The Trans-Pythagorean Nature of Prime Numbers

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Abstract

In this paper, prime an composite numbers are studied from the point of view of the two modes of consciousness: 1) the deep, intuitive and synthetic consciousness; 2) the superficial, rational and analytical consciousness. This universal duality manifests respectively in prime and composite numbers.

According to the downwarding causation principle, the superficial level is always a manifestation of the deep one, and it is imposible to express the deep level from the superficial one. Therefore, composite numbers are manifestations of prime numbers, and prime numbers are inexpressible. This inexpressibility of the primes is manifested, justified and fundamented in the sum and difference of squares, i.e., in the dual Pythagorean expressions a^2+b^2 and $a^2–b^2$:

- Odd prime numbers –all prime numbers except 2– are divided into two classes: type $4k–1$ and type $4k+1$. According to the Fermat's Christmas theorem, all primes of type $4k+1$ are expressible as sum of squares in a unique way. Primes of type $4k–1$ are not expressible from the Pythagorean point of view, i.e., are not expressible as sum or difference of squares, except in a trivial way as difference of the squares of two consecutive numbers (as all the odd numbers).

- There are many reasons to consider the primes of type $4k+1$ as not real primes, among them –mainly– because they are not gaussian primes. From this point of view, the inexpresibility of prime numbers –its trans-Pythagorean nature– agree with the idea that from the superficial level (composite numbers) is not possible express the deep (prime numbers).

In conclusion, there is nothing mysterious or strange or complex in the subject of prime numbers. On the contrary, it is something fundamentally simple due to the close relation with the Pythagoras' theorem. The key to understanding prime numbers lies in the sum/difference of squares, i.e., the dual forms of the Pythagoras' theorem. Pythagoras's theorem is the most fundamental theorem of mathematics. Pythagoras's theorem is a theorem of consciousness, the Holy Grail of mathematics.

Additionally, this study has led to the discovery of the existence of 7 types of odd numbers from the point of view of the Pythagorean expressions: 2 in the branch $4k–1$ and 5 in the branch $4k+1$. The inexpressible type of the branch $4k–1$ corresponds to the class of the "real" prime numbers.

Previous Basic Concepts

Prime numbers

A prime number is a natural number that only have as divisors itself and 1. The first prime numbers are:

2, 3, 5, 7, 11, 13, 17, 19, 23, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, ….

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The fundamental theorem of arithmetic states that every natural number greater than 1 either is prime itself or is the product of two or more primes (alike or distinct ones). This theorem gives rise to what might be called “the fundamental problem of arithmetic”: given an integer $n>1$, find its prime factorization.

Natural numbers that are not prime are called “composites”, that is to say, they are the product of two or more primes. Therefore every natural number is prime or composite. A composite number set a relation (product) among prime numbers. For example, $72 = 2^3 \times 3^2$.

A set of natural numbers are coprimes (or relative primes) if they do not have a common factor, i.e., its g.c.d. is 1. For example, 5, 12 and 17 are coprimes.

Prime numbers are the keys of the modern encription systems used for security in the communications. Large numbers are used to be decomposed into prime factors. This task requires many computational and timing resources. For example, it is used by RSA, the asymmetric system of public key for secure data transmisión. The public key is based on the product of two large prime numbers. Everybody can use the public key to encrypt a message. If the recipient know the prime factors, then it is posible decipher easily the message.

Prime numbers are considered “the crown” of number theory. This theory, which deals with the properties of natural numbers and their relations, is the purest and oldest area of mathematics. It may seem something elementary, but really number theory is one of the deepest and most difficult areas of mathematics. It is also called “higher arithmetic”, an arithmetic addressing complex problems like the Fermat’s last theorem, the Riemann hypothesis, the twin primes conjecture, the Goldbach conjecture, etc. “Mathematics is the queen of sciences and number theory is the queen of mathematics” (Gauss) [1].

