The Simple Mersenne Conjecture

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Abstract

In this paper we conjecture that there is no Mersenne number $M_p = 2^p - 1$ to be prime for $p = 2^k \pm 1, \pm 3$ when $k > 7$, where p is positive integer and k is natural number. It is called the simple Mersenne conjecture and holds till $p \leq 30402457$ from status of this conjecture. If the conjecture is true then there are no more double Mersenne primes besides known double Mersenne primes MM_2, MM_3, MM_5, MM_7.

Keywords: Mersenne prime; double Mersenne prime; new Mersenne conjecture; strong law of small numbers; simple Mersenne conjecture.

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How did Mersenne form his list p = 2,3,5,7,13,17,19,31,67,127,257 to make 2^p-1 become primes (original Mersenne conjecture) and why did the list have five errors (67 and 257 were wrong but 61,89,107 did not appear here)? Some of mathematicians have studied this problem carefully[1]. From verification results of new Mersenne conjecture we see three conditions in the conjecture all hold only for p = 3,5,7,13,17,19,31,61,127 though new Mersenne conjecture has been verified to be true for all primes $p < 20000000$[2,3]. If we only consider Mersenne primes and p is positive integer then we will discovery there is at least one prime 2^p-1 for $p = 2^k \pm 1, \pm 3$ when $k \le 7$ (k is natural number 0,1,2,3,…), however, such connections will disappear completely from known Mersenne primes when $k > 7$. Therefore, a new conjecture about Mersenne primes, the simple Mersenne conjecture, will be presented.

1. Mersenne Primes for $k \le 8$

Definition 1.1 If p is a prime number then $M_p = 2^p-1$ is called a Mersenne number.

Definition 1.2 If Mersenne number $M_p = 2^p-1$ is prime then the number $M_p = 2^p-1$ is called Mersenne prime.

When $k \le 7$, there exists at least one Mersenne prime $M_p = 2^p-1$ to be able to be found for $p = 2^k \pm 1, \pm 3$.

Case 1.1 $k = 0$: $p = 2^0+1 = 2$. $M_2 = 2^2-1$ is Mersenne prime.

Case 1.2 $k = 1$: $p = 2^1+1 = 3$, $p = 2^1+3 = 5$. $M_3 = 2^3–1$, $M_5 = 2^5–1$ are Mersenne primes.

Case 1.3 $k = 2$: $p = 2^2+1 = 5$, $p = 2^2–1 = 3$, $p = 2^2+3 = 7$. $M_5 = 2^5–1$, $M_3 = 2^3–1$, $M_7 = 2^7–1$ are Mersenne primes.

Case 1.4 $k = 3$: $p = 2^3–1 = 7$, $p = 2^3–3 = 5$. $M_7 = 2^7–1$, $M_5 = 2^5–1$ are Mersenne primes.

Case 1.5 $k = 4$: $p = 2^4+1 = 17$, $p = 2^4+3 = 19$, $p = 2^4–3 = 13$. $M_{17} = 2^{17}–1$, $M_{19} = 2^{19}–1$, $M_{13} = 2^{13}–1$ are Mersenne primes.

Case 1.6 $k = 5$: $p = 2^5–1 = 31$. $M_{31} = 2^{31}–1$ is Mersenne prime.

Case 1.7 $k = 6$: $p = 2^6–3 = 61$. $M_{61} = 2^{61}–1$ is Mersenne prime.

Case 1.8 $k = 7$: $p =2^7–1 = 127$. $M_{127} = 2^{127}–1$ is Mersenne prime.

These results indicate that most of Mersenne primes existing for $p \leq 127$ (M_2, M_3, M_5, M_7, M_{13}, M_{17}, M_{19}, M_{31}, M_{61}, M_{127}) can be found by way of this pattern because only two Mersenne primes M_{89}, M_{107} do not appear in these cases. Considering at least one Mersenne prime to appear in the first eight such cases, it seems to be a reasonable pattern in searching for a part of Mersenne primes. However, such connections will disappear completely when $k = 8$ as follows

Case 1.9 $k = 8$: there are no any known Mersenne primes M_p for $p = 2^8 \pm 1, \pm 3$.

It is no doubt another example of Guy's strong law of small numbers[4].

