Strong uniform continuity and filter exhaustiveness of nets of cone metric space-valued functions

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A *cone metric space* is a nonempty set R endowed with a "distance" function $\rho : R \times R \to Y$, where Y is a Dedekind complete lattice group.

A function $f : X \to R$ is *globally continuous on* X iff there exists an (O)-sequence $(\sigma_p)_p$ in Y such that for any $p \in \mathbb{N}$ and $x \in X$ there is a neighborhood U_x of x with $\rho(f(x), f(z)) \leq \sigma_p$ for each $z \in U_x$.

Let X be a uniform space and $\emptyset \neq B \subset X$. A function $f : X \to R$ is *strongly uniformly continuous on* B iff there is an (O)-sequence $(\sigma_p)_p$ in Y such that for every $p \in \mathbb{N}$ there exists an entourage D with $\rho(f(\beta), f(x)) \leq \sigma_p$ whenever $x \in X$, $\beta \in B$ and $(x, \beta) \in D$.

Let $\mathcal{B}$ be a bornology on X. We say that a function $f : X \to R$ is *strongly uniformly continuous on* $\mathcal{B}$ iff it is strongly uniformly continuous on B for every $B \in \mathcal{B}$, with respect to a single (O)-sequence independent of B.

Let $(\Lambda, \geq)$ be a directed set. A filter $\mathcal{F}$ of Λ is said to be (Λ)*-free* iff $M_\lambda \in \mathcal{F}$ for each $\lambda \in \Lambda$, where $M_\lambda := \{\zeta \in \Lambda : \zeta \geq \lambda\}$. The filter $\mathcal{F}_{\text{cofin}}$ is the filter of all subsets of $\mathbb{N}$ whose complement is finite.

If X is any Hausdorff topological space, $\mathcal{F}$ is a (Λ)-free filter of Λ, $x \in X$ and $(x_\lambda)_{\lambda \in \Lambda}$ is a net in X, then we say that $(x_\lambda)_\lambda$ $\mathcal{F}$*-converges to* $x \in X$ (in brief, $(\mathcal{F}) \lim_\lambda x_\lambda = x$) iff $\{\lambda \in \Lambda : x_\lambda \in U\} \in \mathcal{F}$ for each neighborhood U of x.

Let Ξ be any nonempty set. A family $\{(x_{\lambda,\xi})_\lambda : \xi \in \Xi\}$ $(RO\mathcal{F})$*-converges* to $x_\xi \in R$ (as λ varies in Λ) iff there exists an (O)-sequence $(\sigma_p)_p$ in Y with $\{\lambda \in \Lambda : \rho(x_{\lambda,\xi}, x_\xi) \leq \sigma_p\} \in \mathcal{F}$ for each $p \in \mathbb{N}$ and $\xi \in \Xi$.

We say that a net $f_\lambda : X \to R$, $\lambda \in \Lambda$, is *strongly weakly* $\mathcal{F}$*-exhaustive on* B iff there is an (O)-sequence $(\sigma_p)_p$ such that for each $p \in \mathbb{N}$ there is an entourage D such that, for every $x \in X$ and $\beta \in B$ with $(x, \beta) \in D$, there is $F \in \mathcal{F}$ (depending on x and β) with $\rho(f_\lambda(x), f_\lambda(\beta)) \leq \sigma_p$ whenever $\lambda \in F$.

Given a bornology $\mathcal{B}$ on X, we say that $f_\lambda : X \to R$, $\lambda \in \Lambda$, is said to be *strongly (weakly)* $\mathcal{F}$*-exhaustive on* $\mathcal{B}$ iff it is strongly (weakly) $\mathcal{F}$-exhaustive on every $B \in \mathcal{B}$ with respect to a single (O)-sequence, independent of B.

We say that $f_\lambda : X \to R$, $\lambda \in \Lambda$, is *(weakly)* $\mathcal{F}$*-exhaustive on* X iff it is (weakly) $\mathcal{F}$-exhaustive at every $x \in X$ with respect to a single (O)-sequence, independent of $x \in X$.