Number theory have two branches: additive and multiplicative. Paradoxically, the multiplicative branch –which is more complex than the additive one– is the one that has been developed more, and has an antiquity that dates back to Pythagoras. The additive branch is much younger. It began to be developped with Euler, and deals with how a number can be expressed as a sum of other natural numbers.

Number theory –despite its name– has also an experimental part. Theory and practice complement each other. Normally the experimental part comes first and it leads to questions that have to be answered at theoretical level. Nowadays, the experimental part is usually supported by computer applications.

Properties of prime numbers

Prime numbers have many properties. Here we are interest only in the following ones:

- It is known from Euclid that there are infinitely many prime numbers, likewise the natural numbers.

- Tradicionally, 1 is not considered prime, although it fulfills the primality criterion.

Neither it is composite because it cannot be decomposed in smaller primes, unless we include negative integers, in whose case 1 would be composite because $1 = (-1)\times(-1)$. In turn, -1 would not be prime if we consider complex numbers, because $-1 = i\times i$, where i is the imaginary unit. The number 1 is the neutral element of the multiplication and it is implicit in all numbers (primes and composites ones). Although 1 is considered the first natural number, really it represents the absolute, the previous unit to the creation of the duality. So, 2 is considered the first prime number.

- Except 2 –which is the only even and the smallest prime–, all other prime numbers are odd.

- Divisor 1 and number n itself are considered “trivials”. All other divisors are called “proper”. Therefore every proper divisor d of n satisfies $1 < d < n$.

- With the exception of 3, the sum of the digits of a prime number cannot be a multiple of 3.

- Prime numbers –with the exception of 2 and 5– end in 1, 3, 7 or 9. Prime numbers ending in 5 are not primes because they are multiple of 5. In this sense we can say that there are only 4 types or classes of prime numbers.

- Prime numbers are gradually separating among them. The so-called “Prime Number Theorem” states that the number of primes between 1 and n tends to $n/\ln(n)$ when n tends to infinite, where $ln(n)$ is the natural logarithm of n.

  The separation between two consecutive prime numbers is as large as we wish. Indeed, between $n!+2$ and $n!+n$ there are no primes, since $n!+2, n!+3, \ldots, n!+n$ are divisible respectively by $2, 3, \ldots, n$. For example, if $n = 5$, $n! = 120$, and we have the sequence 120+2, 120+3, 120+4, 120+5 = 122, 123, 124, 125, which are all composite numbers. Choosing n large enough, we have $n-1$ consecutive composite numbers.

Types of prime numbers

There are many types of prime numbers. Here we are only interested in the following ones:

- Twin primes are two consecutive primes that differ in 2. The first twin primes are:

  (3, 5), (5, 7), (11, 13), (17, 19), (29, 31), (41, 43), ...

  5 is the only prime number that belongs to two twin primes. It is conjectured that there are infinite twin primes. Most mathematicians believe that this is true, but it has not been proved yet.

- A Gaussian number is a complex number of the form $a+bi$, where a and b are integer numbers (one of them can be zero) and i is he imaginary unit. A Gaussian prime umber is a number that cannot be factorized in other Gaussian numbers.

Some prime numbers are not gaussian primes because they can be factorized in the complex plane. For example, prime number 17 is not gaussian prime because it can be factorized as $(4+i)(4-i) = 4^2+1 = 17$. In general, a natural number that can be expressed as sum of two squares (a^2+b^2) is not a gaussian prime because it can be factorized as $(a+bi)(a+bi)$.

Examples of gaussian primes are: $1+i$, $1-i$, $10+9i$, $14+i$, $17+2i$. In general, if $a+bi$ is gaussian prime, it is also its conjugate ($a-bi$).

The natural numbers that are gaussian primes are: 3, 7, 11, 19, 23 … They are all of the form $4k-1$ (k = 1, 2, …).

