Image Comparisons of Black Hole vs. Neutron Dark Star by Ray Tracing

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ABSTRACT

In previous papers we have discussed the concept of a theory of gravitation with local energy conservation, and the properties of a large neutron star resulting when the energy of gravitation resides locally with the particle mass and not in the gravitational field [1][2][3]. A large neutron star's surface radius grows closer to the gravitational radius as the mass increases, but is always slightly larger. As the mass increases there is a continuously greater mass defect for incoming particles. Since the localization of energy applies to the photon, a photon does not decrease in energy rising in a gravitational field, and can thus escape. Photon trajectories in a strong gravitational field have some peculiar features that are not immediately obvious, but can be investigated by the use of ray tracing procedures. The most notable is the fact that only a fraction of the blackbody radiation emitted from the surface escapes into space (about 0.00004% for Sag A*). The remainder enters orbit below the maximum photon orbit. Because of the low percent of escaping blackbody radiation, the heavy neutron stars considered in this paper will be referred to as a Neutron Dark Star (NDS). In contrast to the Black Hole (BH) which should be totally dark inside the photon shadow, the NDS will appear as a fuzzy low luminosity ball with a Full Width Half Maximum intensity diameter of about 3.85 Schwarzschild radii inside the shadow. This paper will investigate the difference in the appearance of a Neutron Dark Star and a Black Hole by using ray tracing techniques. The Event Horizon Telescope currently under development should be able to distinguish the difference between the theories.

Introduction

Early on in the development of GR Hilbert recognized that the theory had an "improper energy theorem" that is, one could define a divergence free quantity, analogous to the momentum density of Special Relativity, but it is quite arbitrary and is gauge dependent. It is not covariant under a general coordinate transformation, or more simply there is no local conservation of energy. In a defined volume of space the change of energy inside, is not the sum of the energy entering and leaving through the surface.

Emmy Noether formalized the issue 1918 in a definitive paper "Invariante Varlations Probleme" illustrating the problem. Noether's theorem definitively shows that contrary to all other forces, energy cannot be conserved nor localized in a Riemannian gauge field representation. It is presumed here that this is a flaw in GR, and it is asserted here that Noether's theorem is not an indicator of a physical reality, but an indicator of the approximate nature of GR. This can best be tested in the observation of the properties of objects cited as being black holes.

From "Scalar Gravitational Theory with Variable Rest Mass" [1] the conservation of energy requires the relation between the relativistic mass and the rest mass is as a function of gravitational potential to be:

$$M^2\left(1-\frac{v^2}{c^2}\right)=M_0^2\left(1-\frac{\mu}{r}\right)^2 \tag{1.1}$$

Method

The comparison of the appearance of a Neutron Dark Star (NDS) to a Black Hole (BH) will be made by the use of ray tracing techniques of the photon trajectories. For GR, the index of refraction for photons traversing space can be made using the general form of a static and spherically symmetric metric [4][5],

$$\begin{gathered} ds^2 - B(r)\,dt - A(r)\,dr^2 - r^2\left(d\theta^2 + \mathbf{sin}^2\,\theta d\phi\right) \\ A=\left(1-\frac{2\mu}{r}\right)^{-1} \quad B=\frac{2\mu}{r^2}\left(1-\frac{2\mu}{r}\right)^{-1} \end{gathered} \tag{1.2}$$

That has an index of refraction (c/c_0) value in flat space of [6]:

$$n=\left(1-\frac{2\mu}{r}\right) \tag{1.3}$$

Or the more detailed analytic expression of Karimi1&Khorasani [7]:

$$n = \left\{ 1 - \frac{1}{2}\left[\frac{r}{r_s} - \frac{1}{2} + \sqrt{\left(\frac{r}{r_s}\right)^2 - \frac{r}{r_s}} \right]^{-1} \right\}$$
$$* \left\{ 1 + \frac{1}{2}\left[\frac{r}{r_s} - \frac{1}{2} + \sqrt{\left(\frac{r}{r_s}\right)^2 - \frac{r}{r_s}} \right]^{-1} \right\}^{-3} \tag{1.4}$$

The BH has no solid surface, and as shown analytically by Lacroix & Silk [4] to have a black gravitational shadow radius of about $2.6 r_S$. (Note the Schwarzschild radius is twice the gravitational radius. $r_S = 2\mu$)

For the Neutron Dark Star (NDS), the index of refraction for the photon deducible from the conservation of energy [12], and suitable for ray tracing ray tracing from is:

$$n = \left(1 - \frac{\mu}{r}\right)^2 \tag{1.5}$$

The difference in the index of refraction for the BH and the NDS is slight for large radius, but for small radius the effect on the trajectory is substantial. A ray tracing program has been developed for generating comparison trajectories. The program, a comparison with other procedures, and the Quick Basic version, are included in the appendix.

From calculations in [2], an NDS the size of Sag A* has a solid surface about 1.025 times the gravitational radius, and it will be shown that the shadow radius for interloping photons for the NDS will be about 2.93 r_S compared with the 2.6 r_S analytic solution for GR[7]. (see figure [1]).

Photons leaving the surface of an NDS vertically, escape into space, however if photon leave the source at a slight angle, gravitation can bend the trajectory into an orbit at a level below the maximum photon orbit. For a neutron star the size of Sag A*, the maximum angle from vertical for a photon to escape is about 0.004 rad. At that angle the photon will go into the maximum photon orbit, but if the angle is greater, the orbit lies between the surface and the maximum orbit. Although there may be a stability issue, as the photons trajectory curves perpendicular to the radius vector, the structure of Snell's law will not allow the radial velocity to become negative, therefore the trajectory at any elevation below the maximum photon orbit will become a circular orbit.

The Neutron Dark Star should, produce blackbody radiation however feeble, and should be observable within the gravitational shadow. Because of the mass defect for particles on the surface, and the slower velocity of light, the apparent temperature of the blackbody radiation will be shifted downward proportional to the mass defect, and thus the luminosity will be further reduced

Ray Tracing

Ray tracing of photon trajectories for the NDS is straightforward using Eq.(1.5), which is deduced from the conservation of energy and the equivalence principle [1],[12]. Comparative projections by F. Karimi, and S. Khorasani [7], [8],for GR are used to test the projection algorithm.

From T. Lacroix & J. Silk,[4] a semi analytical derivation of the Shadow of a GR Black Hole from the field equations is illustrated in Figure 1.

