Semicontinuous filter limits of nets of lattice group-valued functions

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ABSTRACT: Some conditions for semicontinuity of the limit function of a pointwise convergent net of lattice group-valued functions with respect to filter convergence are given. In this framework we consider some kinds of filter exhaustiveness.

Definitions 1 Let X be a Hausdorff topological space.

(a) A function $f : X \to R$ is *upper semicontinuous* at a point $x \in X$ iff there is an (O)-sequence $(\sigma_p)_p$ (depending on x) such that for each $p \in \mathbb{N}$ there is a neighborhood U_x of x with $f(z) \le f(x) + \sigma_p$ whenever $z \in U_x$. We say that $f \in R^X$ is *lower semicontinuous* at x iff $-f$ is upper semicontinuous at x. A function $f \in R^X$ is *continuous* at x iff it is both upper and lower semicontinuous at x.If the (O)-sequence $(\sigma_p)_p$ can be chosen independently of $x \in X$, then we say that f is *globally upper semicontinuous* (resp. *globally lower semicontinuous*) on X. A function $f \in R^X$ is *globally continuous* on X iff it is both globally upper and globally lower semicontinuous on X.

(b) Let $x \in X$. We say that a net $f_\lambda : X \to R$, $\lambda \in \Lambda$, is $\mathcal{F}$*-upper exhaustive* (or $\mathcal{F}$*-backward exhaustive*) *at* x iff there is an (O)-sequence $(\sigma_p)_p$ such that for any $p \in \mathbb{N}$ there exist a neighborhood U of x and a set $F \in \mathcal{F}$ such that for each $\lambda \in F$ and $z \in U$ we have $f_\lambda(z) \le f_\lambda(x) + \sigma_p$. We say that $(f_\lambda)_\lambda$ is $\mathcal{F}$*-lower exhaustive* (or $\mathcal{F}$*-forward exhaustive*) *at* x iff $(-f_\lambda)_\lambda$ is $\mathcal{F}$-upper exhaustive at x.

(c) A net $f_\lambda : X \to R$, $\lambda \in \Lambda$, is *weakly* $\mathcal{F}$*-upper exhaustive at* x iff there is an (O)-sequence $(\sigma_p)_p$ such that for each $p \in \mathbb{N}$ there is a neighborhood U of x such that for every $z \in U$ there is $F_z \in \mathcal{F}$ with $f_\lambda(z) \le f_\lambda(x) + \sigma_p$ whenever $\lambda \in F_z$. We say that $(f_\lambda)_\lambda$ is *weakly $\mathcal{F}$-lower exhaustive at x* iff $(-f_\lambda)_\lambda$ is weakly $\mathcal{F}$-upper exhaustive at x.

(d) We say that $f_\lambda : X \to R$, $\lambda \in \Lambda$, is *weakly $\mathcal{F}$-upper (lower) exhaustive on X* iff it is weakly $\mathcal{F}$-upper (lower) exhaustive at every $x \in X$ with respect to a single (O)-sequence, independent of $x \in X$.

(e) We say that $(f_\lambda)_\lambda$ is weakly $\mathcal{F}$-exhaustive at x (resp. on X) iff it is both weakly $\mathcal{F}$-upper and weakly $\mathcal{F}$-lower exhaustive at x (resp. on X).

(f) We say that the net $f_\lambda : X \to R$, $\lambda \in \Lambda$, is *$\mathcal{F}$-almost below a function $f : X \to R$ around a point $x \in X$* iff there exists an (O)-sequence $(\sigma_p)_p$ such that for every $p \in \mathbb{N}$ there is a neighborhood U of x such that for each $z \in U$ there is a set $F_z \in \mathcal{F}$ with $f_\lambda(z) \le f(x) + \sigma_p$ whenever $\lambda \in F_z$. We say that $(f_\lambda)_\lambda$ is *$\mathcal{F}$-almost above f around x* iff $(-f_\lambda)_\lambda$ is $\mathcal{F}$-almost below $-f$ around x.

Theorem 2 *Let $\mathcal{F}$ be a (Λ)-free filter of Λ, X be a Hausdorff topological space, $f_\lambda : X \to R$, $\lambda \in \Lambda$, be a net of functions, $(RO\mathcal{F})$-convergent to $f : X \to R$ with respect to a single (O)-sequence $(\sigma^*_p)_p$, and $x \in X$ be a fixed point. Then the following are equivalent:*

(i) $(f_\lambda)_\lambda$ is weakly $\mathcal{F}$-upper exhaustive at x;

(ii) f is upper semicontinuous at x;

(iii) $(f_\lambda)_\lambda$ is $\mathcal{F}$-almost below f around x.