- The Pythagorean primes –discovered by Diophantus of Alexandria – are those that can be expressed as sum of two squares: $p = a^2 + b^2$. For example:

  $5 = 1^2 + 2^2$ $13 = 2^2 + 3^2$ $17 = 1^2 + 4^2$ $29 = 2^2 + 5^2$ $37 = 1^2 + 6^2$

  It is known that the numbers of Pythagorean and non-Pythagorean primes up to n are aproximately equal.

  There are numbers that can be expressed as sum of squares but they are not prime. For example,

  $25 = 3^2 + 4^2$ $45 = 3^2 + 6^2$ $65 = 1^2 + 8^2$

  The first Pythagorean primes are:

  2, 5, 13, 17, 29, 37, 41, 53, 61, 73, 89, 97, 101, 109, 113, …

  Number 2 corresponds to the expression $1^2+1^2 = 2$, and it is the only even Pythagorean prime. Fermat discovered that the odd Pythagorean primes p are of the form $4k+1$, i.e., are congruents with 1 module 4: $p \equiv 1$ (module 4).

Pythagorean triples

A Pythagorean triple is an ordered set of three natural numbers (x, y, z) which corresponds to the three sides of a right triangle, where x and y are the legs and z is the hypotenuse. That is to say, the Pythagoras' theorem is fulfilled: $x^2+y^2 = z^2$. A right triangle whose sides form a Pythagorean triple is called a Pythagorean triangle.

Excluding the trivial triple (1, 1, 2), the Pythagorean triples have the following properties:

- Since x and y have different parity, z must be odd. Since x and y are exchangeable, we can suposse that x is odd. Therefore, a Pythagorean triple has a symmetric form: (odd, even, odd).

- Pythagorean triples have the form $(a^2-b^2, 2ab, a^2+b^2)$, with $a>b$, since $(a^2-b^2)^2 + (2ab)^2$

$= (a^2+b^2)^2$ is fulfilled. Since $a^2–b^2$ and a^2+b^2 are odd, a and b have different parity. Given the Pythagorean triple (x, y, z), the corresponding values of a and b are:

$$a = \sqrt{((z+x)/2)} \qquad b = \sqrt{((z–x)/2)}$$

- Lets call "positive Pythagorean expression" to an expression of the type a^2+b^2, where a and b are natural numbers of different parity. At geometric level, the corresponding number ($n = a^2+b^2$) has the property that its square root ($\sqrt{n}$) is the hypotenuse of a right triangle of legs a and b. The numbers of this type may or not be primes, but they are not gaussian primes because they can be decomposed in factors: $a^2+b^2 = (a + bi)(a – bi)$.

- Lets call "negative Pythagorean expression" to an expression of the type $a^2–b^2$, where a and b are natural numbers of different parity and $a>b$. Naming n the correspondent number ($n = a^2–b^2$), then at geometric level, a is the hypotenuse of a right triangle of legs $\sqrt{n}$ and b.

- Pythagorean triples connect the positive and negative Pythagorean expressions. That is to say, every Pythagorean triple have duality at the sign level between the extreme terms ($a^2–b^2$ and a^2+b^2). For example, $(4^2–1^2, 8, 4^2+1^2) = (15, 8, 17)$.

- There is a relationship between the gaussian numbers and the Pythagorean triples. A gaussian number, which is a complex number $c = a+bi$, with a and b integers, has as norm $|c| = \sqrt{(a^2+b^2)}$. If we multiply c by itself, we have $c^2 = a^2–b^2+2abi$, where the real part and the imaginary part are the two first terms of a Pythagorean triple (the two legs) and the square of its norm is the third term (the hypothenuse): $|c^2| = a^2+b^2$.