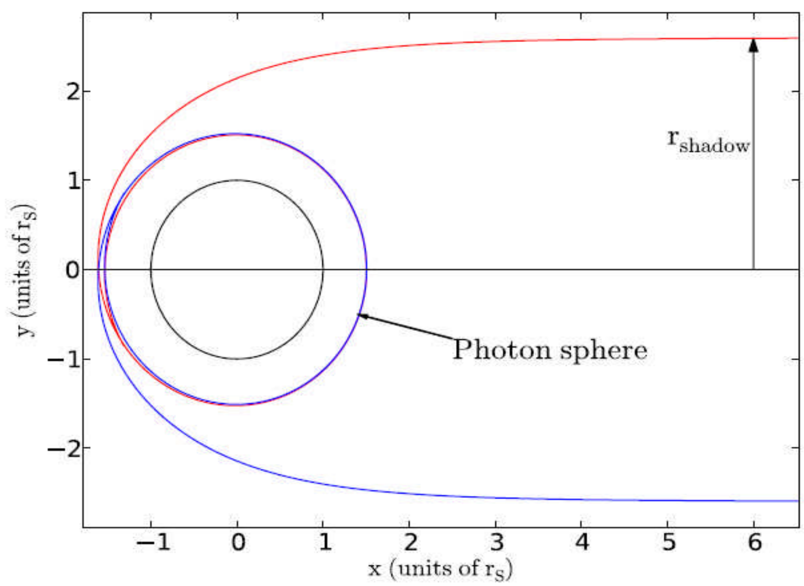

Figure 1. Fig. 1 of T. Lacroix & J. Silk[4]. Shadow of a black hole. The radius of the shadow is the minimum impact parameter of a light ray escaping the black hole, so the shadow is a disk representing the black hole as seen by the observer. The circular orbit lies on the so-called photon sphere. The black circle represents the horizon.

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The black hole has a minimum impact parameter of 1.5 r_S , Schwarzschild radii, equivalent to the photon orbit, and a shadow radius of 2.6 r_S . The ray tracing program for the NDS yields the same minimum impact parameter 1.5 r_S , but a shadow radius of about 2.93 r_S .

Overlaying the NDS projections on the Lacroix & Silk graph is shown in Figure 2:

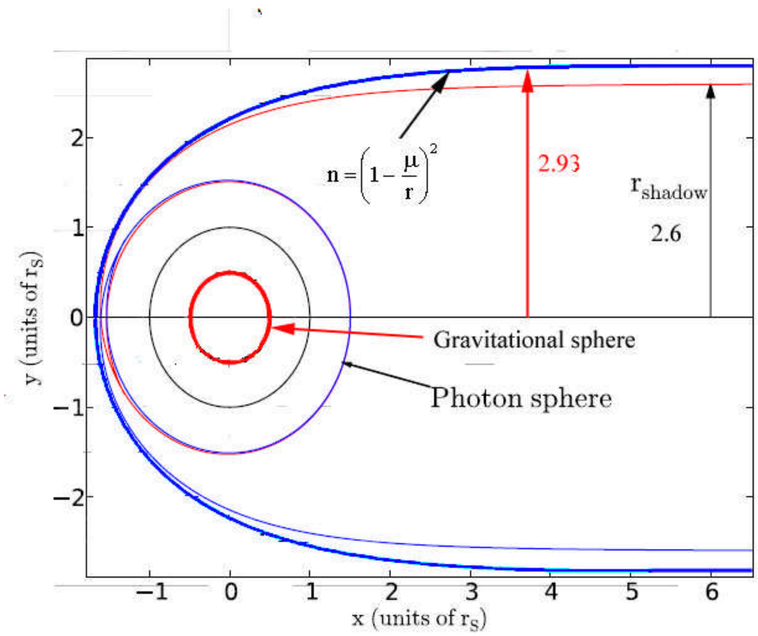

Figure 2. *The NDS photon shadow radius of (2.93* r_S *) is slightly larger than its BH equivalent (2.6* r_S *).*

Results

Photons leaving the surface of an NDS vertically, escape into space, however if photon leave the source at a slight angle, gravitation can bend the trajectory into an orbit at a level below the maximum photon orbit. For a neutron star the size of Sag A*, the maximum angle from vertical for a photon to escape is about 0.004 rad. At that angle the photon will go into the maximum photon orbit, but if the angle is greater, the orbit lies between the surface and the maximum orbit, as shown in Figure 3, Although there may be a stability issue, as the photons trajectory curves perpendicular to the radius vector, the structure of Snell's law will not allow the radial velocity to become negative, therefore the trajectory at any elevation below the maximum photon orbit will become a circular orbit.

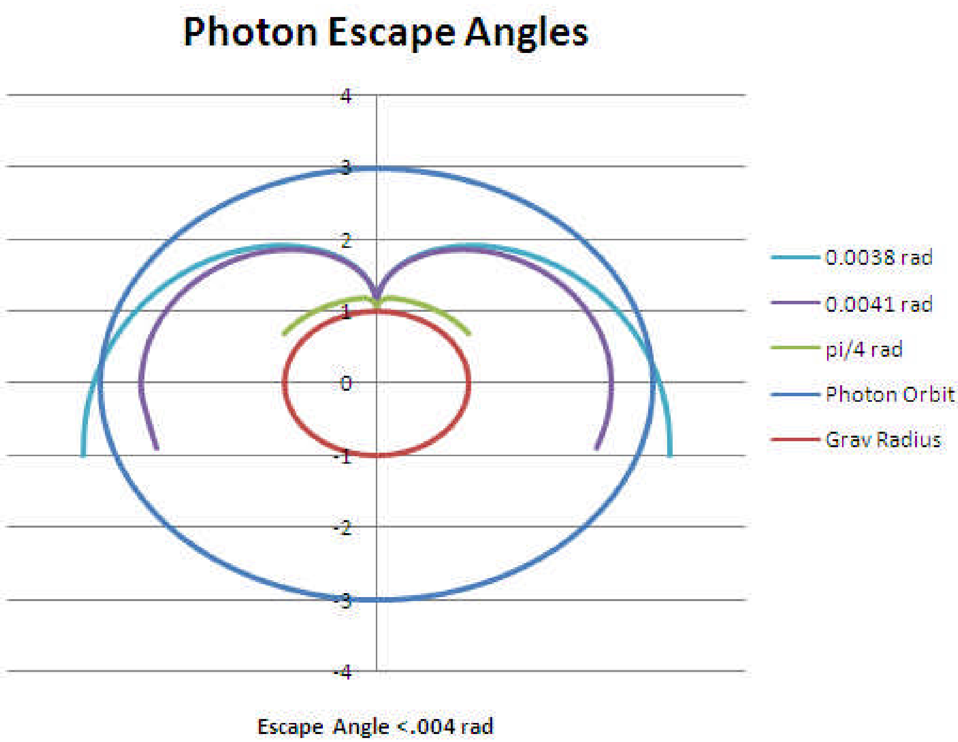

Figure 3. *Illustration of the photon emission angle for photons that escape, and those photons that would be captured in orbit.*

From ray tracing of vertical moving photons leaving the surface of a NDS star the size of Sag A*, at an elevation of 1.0025 μ, the time out is about 1030 seconds longer than a photon traveling the same distance in flat space.

Photon Trajectories

Escape Angle 0 - 0.004 Rad

It is noted that in the absence of intervening or accretion material, GR predicts that there will be a photon void emanating inside the shadow for a BH, thus a dark image radius of about 2.6 r_S . This is because there are no photons originating from the sphere of the black hole, and any interloping photon would be captured.