A family $\mathcal{V}$ of subsets of X is a *cover* of a subset $A \subset X$ iff $A \subset \bigcup_{V \in \mathcal{V}} V$. We say that a family $\mathcal{Z}$ of subsets of X *refines* $\mathcal{V}$ iff for every $Z \in \mathcal{Z}$ there is $V \in \mathcal{V}$ with $Z \subset V$.

An open cover $\mathcal{V}$ of X is called a *$\mathcal{B}$-uniform cover of X* iff for every $B \in \mathcal{B}$ there is an entourage D such that the family $\{D(x) : x \in B\}$ refines $\mathcal{V}$. If it is possible to choose D in such a way that $\{D(x) : x \in B\}$ refines a finite subfamily of $\mathcal{V}$, then we say that $\mathcal{V}$ is a *$\mathcal{B}$-finitely uniform cover of X*.

A net $f_\lambda : X \to R$, $\lambda \in \Lambda$, $(\mathcal{FB})$*-converges* to $f : X \to R$ iff there exists an (O)-sequence $(\sigma_p)_p$ in Y such that $(f_\lambda)_\lambda$ is $(RO\mathcal{F})$-convergent to f with respect to $(\sigma_p)_p$, and for every $B \in \mathcal{B}$ and $p \in \mathbb{N}$ there is $F \in \mathcal{F}$ with $\rho(f_\lambda(x), f(x)) \leq \sigma_p$ for each $x \in B$ and $\lambda \in F$.

A net $f_\lambda : X \to R$, $\lambda \in \Lambda$, *converges $\mathcal{F}$-strongly uniformly to f on $\mathcal{B}$* (and we write $f_\lambda \overset{\mathcal{F}-\mathcal{T}^s_{\mathcal{B}}}{\to} f$), iff there is an (O)-sequence $(\sigma_p)_p$ with the property that for every $p \in \mathbb{N}$ and $B \in \mathcal{B}$ there exists $F \in \mathcal{F}$ such that for each $\lambda \in F$ there is an entourage D with $\rho(f_\lambda(z), f(z)) \leq \sigma_p$ whenever $z \in D(B)$.

We say that $(f_\lambda)_\lambda$ *converges $\mathcal{F}$-strongly uniformly to f* ($f_\lambda \overset{\mathcal{F}-\mathcal{T}^s}{\to} f$) iff there is an (O)-sequence $(\sigma_p)_p$ such that for each $p \in \mathbb{N}$ and $x \in X$ there is a set $F \in \mathcal{F}$ such that for every $\lambda \in F$ there is a neighborhood U of x with $\rho(f_\lambda(z), f(z)) \leq \sigma_p$ for each $z \in U$.

We say that a net $f_\lambda : X \to R$, $\lambda \in \Lambda$, is $(\mathcal{FB})$*-Alexandroff convergent* to $f : X \to R$ (shortly $f_\lambda \overset{(\mathcal{FB})-Al.}{\to} f$) iff there is an (O)-sequence $(\sigma_p)_p$ in Y such that for every $p \in \mathbb{N}$ and $F \in \mathcal{F}$ there are an infinite set $\Lambda_0 \subset F$ and a $\mathcal{B}$-finitely uniform open cover $\{U_\lambda : \lambda \in \Lambda_0\}$ of X with $\rho(f_\lambda(z), f(z)) \leq \sigma_p$ for each $\lambda \in \Lambda_0$ and $z \in U_\lambda$.

We say that $(f_\lambda)_\lambda$ is *$\mathcal{F}$-Alexandroff convergent* to f ($f_\lambda \overset{\mathcal{F}-Al.}{\to} f$) iff there exists an (O)-sequence $(\sigma_p)_p$ such that for every $p \in \mathbb{N}$ and $F \in \mathcal{F}$ there exist a set $\Lambda_0 \subset F$ and an open cover $\{U_\lambda\colon \lambda \in \Lambda_0\}$ of X such that for every $\lambda \in \Lambda_0$ and $z \in U_\lambda$ we have $\rho(f_\lambda(z), f(z)) \leq \sigma_p$.