- There are two types of Pythagorean triples: primitive and derivative. The primitive triples (x, y, z) fulfill g.c.d.$(x, y, z) = 1$, i.e., they do not have any common factor (they are coprime). The first primitive Pythagorean triples with $x<100$ are:

  (3, 4, 5), (5, 12, 13), (7, 24, 25), (8, 15, 17), (9, 40, 41), (11, 60, 61), (12, 35, 37), (13, 84, 85), (16, 63, 65), (20, 21, 29), (28, 45, 53), (33, 56, 65), (36, 37, 85), (39, 80, 89), (48, 55, 73), (65, 72, 97)

  There are infinite primitive Pythagorean triples. Euclid proved this the following way. Every odd number can be expressed as a difference between the squares of two consecutive numbers. The formula is: $n = a^2–b^2$, where $a = (n+1)/2$ and $b = (n–1)/2$. From this formula, we can build the triple $(a^2–b^2, 2ab, a^2+b^2)$. For example, for n=7, we have a=4 and b=3, and the triple is $(4^2–3^2, 24, 4^2+3^2) = (7, 24, 25)$. Since there are infinite odd numbers, there are infinite primitive Pythagorean triples.

  The derivative triples of a primitive triple (x, y, z) are multiple of the type (nx, ny, nz), with $n>1$. There are, therefore, infinite derivative Pythagorean triples of a primitive Pythagorean triple. For example, the Pythagorean triple (3, 4, 5) have as derivative triples: (6, 8, 10), (9, 12, 15), (12, 16, 20), etc.

- The smallest Pythagorean triple is (3, 4, 5), corresponding to the so-called "egyptian triangle", where 3 and 5 are the first odd prime numbers.

- Every natural number a is part of a positive Pythagorean expression a^2+b^2, and this in turn is part of a Pythagorean triple. For example, $1^2+2^2 = 5$, $2^2+3^2 = 13$, $3^2+4^2 = 25$, etc.

- In a Pythagorean triple $(a^2-b^2, 2ab, a^2+b^2)$, a^2-b^2 can be prime or composite, $2ab$ is composite (is even) and a^2+b^2 can be prime or composite. Examples:

  $(3, 4, 5) = (2^2-1^2, 4, 2^2+1^2)$ (3 and 5 are primes)
  $(15, 8, 17) = (4^2-1^2, 8, 4^2+1^2)$ (15 is composite, 17 is prime)
  $(23, 264, 265) = (12^2-11^2, 264, 11^2+12^2)$ (23 is prime, 265 is composite)
  $(63, 16, 65) = (8^2-1^2, 16, 8^2+1^2)$ (63 and 65 are composites)

- The so-called "H hypothesis" of Schinzel-Sierpinski [Ribenboim, 1996] states that there are infinite Pythagorean triples containing 2 primes (the first and third elements, since the second one is even).

  A prime Pythagorean triple is a triple whose extreme elements are primes. It is a primitive triple. The corresponding right triangle is called "prime Pythagorean triangle". The 10 first prime Pythagorean triples are:

  $(3, 4, 5) = (2^2-1^2, 4, 2^2+1^2)$
  $(5, 12, 13) = (3^2-2^2, 12, 3^2+2^2)$
  $(11, 60, 61) = (6^2-5^2, 60, 6^2+5^2)$
  $(19, 180, 181) = (10^2-9^2, 180, 10^2+9^2)$
  $(29, 420, 421) = (15^2-14^2, 420, 15^2+14^2)$
  $(59, 1740, 1741) = (30^2-29^2, 1740, 30^2+29^2)$
  $(61, 1860, 1861) = (31^2-30^2, 1860, 31^2+30^2)$
  $(71, 2520, 2521) = (36^2-35^2, 2520, 36^2+35^2)$
  $(79, 3120, 3121) = (40^2-39^2, 3120, 40^2+39^2)$
  $(101, 5100, 5101) = (51^2-50^2, 5100, 51^2+50^2)$

  There is only one prime Pythagorean triple containing two twin primes (p and $p+2$): the archetypal Pythagorean triple (3, 4, 5).

  It is conjectured, as in the case of the twin primes, that there are infinite prime Pythagorean triples.