For the NDS, photons leaving the surface with an angle less than 0.004 rad to the vertical escape into space, beyond that angle the photons are captured into an orbit that is likely unstable but could remain for a period of time.

The trajectories for photons in that range are illustrated in Figure 4:

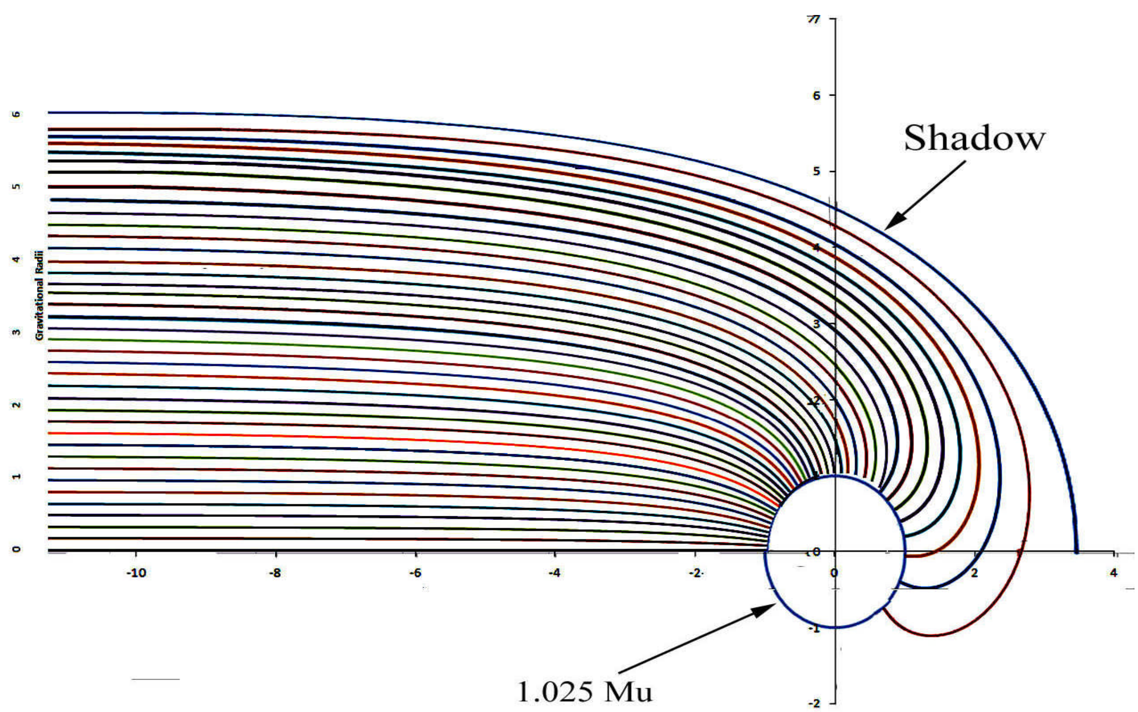

Figure 4. ***The ray trace trajectory of photons leaving the surface with an angle to the vertical between 0.0 and 0.004 rad, and crossing perpendicular to a distant plane.***

Intensity

The narrow escape angle for blackbody photons leaving the surface (~0.004 rad) means that only about 0.00004% of photons emitted from the surface escape. This reduces the luminance of the Sag A* NDS by a factor of factor of more than 1.0e-6 compared to a low mass star of the same temperature.

As photons escape a spot on the surface, those leaving vertically go straight out. As the angle increases from 0 to 0.004 rad, photons wrap further around the sphere before escaping. By taking a ratio of the square of the angle between two photons leaving at a small $(\Delta\theta_s)^2$ at the surface, to the angle difference $(\Delta\theta_p)^2$ at a distant escape plane, the apparent relative luminosity of a spot on the surface to the observed value at a distance in space can be determined.

$$L = L_S \frac{(\Delta\theta_S)^2}{(\Delta\theta_P)^2} \tag{1.6}$$

L is the apparent external luminosity, L_S is the luminosity at the surface , $\Delta\theta_S$ is the angle separating the photons at the surface, and $\Delta\theta_P$ is the angle between the same photons when they escape the system at a distant escape plane. Figure 5, illustrates the angular dispersion difference between photons leaving vertically and those from lower angles wrapping around the star.

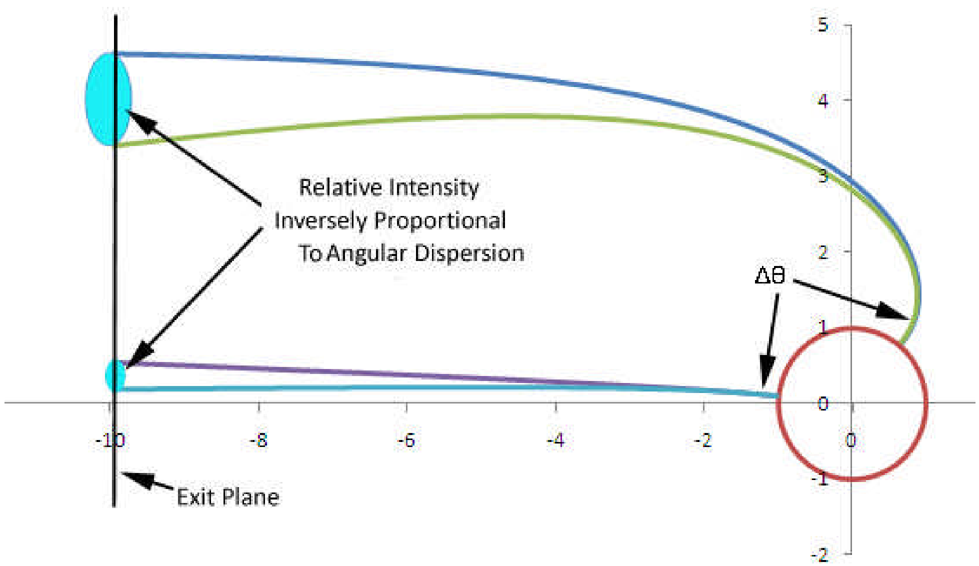

Figure 5. *Illustrating the luminosity at the escape plane as a function of the spherical exit position.* (*Exaggerated*)

By tracking two photons having a small $\Delta\theta$ on the surface (1.0e-6 rad) to their exit from the system the change in the exit angle can be determined, and the relative apparent luminosity can be determined. By taking a series of these tracks for all the escaping angles from 0 to 0.004 rad the radial profile of the relative luminosity of the disk can be evaluated.