2. Mersenne Primes for *k* > 7 and Simple Mersenne Conjecture

We have known there are no known Mersenne primes M_p for $p = 2^8 \pm 1, \pm 3$ (Case 1.9), and generally, we discovery that there always are no known Mersenne primes M_p for $p = 2^k \pm 1, \pm 3$ when $k > 7$, so that we have the following conjecture.

Conjecture 2.1 (Simple Mersenne Conjecture). There is no Mersenne number $M_p = 2^p - 1$ to be prime for $p = 2^k \pm 1, \pm 3$ when $k > 7$, where p is positive integer and k is natural number.

Observation 2.1 The 13rd and the 14th Mersenne primes: M_{521} and M_{607} appear between $p = 2^9 + 3 = 515$ and $p = 2^{10} - 3 = 1021$.

Observation 2.2 The 15th Mersenne prime: M_{1279} appears between $p = 2^{10} + 3 = 1027$ and $p = 2^{11} - 3 = 2045$.

Observation 2.3 The 16th to the 18th Mersenne primes: M_{2203}, M_{2281} and M_{3217} appear between $p = 2^{11} + 3 = 2051$ and $p = 2^{12} - 3 = 4093$.

Observation 2.4 The 19th and the 20th Mersenne primes: M_{4253} and M_{4423} appear between $p = 2^{12}+3 = 4099$ and $p = 2^{13}–3 = 8189$.

Observation 2.5 The 21st to the 23rd Mersenne primes: M_{9689}, M_{9941} and M_{11213} appear between $p = 2^{13}+3 = 8195$ and $p = 2^{14}–3 = 16381$.

Observation 2.6 The 24th to the 26th Mersenne primes: M_{19937}, M_{21701} and M_{23209} appear between $p = 2^{14}+3 = 16387$ and $p = 2^{15}–3 = 32765$.

Observation 2.7 The 27th Mersenne prime: M_{44497} appears between $p = 2^{15}+3 = 32771$ and $p = 2^{16}–3 = 65533$.

Observation 2.8 The 28th and the 29th Mersenne primes: M_{86243} and M_{110503} appear between $p = 2^{16}+3 = 65539$ and $p = 2^{17}–3 = 131069$.

Observation 2.9 The 30th and the 31st Mersenne primes: M_{132049} and M_{216091} appear between $p = 2^{17}+3 = 131075$ and $p = 2^{18}–3 = 262141$.

Observation 2.10 The 32nd and the 33rd Mersenne primes: M_{756839} and M_{859433} appear between $p = 2^{19}+3 = 524291$ and $p = 2^{20}–3 = 1048573$.

Observation 2.11 The 34th and the 35th Mersenne primes: $M_{1257787}$ and $M_{1398269}$ appear between $p = 2^{20}+3 =1048579$ and $p = 2^{21}–3 = 2097149$.

Observation 2.12 The 36th and the 37th Mersenne primes: $M_{2976221}$ and $M_{3021377}$ appear between $p = 2^{21}+3 = 2097155$ and $p = 2^{22}–3 = 4194301$.

Observation 2.13 The 38th Mersenne prime: $M_{6972593}$ appears between $p = 2^{22}+3 = 4194307$ and $p = 2^{23}–3= 8388605$.

Observation 2.14 The 39th Mersenne prime: $M_{13466917}$ appears between $p = 2^{23}+3 = 8388611$ and $p = 2^{24}–3 = 16777213$.

Observation 2.15 The 40th to the 44th Mersenne primes: $M_{20996011}$, $M_{24036583}$, $M_{25964951}$, $M_{30402457}$ and $M_{32582657}$ appear between $p = 2^{24}+3 = 16777219$ and $p = 2^{25}–3 = 33554429$.

Observation 2.16 The 45th to the 48th Mersenne primes: $M_{37156667}$, $M_{42643801}$, $M_{43112609}$ and $M_{57885161}$ appear between $p = 2^{25}+3 = 33554435$ and $p = 2^{26}–3 = 67108861$.

It has been confirmed that there exist no any undiscovered Mersenne primes between the 1st and the 43rd Mersenne primes[5]. Hence the simple Mersenne conjecture holds till $p \leq 30402457$. Although we have not known whether there exist undiscovered Mersenne primes between the 43rd and the 48th Mersenne primes, the 44th to the 48th known Mersenne primes do not contradict the conjecture.