Theorem 3 *Let $\mathcal{F}$, Λ, X be as in Theorem 2, $f_\lambda : X \to R$, $\lambda \in \Lambda$, be a function net, $(RO\mathcal{F})$-convergent to $f : X \to R$, and $x \in X$. Then the following are equivalent:*

(i) $(f_\lambda)_\lambda$ is weakly $\mathcal{F}$-lower exhaustive at x;

(ii) f is lower semicontinuous at x;

(iii) $(f_\lambda)_\lambda$ is $\mathcal{F}$-almost above f around x.

Definitions 4 (a) A net $(f_\lambda)_\lambda$ is $(RO\mathcal{F})$*-upper convergent to* f iff there is an (O)-sequence $(\sigma_p)_p$ such that for each $p \in \mathbb{N}$ and $x \in X$ there is a set $F \in \mathcal{F}$ with $f(x) \le f_\lambda(x) + \sigma_p$ whenever $\lambda \in F$. We say that $(f_\lambda)_\lambda$ is $(RO\mathcal{F})$*-lower convergent to* f iff $(-f_\lambda)_\lambda$ is $(RO\mathcal{F})$-upper convergent to $-f$, and that $(f_\lambda)_\lambda$ is $(RO\mathcal{F})$*-convergent to* iff it is both $(RO\mathcal{F})$-upper and $(RO\mathcal{F})$-lower convergent to f.

(b) A net $(f_\lambda)_\lambda$ is said to be $(\mathcal{F}\text{-}\mathcal{T}^s)$*-upper convergent to* f iff there is an (O)-sequence $(\sigma_p)_p$ such that $(f_\lambda)_\lambda$ $(RO\mathcal{F})$-converges to f with respect to $(\sigma_p)_p$ and for each $p \in \mathbb{N}$ and $x \in X$ there is a set $F \in \mathcal{F}$ such that for every $\lambda \in F$ there is a neighborhood U of x such that $f(z) \le f_\lambda(z) + \sigma_p$ whenever $z \in U$. We say that $(f_\lambda)_\lambda$ is $(\mathcal{F}\text{-}\mathcal{T}^s)$*-lower convergent to* f iff $(-f_\lambda)_\lambda$ is $(\mathcal{F}\text{-}\mathcal{T}^s)$-upper convergent to $-f$, and that $(f_\lambda)_\lambda$ is $(\mathcal{F}\text{-}\mathcal{T}^s)$-*convergent to* f iff $(f_\lambda)_\lambda$ is both $(\mathcal{F}\text{-}\mathcal{T}^s)$-upper and $(\mathcal{F}\text{-}\mathcal{T}^s)$-lower convergent to f,

Theorem 5 *Let X be a Hausdorff topological space, $f: X \to R$ be globally upper semicontinuous on X, $f_\lambda : X \to R$, $\lambda \in \Lambda$, be a net of functions globally lower semicontinuous on X with respect to a single (O)-sequence independent of λ, and $(RO\mathcal{F})$-upper convergent to f. Suppose that*

(i) $(f_\lambda)_\lambda$ is $\mathcal{F}$-almost below f around every point $x \in X$ with respect to a single (O)-sequence, independent of x.

Then $(f_\lambda)_\lambda$ is $(\mathcal{F}\text{-}\mathcal{T}^s)$-upper convergent to f.

Corollary 6 *Let X be a Hausdorff topological space, $f: X \to R$ be globally upper semicontinuous on X, $f_\lambda : X \to R$, $\lambda \in \Lambda$, be a net of functions, globally lower semicontinuous on X with respect to a single (O)-sequence independent of λ, and $(RO\mathcal{F})$-convergent to f.*

Then $(f_\lambda)_\lambda$ is $(\mathcal{F}\text{-}\mathcal{T}^s)$-upper convergent to f.

Theorem 7 *Let X, R and f be as in Corollary.6, $f_\lambda : X \to R$,*

$\lambda \in \Lambda$, *be a net of functions, globally continuous on* X *with respect to a single* (O)*-sequence independent of* λ, $(RO\mathcal{F})$*-convergent to* f *and* $(\mathcal{F}\text{-}\mathcal{T}^s)$*-upper convergent to* f.

Then $(f_\lambda)_\lambda$ *and* f *satisfy condition* (i) *of Theorem 5.*