For example two photons leaving the surface, one vertically $(\theta = 0)$, and one at an angle of 0.000001 rad, by ray trace calculations, arrive at a distant perpendicular plane, out of the system at an angle separation of 0.000566 rad. This gives by Eq.(1.6), the apparent luminosity by an external observer to be a factor of 3.12e-6 less than at the surface. For a pair of photons leaving the surface at.0022 rad and 0.002201 rad, ray trace calculations, show them turning by 1.38 rad, and arriving at the distant escape plane at an angle separation of 0.000692 rad, this gives an apparent relative luminosity between a spot in the center of the star to a spot at about 80 degrees around sphere of 0.66. Two photons in the absence of gravitation maintain the same angle, and the source maintains the same luminosity.

For the purpose of illustrating the image of the NDS, an imaginary plane (escape plane) perpendicular to the radius vector is presumed, at such distance that an arriving photon is moving in the direction toward a far distant observer, and is parallel to the radius vector.

Figure 6 and 7 illustrate the results of these calculations. For the purpose of replication of the results here, the ray trace program routine, as well as the calculated values for the angles around the sphere is included in Appendix II. Figure 6 is the relative luminosity as a function of the distance from the center of the escape plane, showing a half maximum intensity at a diameter of about 3.85 r_S

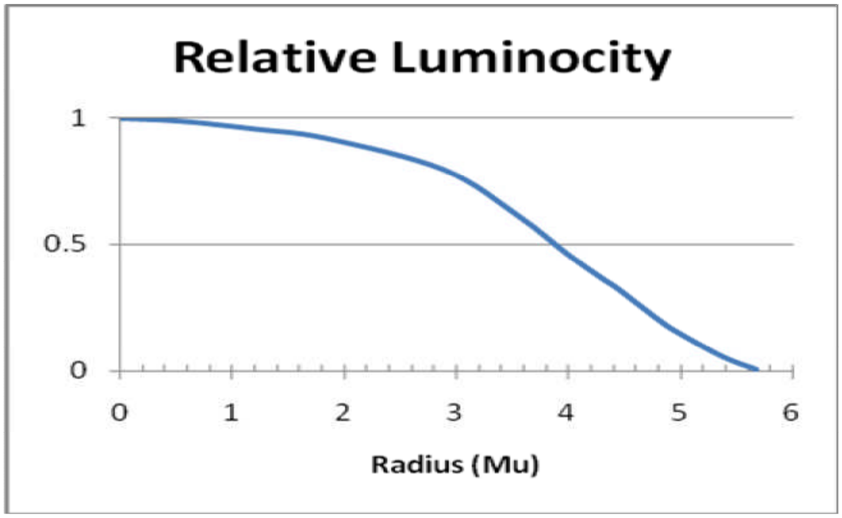

Figure 6, *This is the plot of the intensity profile of photons exiting perpendicular to a distant plane as measured from the center of the image*

Note the half level intensity diameter for the NDS of 3.85 r_s , is very close to the current observed Gaussian Full Width at Half Maximum measured diameter (FWHM), of Sag A*, of about 3.7 r_s . [10]. There has been some discussion as to whether the current measurement is the star or the MHD accretion offset from the star, so at this time the measurements are not definitive. Planned higher resolution data will clarify this issue.

Figure 7, illustrates the exit position on the exit plane of a photon trajectory originating at positions on the surface. The relative intensity of points on the surface as viewed at a large distance perpendicular to that plane is plotted on the left side of the trajectory.

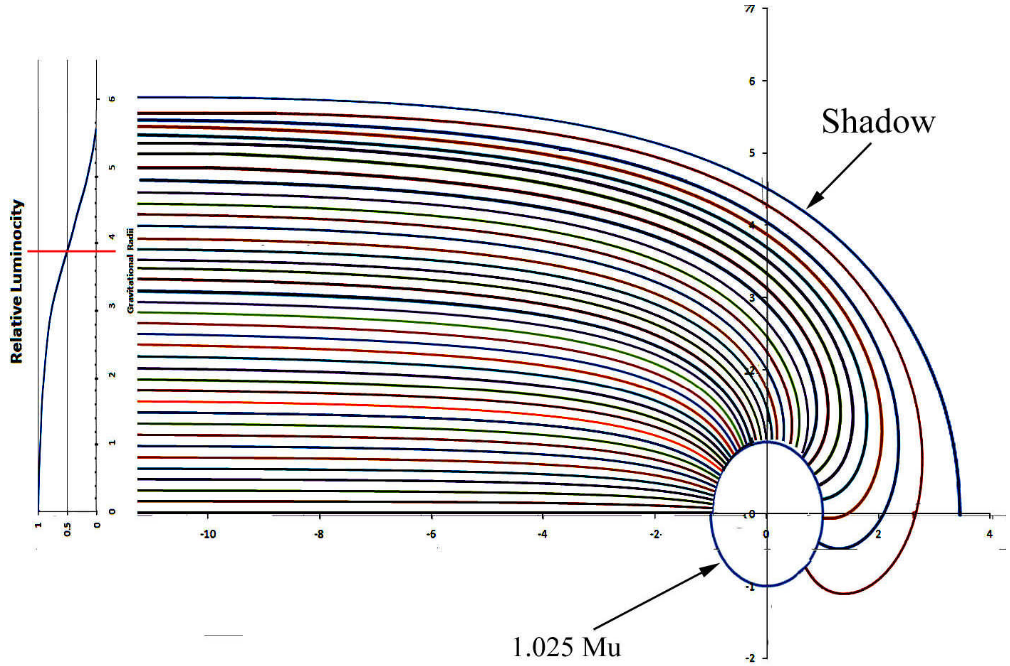

Figure 7. *This illustrates the exit position on the exit plane of a photon trajectory originating at positions on the surface. The relative intensity of points on the surface as viewed at a large distance perpendicular to that plane is plotted on the left side of the trajectory. The red line is the half maximum intensity diameter.*

Comparisons with Current Measurements

Figure 8 is the graph from Doeleman et al [9], illustrating the apparent diameter vs. Black Hole diameter. The red line represents the observed size (FWHM) using the data from 2007 and 2009. The intensity vs. radius for the NDS from Figure 8 is plotted to the right of the graphs showing the Full Width Half Maximum diameter to be about the same as the current low resolution measurements.