3. Connections between Double Mersenne Primes and Simple Mersenne Conjecture

Definition 3.1 If $M_p = 2^p–1$ is a Mersenne prime then $MM_p = 2^{M_p} - 1$ is called a double Mersenne number.

Definition 3.2 If a double Mersenne number MM_p is prime then the number MM_p is called double Mersenne prime.

Corollary 3.1 If the simple Mersenne conjecture is true, then there are no more double Mersenne primes besides known double Mersenne primes MM_2, MM_3, MM_5, MM_7.

Proof. We see known double Mersenne primes are MM_2, MM_3, MM_5, MM_7. By the simple Mersenne conjecture, there is no $M_p = 2^p–1$ to be prime for $p = 2^k \pm 1, \pm 3$ when $k > 7$, which includes the case that $M_p = 2^p–1$ is composite for $p = 2^k–1$ when $k > 7$. If k is a composite number then $p = 2^k–1$ is a composite number but is not a Mersenne number and if k is a prime number then $p = 2^k–1$ is a Mersenne number by Definition 2.1. If $p = 2^k–1$ is a composite Mersenne number then $M_p = 2^p–1$ is a composite number but is not a double Mersenne number and if $p = 2^k–1$ is a Mersenne prime then $M_p = MM_k = 2^p–1$ is a double Mersenne number by Definition 3.1. Since $M_p = MM_k = 2^p–1$ is composite for $p = 2^k–1$ when $k > 7$. Hence there are no double Mersenne primes $M_p = MM_k$ for $p = 2^k-1$ when $k > 7$ by Definition 3.2, which implies there are no more double Mersenne primes besides known double Mersenne primes MM_2, MM_3, MM_5, MM_7.

In fact, we have known that MM_{13} is composite double Mersenne number (for $k = 13 > 7$), MM_{17} is composite double Mersenne number (for $k = 17 > 7$), MM_{19} is composite double Mersenne number (for $k = 19 > 7$) and MM_{31} is composite double Mersenne number (for $k = 31 > 7$)[6]. It seems to be also another example of strong law of small numbers that double Mersenne primes $M_p = MM_k$ appear for $p = 2^k-1$ when k being exponent of Mersenne prime, because double Mersenne numbers $M_p = MM_k$ are all prime for the first four exponents of Mersenne primes $k = 2,3,5,7$ but there are no known double Mersenne primes for $k > 7$.

It is generally believed that Mersenne primes are infinite but it is very difficult to give a proof for the problem. Euler gived a part of proof for existence of infinitely many composite Mersenne numbers[7] but there has not been any part of proof for existence of infinitely many Mersenne primes to this day. However, if the simple Mersenne conjecture is true then the following two subsets of Mersenne primes are finite, that is, Mersenne primes $M_p = 2^p-1$ for $p = 2^k \pm 1, \pm 3$ are finite and Mersenne primes to be able to be written double Mersenne primes $M_p = MM_k$ are finite.

References

[1]. P. T. Bateman, J. L. Selfridge and S. S. Wagstaff, The New Mersenne Conjecture, Amer. Math. Monthly, 96(1989), 125-128.

[2]. The New Mersenne Prime Conjecture in The On-Line The Prime Pages. http://primes.utm.edu/mersenne/NewMersenneConjecture.html

[3]. New Mersenne Prime Conjecture in The On-Line Wolfram Mathworld. http://mathworld.wolfram.com/NewMersennePrimeConjecture.html

[4]. Richard K. Guy, The Strong Law of Small Numbers, Amer. Math. Monthly, 95(1988), 697-712.

[5]. Mersenne prime in The On-Line Wikipedia. http://en.wikipedia.org/wiki/Mersenne_prime

[6]. Double Mersenne Number in The On-Line Wolfram Mathworld. http://mathworld.wolfram.com/DoubleMersenneNumber.html

[7]. Pingyuan Zhou, On the Composite Terms in Sequence Generated from Mersenne-type Recurrence Relations in The On-Line viXra. http://vixra.org/pdf/1046.0181v1.pdf