

When we multiply Eq.(28) by \( m \), we find

\[ (ma)t = 2m \left( \frac{x}{t} \right) \]

\[ Ft = 2mv \quad \text{and} \]

\[ J = 2mv \]  \hspace{1cm} (47)

Now equating Eq.(44) and (47), we get

\[ mv = 2mv \quad \text{and} \]

Again, that leads to the inescapable mathematical conclusion that \( 1 = 2 \) which is a contradiction because \( 1 \neq 2 \).

CONCLUSION

We conclude that three different definitions have been assigned to one physical quantity in the equations of motion with constant acceleration, and that quantity is the constant acceleration (either in straight-line motion or rotational motion).

Also we conclude that every equation from the equations of motion with constant acceleration has a different definition to acceleration consequently contradictions have been arisen between these equations either in the equations of straight-line motion or rotational motion.

Also we conclude that the algebraic proof that has been used to derive the equations of motion with constant acceleration which involve using different types of formulae to evaluate the average velocity has been shown to be invalid and violating the sound mathematical rules for set up correct equations.

Also we conclude that the theorems and formulae that have been derived from the equations of motion with constant acceleration inherits the same discrepancies that the equations of motion with constant acceleration have which leads to different definitions to the same physical quantities that have been used in Newtonian mechanics.