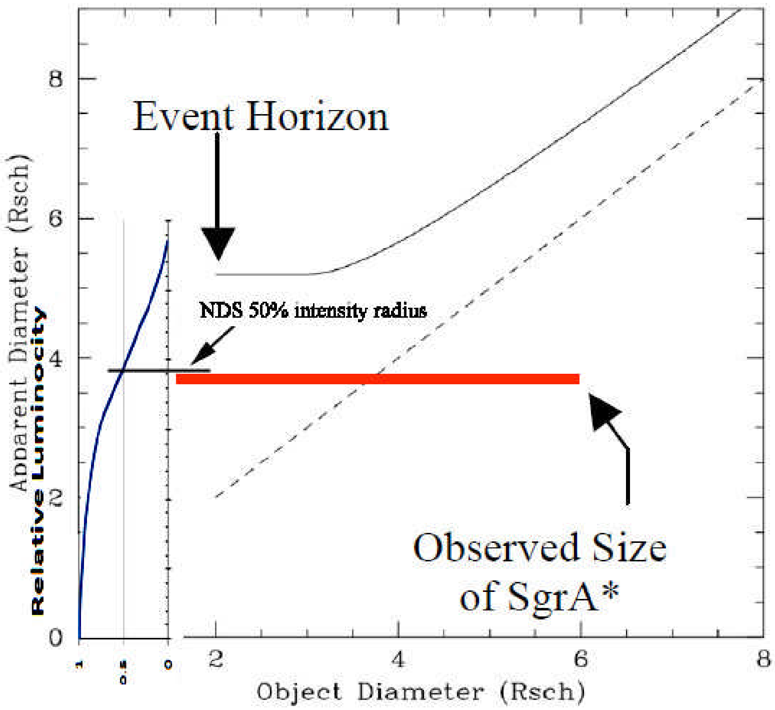

Figure 8. *(* Fig. 1 of Doeleman et al [9]*) A symmetric emitting surface surrounding a black hole is gravitationally lensed to appear larger than its true diameter. Here the apparent size is plotted as a function of the actual object size. The solid black line shows the apparent diameter with lensing by a non-spinning black hole, and the dashed line with no lensing effects included. The intrinsic size of Sgr A* observed with1.3mm VLBI. (horizontal red line), is smaller than the minimum apparent size of the black hole event horizon (labeled 'Event Horizon')*[9]

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EHT Simulated Image
Dimitrios Psalti , et al, [11].

The explanation by the EHT team for the small size of the currently observed low resolution image is that the image is of the edge on view on the accretion disk, and not the rim of the Black Hole.

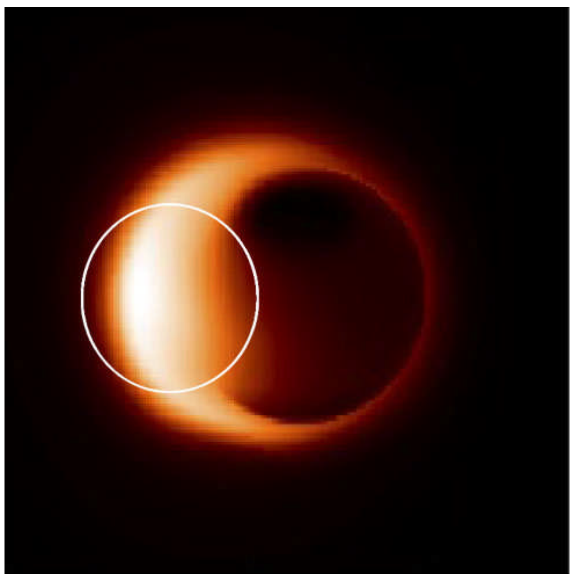

Figure 9. *Black Hole: EHT simulated image of the accretion generated radiation of Sag A**[11]. *The black Hole shadow is black area on the right with the accretion generated radiation to the left. The white 3.7*μ *radius circle (added by this author) is the EHT teams proposed source of the current measurements*

Neutron Dark Star Image

The expected image of the NDS star (figure 10), should be centered at the center of mass, and spherically symmetric, having a Full Width Half Maximum intensity diameter of about 3.85 r_S and a shadow radius of about 5.86 r_S

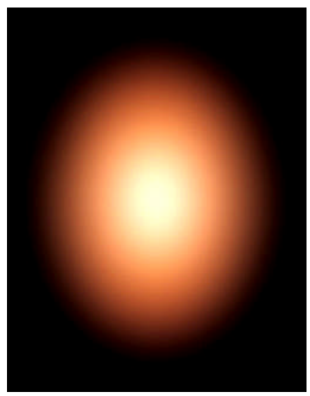

Figure 10. *NDS: Simulated image of the escaping radiation from the Neutron Dark Star. The Full Width Half Maximum diameter is about 3.85 Schwarzschild radii.*

The emitted radiation from the NDS should be thermal and have a constant thermal profile across the observed disk representing an even surface temperature. Outside the rim of the gravitational shadow the thermal profile should be more reflective of the average of galactic stars or any accretion generated radiation. (not simulated)

Particles at the surface have a lower mass, a lower speed of light, and thus a lower rate of thermal photon emission. In the case of Sag A* the ratio is about 100 to 1 [2], and the apparent black body radiation for a given temperature is decreased by that same factor. Thus an external temperature observation will appear about 100 times less than a surface observation.

The substantial decrease in black body radiation resulting from gravitation geometry, and mass defect compared to a normal star provides the justification to refer to The NDS as a dark star.

Conclusion

This paper has presented an the contrast between the image expected of a standard black hole, with a non-locally conserved Riemannian gauge field, and an image that would exist for a heavy neutron star created under the presumption of local conservation of energy. The exact numbers shown in this paper may have a degree of error due to the numerical calculated tracings, but the qualitative descriptions should be close. The fact that the calculated image diameter is near the value determined by current measurements for Sag A*, and those measurement are outside the bounds of current GR theory, gives reason to inspect the theory in more depth. The forthcoming completion of the Event Horizon Telescope currently under development should be able to distinguish the difference, and test the validity of this theory.

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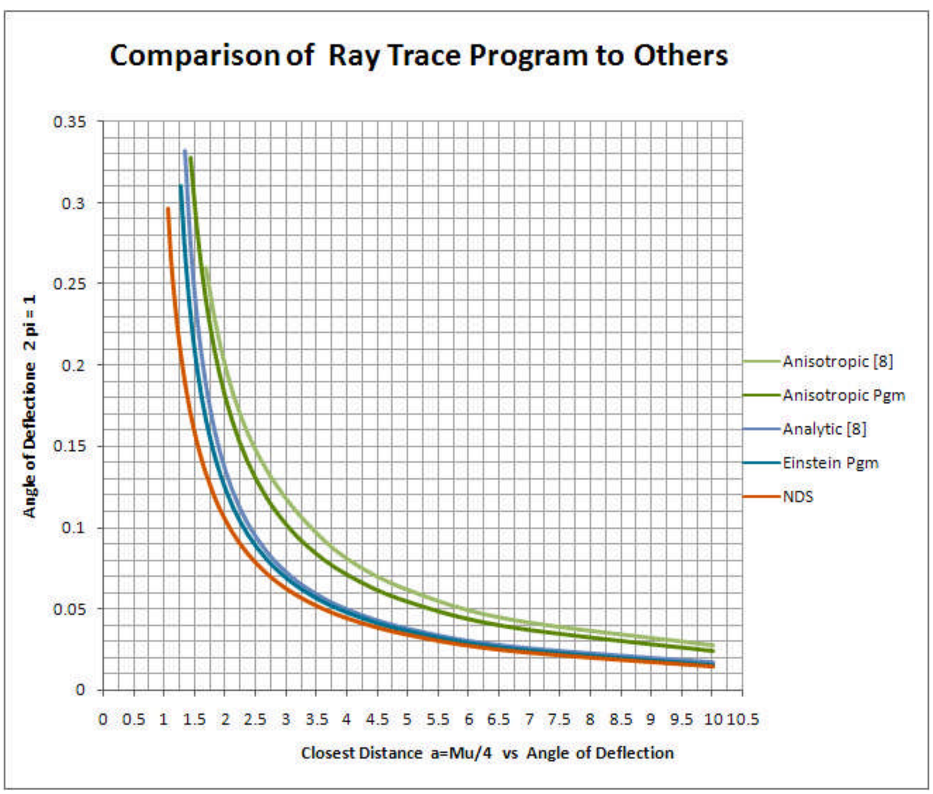
Appendix I