A net $f_\lambda : X \to R$, $\lambda \in \Lambda$, is said to be $(\mathcal{FB})$*-Arzelà convergent* to $f : X \to R$ (shortly $f_\lambda \overset{(\mathcal{FB})-Arz.}{\to} f$) iff there exists an (O)-sequence $(\sigma_p)_p$ in Y such that $(f_\lambda)_\lambda$ $(\mathcal{FB})$-converges to f with respect to $(\sigma_p)_p$ and for every $B \in \mathcal{B}$, $p \in \mathbb{N}$ and $F \in \mathcal{F}$ there are a finite set $\{\lambda_1, \dots, \lambda_k\} \subset F$ and an entourage D such that for each $z \in D(B)$ there is $j \in [1, k]$ with $\rho(f_{\lambda_j}(z), f(z)) \leq \sigma_p$.

We say that $(f_\lambda)_\lambda$ is *$\mathcal{F}$-Arzelà convergent* to f (shortly $f_\lambda \overset{\mathcal{F}-Arz.}{\to} f$) iff there exists an (O)-sequence $(\sigma_p)_p$ such that $(f_\lambda)_\lambda$ $(RO\mathcal{F})$-converges to f with respect to $(\sigma_p)_p$, and for each $x \in X$, $p \in \mathbb{N}$ and $F \in \mathcal{F}$ there exist a finite set $\{\lambda_1, \lambda_2, \dots, \lambda_k\} \subset F$ and an open neighborhood U_x of x such that for every $z \in U_x$ there is $j \in [1, k]$ with $\rho(f_{\lambda_j}(z), f(z)) \leq \sigma_p$.

Theorem 0.1. *Let $\mathcal{F}$ be a (Λ)-free filter of Λ, X be a uniform space, $\mathcal{B}$ be a bornology on X, $f_\lambda : X \to R$, $\lambda \in \Lambda$, be a net of functions, strongly uniformly continuous on $\mathcal{B}$ with respect to a single (O)-sequence independent of $\lambda \in \Lambda$, and $(\mathcal{FB})$-convergent to $f : X \to R$. Then the following are equivalent:*

(i) $(f_\lambda)_\lambda$ is strongly weakly $\mathcal{F}$-exhaustive on $\mathcal{B}$;

(ii) f is strongly uniformly continuous on $\mathcal{B}$;

(iii) $f_\lambda \overset{\mathcal{F}-\mathcal{T}^s_{\mathcal{B}}}{\to} f$;

(iv) $f_\lambda \overset{(\mathcal{FB})-Al.}{\to} f$;

(v) $f_\lambda \overset{(\mathcal{FB})-Arz.}{\to} f$.

Theorem 0.2. *Let Λ, $\mathcal{F}$, X, R be as above, $f_\lambda : X \to R$, $\lambda \in \Lambda$, be a net of functions, $(RO\mathcal{F})$-convergent to $f : X \to R$ and such that the f_λ's are globally continuous with respect to a single (O)-sequence independent of λ. Then the following are equivalent:*

(i) $(f_\lambda)_\lambda$ *is weakly $\mathcal{F}$-exhaustive on X;*

(ii) f *is globally continuous on X;*

(iii) $f_\lambda \overset{\mathcal{F}-\mathcal{T}^s}{\to} f$;

(iv) $f_\lambda \overset{\mathcal{F}-Al.}{\to} f$;

(v) *there exists an (O)-sequence $(\sigma_p)_p$ in Y such that for every nonempty compact subset $C \subset X$, for each $p \in \mathbb{N}$ and $F \in \mathcal{F}$ there are a finite set $\{\lambda_1, \lambda_2, \ldots, \lambda_k\} \subset F$ and an open set $U \supset C$, such that for every $z \in U$ there is $j \in [1, k]$ with $\rho(f_{\lambda_j}(z), f(z)) \leq \sigma_p$;*

(vi) $f_\lambda \overset{\mathcal{F}-Arz.}{\to} f$.