Divergence in the Stefan-Boltzmann law at High Energy Density Conditions

Fran De Aquino
Professor Emeritus of Physics, Maranhao State University, S.Luis/MA, Brazil.
Copyright © 2014 by Fran De Aquino. All Rights Reserved.

It was recently detected an unidentified emission line in the stacked X-ray spectrum of galaxy clusters. Since this line is not catalogued as being the emission of a known chemical element, several hypotheses have been proposed, for example that it is of a known chemical element but with an emissivity of 10 or 20 times the expected theoretical value. Here we show that there is a divergence in the Stefan-Boltzmann equation at high energy density conditions. This divergence is related to the correlation between gravitational mass and inertial mass, and it can explain the increment in the observed emissivity.

Key words: Stefan-Boltzmann law, Thermal radiation, Emissivity, gravitational mass and inertial mass.

1. Introduction

The recent detection of an unidentified emission line in the stacked X-ray spectrum of galaxy clusters \(^{1}\) originated several explanations for the phenomenon. It was proposed, for example that the unidentified emission line, spite to be non-catalogued, it is of a known chemical element but with intensity (emissivity) of 10 to 20 times the expected value.

Here we show that there is a divergence in the Stefan-Boltzmann equation at high energy density conditions. This divergence is related to the correlation between gravitational mass and inertial mass, and it can explain the increment in the observed emissivity.

2. Theory

The quantization of gravity shows that the gravitational mass \(m_g\) and inertial mass \(m_i\) are not equivalents, but correlated by means of a factor \(\chi\), which, under certain circumstances can be negative. The correlation equation is \(^{2}\)

\[
m_g = \chi m_i
\]

where \(m_i\) is the rest inertial mass of the particle.

The expression of \(\chi\) can be put in the following forms \(^{2}\):

\[
\chi = \frac{m_g}{m_i} = \left\{1 - 2 \left[1 + \left(\frac{D}{\rho c^3 n_r c^2} \frac{n_r c^2}{\frac{D n_r c^2}{\rho c^3} - 1}\right)^{\frac{1}{2}}\right]\right\} \quad (3)
\]

where \(W\) is the density of electromagnetic energy on the particle \((J/kg)\); \(D\) is the radiation power density; \(\rho\) is the matter density of the particle \((kg/m^3)\); \(n_r\) is the index of refraction, and \(c\) is the speed of light.

Equations (2) and (3) show that only for \(W = 0\) or \(D = 0\) the gravitational mass is equivalent to the inertial mass \((\chi = 1)\). Also, these equations show that the gravitational mass of a particle can be significatively reduced or made strongly negative when the particle is subjected to high-densities of electromagnetic energy.

Another important equations obtained in the quantization theory of gravity is the new expression for the kinetic energy of a particle with gravitational mass \(m_g\) and velocity \(V\), which is given by \(^{2}\)

\[
E_{	ext{kinetic}} = \frac{1}{2} m_g V^2 = \frac{1}{2} m_i V^2 \quad (4)
\]

Only for \(\chi = 1\) the equation above reduces to the well-known expression \(E_{	ext{kinetic}} = \frac{1}{2} m_i V^2\).

The thermal energy for a single particle calculated starting from this equation is \(k_T = \frac{1}{2} m_i V^2\) \(^{3}\), where the line over the velocity term indicates that the average value
is calculated over the entire ensemble; 
\[ k_B = 1.38 \times 10^{-23} J / K \] is the Boltzmann constant.

Now, this expression can be rewritten as follows.

\[ \langle E \rangle = \frac{\sum_{n=0}^{\infty} n hf P(n hf)}{\sum_{n=0}^{\infty} P(n hf)} \frac{\sum_{n=0}^{\infty} n hf \exp\left(\frac{-n hf}{k_B T}\right)}{\sum_{n=0}^{\infty} \exp\left(\frac{-n hf}{k_B T}\right)} \]

whose result is

\[ \langle E \rangle = \frac{hf}{k_B T} \left(\frac{hf}{e^{k_B T}} - 1\right) \]

or

\[ \frac{\langle E \rangle}{k_B T} = \frac{hf}{e^{k_B T}} \left(\frac{hf}{e^{k_B T}} - 1\right) \]

Note that only for \( hf \ll k_B T \), this expression reduces to \( \langle E \rangle = k_B T \) (the classical assumption that breaks down at high frequencies). Equation (9) is therefore the quantum correction factor, which transforms the Rayleigh-Jeans equation \( \frac{2k_B T^2}{c^2} \) into the Planck’s equation, i.e.,

\[ I(f, T) = \frac{2k_B T^2}{c^2} \left[\frac{hf}{e^{k_B T}} - 1\right] = \frac{2hf^3}{c^2} \frac{1}{e^{k_B T}} - 1 \]

However, in the derivation of the Planck’s law the wrong assumption that \( E_{\text{thermal}} = k_B T \) was maintained. Now, Eq. (5) tells us that we must replace \( k_B T \) for \( \chi k_B T \). Then the Planck’s equation must be rewritten as

\[ I(f, T) = \frac{2hf^2}{c^2} \frac{1}{e^{hf/k_B T}} - 1 \]

\( I(f, T) \) is the amount of energy per unit surface area per unit time per unit solid angle emitted at a frequency \( f \) by a black body at temperature \( T \).