Ray Trace Algorithm

The algorithm used in this paper is a point to point implementation of Fermat's principle, using Snell's law and the index of refraction. The routine is constructed in Quick Basic and has been tested against other ray trace routines, and analytical solutions. The difference has been found to be 1-2 percent in the deflection angles of particles passing near black holes from a range of 4 to 80 μ (gravitational radii). The results for the NDS calculations could be slightly in error as a result of the inherent errors in the algorithm, but the error must be small, and the substantial difference between the NDS and the GR Black Hole theory results are qualitatively unambiguous.

A comparison of the algorithm output shown in Figure A1, to those of Karimi, Khorasani [7],[8] demonstrates the results to be within acceptable levels of difference.

Figure A1.



The top two curves show the projected angle of deflection resulting from photons subjected to the index of refraction calculated by Karimi Eq. 13, for the anisotropic solution of the Schwarzschild metric. The top (light green) is from the reference and the bottom (dark green) is from the computer trace program, both using the index of refraction

$$n = \left(1 - \frac{2\mu}{r}\right)^{1/2} \left(1 - \frac{2\mu}{r}\mathbf{cos}^2\,\theta\right)^{1/2} \quad , \tag{1.7}$$

The results of the output by Khorasani using a Mathematica differential solution and the current ray trace program have a less than 2% difference.

The two middle curves show the angle of deflection from the analytical solution for the Schwarzschild metric by Khorasani [a] Eq. 12

$$\alpha(a) = a\frac{\pi - A\mathbf{rctan}\sqrt{a^2 - 1}}{\sqrt{a^2 - 1}} - \frac{\pi}{2} \tag{1.8}$$

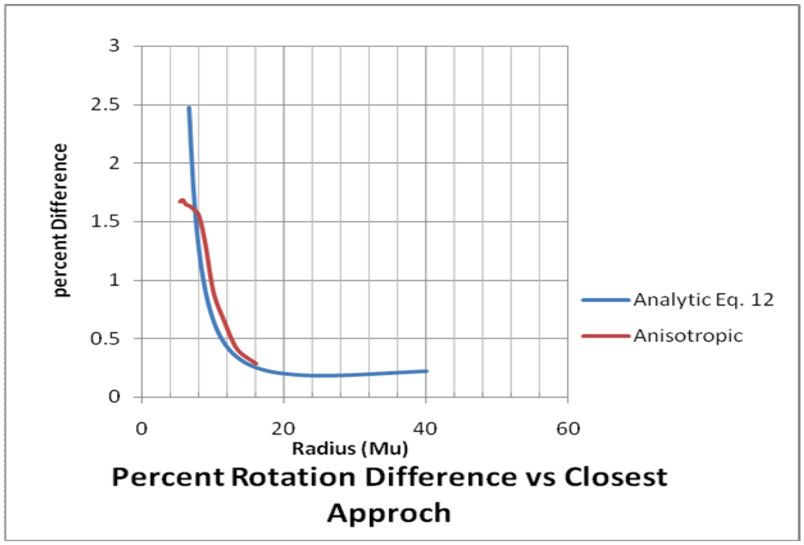
With $a = \mu / 4$, and the current ray trace program using the Einstein index of refraction:

$$n = \left(1 - \frac{2\mu}{r}\right) \quad , \tag{1.9}$$

In both cases the ray trace program is within acceptable agreement with the other calculations.

A plot of the differences is shown in Figure 2A.

Figure 2A.



The blue line is the difference between the ray trace program and the anisotropic trace program of Khorasani, and the red line is the difference between the ray trace program and the analytic solution. The analytic solution diverges below 5 radii which is not unexpected since the author suggests the equation is invalid in that range.

The conclusion can be drawn that the ray trace routine is suitable for the purpose for tracing photon trajectories for a given index of refraction.