- There are Pythagorean triples that share the same leg. For example, (15, 112, 113), (15, 20, 25), (15, 36, 39), (15, 8, 17). However, there are not Pythagorean triples that share the third element (the hypotenuse), since every Pythagorean expression of type $x^2+y^2 = z^2$ is unique.

The Question of the Pattern of Prime Numbers

The current situation

The structure of the sets of prime numbers is aparentently irregular, without any order or specific quantitative pattern, but with a general qualitative pattern: prime numbers are gradually separating among them. Prime numbers are independent of each other, without multiplicative relations at the level of natural numbers. They only have particular additive relations, but without a known general pattern.

All attempts to obtain the pattern of prime numbers have failed. It is often said that prime numbers are what is left when you have taken all the patterns away. Particular patterns have been found, but up to now no general pattern has been found. However, the search continues for two reasons: 1) because it is believed that something so fundamental as the prime numbers should have a pattern; 2) because mathematics is a logical discipline, so the distribution of prime numbers in the real line must be governed by totally deterministic rules of logic.

The search of this pattern has fascinated to profesional and amateur mathematicians along history, but up to now without result. Some authors believe that this problem might be undecidable (in Gödel sense). Some pesimistic opinions on this subject are:

- "Mathematicians have tried in vain to this day to discover some order in the sequence of prime numbers, and we have reason to believe that it is a mistery into which the human mind will never penetrate" (Leonhard Euler) [2].
- "God may not play dice with the universe, but something strange is going on with the prime numbers" (atributted to Paul Erdös) [3].
- "It will be another million years, at least, before we understand the prime numbers" (Paul Erdös) [4].
- "Prime numbers are the greatest mystery in mathematics" (Marcus du Sautoy) [5].

It is believed that if the pattern of prime numbers were discovered:

- It would be the "Holy Grail" of mathematics. It could shed light about the ultimate and deep nature of natural numbers, of mathematics and even of the universe. In short, it may be the key to the "Theory of Everything".

  Understanding prime numbers is understanding all posible worlds, because prime numbers transcend the physical reality. Prime numbers are real and they are in all posible worlds.

- It would have effects on diverse especific fields, especially of the deep type, as genetics and quantum physics.

  According to Igor V. Volovich [6], the fundamental entities of the universe are not the physical particles (electrons, quarks, etc.) or quantic fields or strings, but the natural numbers, the fundamental mathematical entities. And as the natural numbers are built from prime numbers, the fundamental entities of the universe are the prime numbers. He justifies this because, at deep physical level, at distances less than the Plank length (the smallest distance able to be measured) do not apply the normal (macroscopic) concepts of 3D space or linear time or the conventional euclidean geometry.

Galileo asserted that the universe is written in the language of mathematics. But Max Tegmark [7] goes beyond this: the universe is a mathematical structure; the deepest nature of reality is of mathematical type; matemathics is the only universe that exists.

- It would suppose an experience of transcendental and unified consciousness of the internal and external reality. According to Greg Chaitin [8], “comprehension is compression”, i.e., every form of knowledge is based on compression. In the case of finding the pattern of prime numbers, the infinite would compress in the finite.

- The proofs of mathematical theorems of number theory (as the Fermat’s last theorem) would be simplified, and it could resolve pending mathematical problems as the Riemann hypothesis, the Goldbach´s conjecture, the twin primes conjecture, etc. At deep level everything is simplified because from that level it is possible to contemplate with clarity all the manifested things.

- Cryptographic systems of public key, based on the factorization of large prime numbers, would weaken.

Enclosure patterns of prime numbers

Although the pattern of prime numbers has not been found yet, there are attempts to narrow them through different enclosure patterns, i.e., patterns that include all prime numbers. The most important are:

- The pattern $2k+1$.
  With the exception of 2, all prime numbers are odd. Therefore, el pattern $n = 2k+1$, with $k = 1, 2, 3, 4, 5, 6, \ldots$ include all odd prime numbers. That is to say, $n \equiv 1$ (module 2). Thus the 50 % of natural numbers is sieved. Odd prime numbers are always beside a composite number. It is enough to do $p-1$ or $p+1$ for finding a even (composite) number. Every prime $p = 2k+1$ is always “escorted” by the even numbers $2k$ and $2k+2$.