Starting from Eq. (11) we can write the expression of the power density \( D \) (watts/m\(^2\)) for emitted radiation

\[ D = \frac{P}{A} = \int_0^\infty I(f, T) df \int d\Omega \]

To derive the Stefan–Boltzmann law, we must integrate \( \Omega \) over the half-sphere and integrate \( f \) from 0 to \( \infty \). Furthermore,
because black bodies are Lambertian (i.e. they obey Lambert’s cosine law), the intensity observed along the sphere will be the actual intensity times the cosine of the zenith angle $\varphi$, and in spherical coordinates, $d\Omega = \sin \varphi \, d\varphi \, d\theta$. Thus,

$$D = \frac{P}{A} = \int_0^\pi f(f,T) \frac{2\pi}{\theta} \int_0^{\pi/2} \cos \varphi \sin \varphi d\varphi =$$

$$= \pi \int_0^\pi f(f,T) \frac{2\pi}{c^2} \int_0^{\pi/2} \frac{f^3 \, df}{e^{\frac{hf}{k_BT}}}$$

Then, by making

$$u = \frac{hf}{k_BT}, \quad du = \frac{h}{k_BT} \, df$$

Then Eq. (13) gives

$$D = \chi^4 \frac{2\pi}{c^2} \frac{h}{k_BT} \left(\frac{k_BT}{h}\right)^4 \int_0^\infty \frac{u^3 \, du}{e^u - 1}$$

The integral above can be done in several ways. The result is, $\pi^4/15$ [4]. Thus, we get

$$D = \chi^4 \left(\frac{2\pi^5 k_B^4}{15 c^2 h^3}\right) T^4 = \chi^4 \sigma_B T^4$$  \hspace{1cm} (14)

where $\sigma_B = 5.67 \times 10^{-8}$ watts / $m^2 \cdot K^4$ is the Stefan-Boltzmann’s constant.

Note that, for $\chi = 1$ (gravitational mass equal to inertial mass), Eq. (14) reduces to the well-known Stefan-Boltzmann’s equation. However, at high energy density conditions the factor $\chi^4$ can become much greater than 1 (See Eqs. (2) and (3)). This divergence, which is related to the correlation between gravitational mass and inertial mass, can explain the increment of 10 to 20 times in the recently observed emissivity. In this case, we would have $\chi^4 = 10$ to $20 \rightarrow \chi \approx -2$.

If we put $\chi \approx -2$ and $W = B^2 / \mu_0$ into Eq. (2) the result is

$$B = \sqrt{\frac{21 \mu_0 c^2}{2\mu_r}} = 5.1 \times 10^6 \sqrt{\rho / \mu_r}$$

For example, in the case of an intergalactic plasma with $\rho \ll 1 \, kg/m^3$ and $n_r \approx 1$, Eq. (15) gives

$$B \ll 5.3 \times 10^6 \text{ Tesla}$$

Magnetic fields with these intensities are relatively common in the Universe, and even much more intense as for example, the magnetic field of neutron stars ($10^6$ to $10^8$ Tesla) and of the magnetars ($10^8$ to $10^{11}$ Tesla) [5, 6, 7].

In the case of Thermal radiation, considering Eq. (14), we can put Eq. (3) in the following form

$$\chi = 1 - 2 \left[ 1 + \left(\frac{\chi^4 \sigma_B T^4 \frac{n_r^2}{\rho}}{\chi^4 / 15}\right)^2 \right]^{-1}$$  \hspace{1cm} (17)

For $\chi \approx -2$, we get

$$T = 9.08 \times 10^7 \frac{\rho}{n_r^2}$$

For $\rho \ll 1 \, kg/m^3$ and $n_r \approx 1$ Eq. (18) gives

$$T \ll 9.08 \times 10^6 K$$

Temperatures $T \approx 10^6 K$ are relatively common in the Universe (close to a star, for example).

Thus, we can conclude that there are several ways to produce $\chi \approx -2$ in an intergalactic plasma (or interstellar plasma) in the Universe.
References