Basic Program Ray Trace Routine

```
'          Gravitational Ray Trace
'               DT Froedge
'             copyright 2014
'    GR black hole, and VRM Dark Star


'This is a basic program that calculates ray traces
'in the vicinity of black holes and variable mass stars
'calculations are all in double precision.
'Output is a comma delimited text file that can be imported into Excel
'or other spreadsheets for graphing.
'Program calculates in the first quadrant (quad)by rotation
'then corrects output to proper quadrant by reversing this.
'program only handles counter clockwise motion.
'distance is in gravitational radii. c=1, Mu = 1
'There are 10000 iterations between print iterations

fprmat% = 2           'pick here User choice of 1-5 index of refraction formulas

  pi# = 3.14159265358979#
  x1# = 6                 'initial x position in Mu  (to be set by user)
  y1# = 0                 'initial y position in Mu  (to be set by user)
  P1# = pi# / 2           'initial angle in radians  (to be set by user)


CLS
'Sag a*        mass 4.1 e6   0.6 e6 suns           =  8.154572e39  grams
'Gm/c^2=     Mu = 605181439025  cm     =  20.186  light seconds
```

```
quad% = 1                        'start quadrant
Delt# = .00001                   'increments of time
sht% = 0                         'increment counter
sht2% = 0                        'incremental outputs data after Sht% counts out
r0# = (xo# ^ 2 + yo# ^ 2) ^ .5   'initial radius       defined
a#(2) = 0
st% = 2

xxo# = x1#
yyo# = y1#
po# = P1#
fun% = -1

a#(2) = 0
a#(1) = 0
‘                                ‘Some Index of refraction formulas tested
IF fprmat% = 1 THEN Formprt$ = " NDS     v = (1 - 1 / r#) ^ 2 "
IF fprmat% = 2 THEN Formprt$ = "Einstein v = (1 - 2 / r#) "
IF fprmat% = 3 THEN Formprt$ = "Karimi Isotropic   Eq 5b"
IF fprmat% = 4 THEN Formprt$ = "Karimi Anisotropi  Eq 13"
IF fprmat% = 5 THEN Formprt$ = "Karimi Component   Eq.14"


OutFile$ = "Out.txt"
OPEN OutFile$ FOR OUTPUT AS #2


PRINT #2, "x Eq" + STR$(fprmat%) + ",";
PRINT #2, "y,";
PRINT #2, "x grav cir,y grav cir,r, direction, Rad vel,Ang vel, veloc, direction,";
PRINT #2, "deltV, AngularPos,Delta a ,seconds, rotations,";
PRINT #2, Formprt$
                      'Initial set
grad# = 1                            'defines Circle at gravitational radius

p#(1) = P1#                          'initial velocity direction angle from 1 to 2
p#(2) = P1#
x#(1) = x1#                           'initial first point x position
y#(1) = y1#                           'initial first point  y position
r#(1) = (x#(1) ^ 2 + y#(1) ^ 2) ^ .5  'first point r & v values
                                          'velocity at point1
r# = r#(1)
p# = p#(1)

GOSUB selectN
v#(1) = v#

                                                             'initial print
x#(2) = x#(1)
y#(2) = y#(1)
r#(2) = r#(1)
p#(2) = p#(1)
GOSUB procede


                             ' calculate 2nd point  based on initial conditions

v# = v#(1)
x# = x#(1)
y# = y#(1)
p# = p#(1)

GOSUB Nextpos

x#(2) = x#                            '2nd point location running along p1
y#(2) = y#       '
r#(2) = r#

here:                                 ' main loop back point
```

```
r# = r#(2)
p# = p#(2)
GOSUB selectN                         'velocity at 2nd point need r#
v#(2) = v#

                                          'angular position from x & y

        IF x#(2) = 0 THEN a#(2) = pi# / 2: GOTO skip
        a#(2) = ATN(y#(2) / x#(2))
        IF (y#(2) / x#(2)) < 0 THEN                  'second quad
        a#(2) = pi# - ATN(-y#(2) / x#(2))
        END IF
skip:
        dela#(2) = (a#(2) - a#(1))        'change in angular photon position

                            'This rotates coordinates back to first
                            'quadrant when location goes into third quad
                            'and keeps up with rotation to correct output

                 IF y#(2) < 0 THEN  'trigger point 'third quadrant reset to first quad
                 st% = st% + 1      'counter
                 IF st% = 4 THEN
                     st% = 3: a#(2) = a#(2) - pi#
                     GOTO Plast                      'one rotation stop
                 END IF
                     x#(2) = -x#(2)                  'resign 3rd point to second and
                                                   '2nd  point to 1st, recycle
                     y#(2) = -y#(2)
                     p#(1) = p#(2) - pi#             'resets direction
               END IF

                                                              'initial component velocity from 1
                                                        'arriving at second point (3rd pt)

vx# = v#(1) * COS(p#(1))                 'Sub variables
vy# = v#(1) * SIN(p#(1))


GOSUB calcradv                       'rotates coordinates to a = 0 so that x and y velocity
                                     'components align with v angular and v radial.


GOSUB Snell                          ' apply Snell angle change in radial and angular
                                     ' velocities
                                     ' and calculate change in direction
p#(2) = p#(1) + DelRef#(2)           ' apply change in direction to particle direction.


v# = v#(2)
x# = x#(2)
y# = y#(2)
p# = p#(2)
GOSUB Nextpos
v#(3) = v#
x#(3) = x#
y#(3) = y#
r#(3) = r#

  x#(1) = x#(2)                               'resign 2nd point, to 1st for recycle
  y#(1) = y#(2)
  r#(1) = r#(2)
  p#(1) = p#(2)
  a#(1) = a#(2)
  v#(4) = v#(1)                                'hold for print
  v#(1) = v#(2)

  x#(2) = x#(3)                                'resign 3nd point, to 2nd for recycle
  y#(2) = y#(3)
  r#(2) = r#(3)
```

```
GOSUB PUTITOUT5           'data out
GOTO here                 'main loop back

                                          'Subroutines

Nextpos:
         x# = x# + v# * COS(p#) * Delt#           '2nd point location running along p
         y# = y# + v# * SIN(p#) * Delt#
         r# = (x# ^ 2 + y# ^ 2) ^ .5              'r  next point
RETURN


Snell:                                 ' Calculates change in direction
   IF vrad# = 0 THEN refa#(1) = pi# / 2: GOTO skip2
   refa#(1) = pi# / 2 - ATN(vtheta# / vrad#)
skip2:
   vtheta#(2) = ((v#(2) / v#(1)) ^ 2) * vtheta#        ' Snell velocity change
   vrad#(2) = (ABS((v#(2) ^ 2 - vtheta#(2) ^ 2)) ^ .5) 'new velocities components
                                                       ' leaving point 2 to 3 in rad
   IF v#(2) < v#(1) THEN
        vrad#(2) = -vrad#(2)
   END IF
   IF vrad#(2) = 0 THEN refa#(2) = 0: GOTO skip3
   refa#(2) = pi# / 2 - ATN(vtheta#(2) / ABS(vrad#(2)))
skip3:
   DelRef#(2) = refa#(1) - refa#(2)
RETURN


calcradv:   'Rotates coordinates to a=0 so that x,y velocity align with r,Theta velocity
   aa#(2) = a#(2)
   IF aa#(2) > pi# THEN aa#(2) = aa#(2) - pi#  ' cycling, doesn't happen in this version
   vrad# = vx# * COS(aa#(2)) + vy# * SIN(aa#(2))
   vtheta# = vy# * COS(aa#(2)) - vx# * SIN(aa#(2))
RETURN




PUTITOUT5:                                          ' outputs location out file
       sht% = sht% + 1                               'iteration counter
       distance# = distance# + v#(1) * Delt#

                'Stop conditions
       IF r#(2) <= 1 THEN GOTO Plast
       IF x#(2) > 25 THEN GOTO Plast
       IF y#(2) > 25 THEN GOTO Plast
       IF x#(2) < -25 THEN GOTO Plast
       IF y#(2) < -25 THEN GOTO Plast
       K$ = ""
       K$ = INKEY$: IF K$ <> "" THEN CLOSE : STOP
       IF sht% = 10000 THEN sht% = 0: sht2% = sht2% + 1: GOTO procede  'output 1 per 10000
       RETURN

procede:
       Rx#(2) = grad# * COS(sht2% * pi# / 20)    'gravitational radius
       Ry#(2) = grad# * SIN(sht2% * pi# / 20)


       PRINT #2, USING "+##.########"; x#(2) * ((-1) ^ st%);          'x file
       PRINT #2, ",";
       PRINT #2, USING "+##.########"; y#(2) * ((-1) ^ st%);          'y
       PRINT #2, ",";
       PRINT USING "+##.########"; x#(2) * ((-1) ^ st%);              'x screen
       PRINT USING "+##.########"; y#(2) * ((-1) ^ st%);              'y
       PRINT USING "+##.########"; r#(2)                              'r
```

```
PRINT #2, USING "+######.####"; Rx#(2);                              'x  coordinates of
                                                                     ' gravitational radius
      PRINT #2, ",";
PRINT #2, USING "+######.####"; Ry#(2);                              'y
      PRINT #2, ",";
PRINT #2, USING "+######.####"; (x#(2) ^ 2 + y#(2) ^ 2) ^ .5;        'r
      PRINT #2, ",";

 PRINT #2, USING "+###.#########"; p#(2) + pi# * (st% - 2);          'direction
      PRINT #2, ",";
PRINT #2, USING "+###.#########"; vrad#(2);                          'rad vel
      PRINT #2, ",";
PRINT #2, USING "+###.#########"; vtheta#(2);                        ' ang vel
      PRINT #2, ",";
PRINT #2, USING "+###.#########"; v#(2);                             ' mag velocity
      PRINT #2, ",";
PRINT #2, USING "+###.#########"; (p#(2) + pi# * (st% - 2)) * 360 / (2 * pi#);  'direction
      PRINT #2, ",";
PRINT #2, USING "+#.###################"; v#(1) - v#(4);              'change in velocity
                                                                      '  point to point
      PRINT #2, ", ";
PRINT #2, USING "+###.##############"; (a#(2) + pi# * (st% - 2)) * 360 / (2 * pi#);'angular
position
      PRINT #2, ",";
PRINT #2, USING "+##.###############"; DelRef#(2);                    'refraction angle
      PRINT #2, ",";
PRINT #2, sht2% * 20.186;                                             'seconds to position
      PRINT #2, ",";
PRINT #2, (st% - 2) / 2;                                              ' rotations
      PRINT #2, ","

RETURN



selectN:                            'Selected Index of refraction formulas

                                        ' NDS
                                        ' b 3.43 sh 5.87 orbit 3.0    escape angle .004
IF fprmat% = 1 THEN
   v# = (1 - 1 / r#) ^ 2
END IF

                                         ' Einstein (1 - 2 / r#)
                                         ' 13c b 4.3450     shadow 6.93 orbit 4.0
IF fprmat% = 2 THEN
 v# = (1 - 2 / r#)
 END IF
                                        'Karimi, Khorasani Eq 5b
                                        '13c b 4.165 sh 6.72 orbit 3.8100
IF fprmat% = 3 THEN
 v# = ((1 - .5 * (((r# / 2) - .5 + (((r# / 2) ^ 2 - (r# / 2)) ^ .5)) ^ (-1))) ^ (-1))
 v# = v# * ((1 + .5 * (((r# / 2) - .5 + (((r# / 2) ^ 2 - (r# / 2)) ^ .5)) ^ (-1))) ^ (3))
 v# = 1 / v#
 END IF
                                         'Anisotropic Karimi, Khorasani  Eq. 13
                                        '  3b b= 4.80 sh = 7.62.txt  orbit 4.0
IF fprmat% = 4 THEN
 va# = (1 - 2 / r#) ^ (.5)
 vb# = (2 / r#) * COS(p# - a#(2) - pi# / 2) ^ 2 '- pi# / 2
 vc# = (1 - vb#) ^ (.5)
 v# = va# * vc#

END IF
                                        'Component Karimi, Khorasani  Eq.14
                                    ' b=  3.51 >> sh = 5.68  orbit 3.364
```

```
IF fprmat% = 5 THEN
  transv# = (1 - 2 / r#) * COS(p# - a#(2) - pi# / 2)
  radial# = (ABS((1 - 2 / r#)) ^ .5) * SIN(p#(2) - a#(2) - pi# / 2)
  v# = (radial# ^ 2 + transv# ^ 2) ^ (.5)
END IF

RETURN

Plast:
CLOSE : STOP
```