- The pattern $4k\pm1$.
  The odd natural numbers can be divided in two branches:

  1. $n_1 = 4k-1$ (k =1, 2, 3, …): 3, 7, 11, 15, … That is to say, $n_1 \equiv -1$ (module 2).
  2. $n_2 = 4k+1$ (k = 1, 2, 3, …): 5, 9, 13, 17, … That is to say, $n_2 \equiv 1$ (module 2).

  The difference between the numbers of both branches for the same k is $(4k+1) - (4k-1) = 2$.

- The pattern $6k\pm1$.
  With the exception of 2 and 3, all prime numbers “fall” inside this pattern. This pattern results of considering 6 expressions:

$6k–3$, $6k–2$, $6k–1$, $6k$, $6k+1$, $6k+2$

and discarding the multiple of 2 and 3. Thus 4/6 = 2/3 (66,666… %) of the natural numbers is sieved.

- Another possible possible pattern is based in considering the product of the three first prime numbers: $2\times3\times5 = 30$. In this case, there are 30 expressions, and we would have to discard the multiple of 2, 3 and 5, remaining finally $30k\pm1$, $30k\pm7$, $30k\pm11$, $30k\pm13$. Thus (30–8)/30 = 22/30 (73,333… %) of natural numbers is sieved.

  It is also posible to choose more initial prime numbers. With this system are sieved out: the multiple 2 and 3; of 2, 3 and 5; of 2, 3, 5 and 7; etc. The set of the corresponding partial formulas enclose all prime numbers. The greater is the number of initial prime numbers, the greater is the approach to prime numbers, but the set of formulas becomes more complex. A final adjustment with this system is impossible because it implies the infinite. In this sense, there is an analogy with irrational numbers. This system is paradoxical: the formula that enclose all prime numbers is based on the prime numbers themselves.

Factors to consider

In the subject of the search of the posible pattern of prime numbers, it is necessary to take into account several factor sor aspects:

- The product question.
  It is usually said that prime numbers are the “building blocks” of natural numbers, like the atoms are the building blocks of matter, and like cells are the building blocks of living beings. These statements are not true because in these cases the building blocks are based on the sum, not in the product.

  Usually it is also established an analogy between a composite expression as $2^2\times3\times5^3$, and a chemical formula as $C_6H_{12}O_6$ (glucose’s formula), stating that prime numbers are like atoms. This is not true either. Firstly because the number of atoms is finite and the number of primes is infinite. Secondly because the chemical relation is of additive type, not multiplicative one. Thirdly because there are formulas that correspond to a cyclic structure such as the benzene (C_6H_6). The analogy is merely superficial, only based in the form.

  So the difficulty of finding the pattern of prime numbers is due to that all is based on the product of natural numbers. As a matter of fact, we should say that “the primes are the multiplicative building blocks of the natural numbers”. It is imposible to establish a multiplicative relationship among the prime numbers not involving the rational numbers. The right way to find that posible pattern must be based on the additive branch of number theory: the sum, which is a operation simpler than the product. From the sum point of view, there is only a number which can be qualified as “prime” o “primary”, which is the unity (1). The rest of them are composite numbers.

A way to establish relations with the sum is considering the subject of partition numbers. Given a natural number *n*, there are a number of posible partitions of that number. The decomposition of a number in products of primes is unique, but the decomposition in sums is not unique. Calling $P(n)$ to the number of partitions of *n*, we have, for example:

$P(2) = 2$ (2, 1+1)
$P(3) = 3$ (3, 2+1, 1+1+1)
$P(4) = 5$ (4, 3+1, 2+2, 2+1+1, 1+1+1+1)
$P(5) = 7$
$P(10) = 42$
$P(100) = 190.569.292$

Partition numbers were studied by Ramanujan, but Ken Ono and his team discovered the first formula to directly compute $P(n)$ from *n* and they also discovered that these numbers have a fractal structure [Bruiner & Ono, 2011].