Appendix II

Luminosity data for Figures 4 & 5.

The data provided here is from the ray trace projections. The delta angle for two photons leaving the surface radius at 1.025μ was 1e-6 radians and the change in angle at the escape plane in is the Delta escape angle. The proportional luminosity is the proportional luminosity referenced to the value at the vertical angle. The escape radius is the radial distance of the observed point from the center as viewed at a distance.
Note that two free photons traveling at an angle from each maintain the same angle. The delta angle for the initial and final positions would be the same, and the proportional, as well as the relative luminosity, defined here, would be the same and equal to one.

Luminosity Data

Initial Surface Angle	displaced Surface Angle	Delta Initial Angle	Initial Escape Angle	Displaced Escape Angle	Delta Escape Angle
0	0.000001	0.000001	0	0.000565536	0.000565536
0.0001	0.000101	0.000001	0.056517909	0.057083284	0.000565375
0.0002	0.000201	0.000001	0.113220451	0.113787254	0.000566803
0.0003	0.000301	0.000001	0.169814872	0.170382234	0.000567362
0.0004	0.000401	0.000001	0.226871927	0.227441138	0.000569211
0.0005	0.000501	0.000001	0.284164253	0.284735259	0.000571006
0.0006	0.000601	0.000001	0.341681253	0.342253749	0.000572496
0.0009	0.000901	0.000001	0.517805883	0.5183849	0.000579017
0.0013	0.001301	0.000001	0.763015878	0.763608122	0.000592244
0.002	0.002001	0.000001	1.231570139	1.232220188	0.000650049
0.0024	0.002401	0.000001	1.539032941	1.539755511	0.00072257
0.0025	0.002501	0.000001	1.624307298	1.6250584	0.000751102
0.0027	0.002701	0.000001	1.81005838	1.810893616	0.000835236
0.0028	0.002801	0.000001	1.911447195	1.9123348	0.000887605

0.0031	0.003101	0.000001	2.263398951	2.264505018	0.001106067
0.0033	0.003301	0.000001	2.566210905	2.5676693	0.001458395
0.0037	0.003701	0.000001	3.543882473	3.547295329	0.003412856
0.0038	0.003801	0.000001	4.099719907	4.108902673	0.009182766

Remainder of chart columns

Initial Surface Angle	Proportional Luminocity $L \sim \frac{(\Delta\theta_s)^2}{(\Delta\theta_s)^2}$	Relative Luninocity $\mathbf{L / L_0}$	Escape Plane Radius
0	3.12665E-06	1	0
0.0001	3.12843E-06	1.000569615	0.147365031
0.0002	3.11269E-06	0.995534308	0.296405398
0.0003	3.10656E-06	0.993573551	0.442326311
0.0004	3.08641E-06	0.987129073	0.592008235
0.0005	3.06703E-06	0.9809326	0.741544887
0.0006	3.05109E-06	0.975833219	0.88985953
0.0009	2.98275E-06	0.953976951	1.349009536
0.0013	2.85101E-06	0.911841119	1.984376818
0.002	2.36651E-06	0.756882253	3.025269097
0.0024	1.91531E-06	0.612576961	3.556394611
0.0025	1.77256E-06	0.56692117	3.695953764
0.0027	1.43345E-06	0.458460688	4.001035464
0.0028	1.26929E-06	0.405957932	4.15570472
0.0031	8.17405E-07	0.261431335	4.629029083
0.0033	4.70164E-07	0.150373109	4.983267875
0.0037	8.58547E-08	0.027458992	5.455776922
0.0038	1.18591E-08	0.003792918	5.678647639