- The modes of consciousness.
  As it is known, there are two modes of consciousness:

  1. The intuitive, deep, conceptual, synthetic, creative, general, global, imaginative, qualitative, parallel, continuous, etc. It is usually associated to the right hemisphere of the brain. We will call it "RH consciousness" for short.

  2. The rational, superficial, formal, analytical, particular, quantitative, sequential, discrete, etc. It is usually associated to the left hemisphere of the brain. We will call it "LH consciousness" for short.

The whole consciousness arises when both modes of consciousness are connected. This connection is in such a way that every particular thing is a manifestation of something general. In this sense, RH consciousness is higher than LH consciousness. Any particular thing can never be isolated. It must be linked to something general or universal. This connection is precisely the semantics of the particular, what gives it a meaning.

This universal duality is manifested in the particular duality between prime and composite numbers:

- Prime numbers correspond to RH consciousness: they are non-linear, qualitative, descriptive, synthetic, non-analyzable and non-decomposable. They lie in a deep level and they make reference to themselves. Prime numbers are usually called "God code" because it is considered that they are in the most primary and deep level.

- Composite numbers correspond to LH consciousness. They are linear, quantitative, operative, analyzable and decomposable. They are in a superficial level and make reference (explicitly or implicetly) to the prime numbers. They are manifestations of combinations (of type product) of prime numbers. The larger are the composite numbers, the more superficial they are.

Numbers are archetypes. Prime numbers are primary archetypes. The lower is a prime number, the higher is its depth as an archetype and the greater are its manifestations as composite numbers. Composite numbers are projections or manifestations of prime numbers.

Trying to capture the pattern of prime numbers is imposible because that pattern belongs to RH consciousness and its capture requieres the LH consciousness, and this is not posible because the RH consciousness is on a higher level than the LH one.

The pattern of prime numbers can not be of superficial level (type "consciousness LH"), but of deep one (type "RH consciousness"). It can not be operational or quantitative, but it has to be necessarily descriptive and qualitative. As a matter of fact, already exists a descriptive pattern, which is the own definition of prime number.

The superficial level can express in terms of the deep, but not the other way: the deep level can not express in terms of the superficial one. It would be a contradiction that prime numbers to be expressed in terms of aritmetic or algebraic relations from themselves or from the composite numbers, which are manifestations of primes.

- The duality algebra – geometry.
  In mathematics, these two modes of consciousness are reflected in the duality algebra-geometry, where algebra corresponds to LH consciousness and geometry corresponds to RH consciousness. What we can call "mathematical consciousness" arises when algebra and geometry get connected.

  Pythagoras' theorem plays a fundamental role in the connection between algebra and geometry. Indeed, if we have the expression $x+y = z$, then exists a right triangle of legs $\sqrt{x}$, $\sqrt{y}$ and hypothenuse $\sqrt{z}$. That is to say, in the sum –the most fundamental operation of mathematics – is implicit the Pythagoras' theorem. Hence its enormous importance as a universal connector between algebra and geometry:

  – To pass from geometry to arithmetic we have to square the numbers that represents the sides of a right triangle.
  – To pass from arithmetic to geometry we have to transform the numbers in square roots.

  Pythagoras' theorem represents the "mathematical consciousness": the connection between algebra and geometry, i.e., the unión of the two modes of consciousness.

  Since RH consciousness is higher than LH consciousness, and since geometry is RH consciousness and algebra is LH consciousness, geometry is on a higher level than algebra, so algebra should be a particularization or manifestation of geometry.

  Therefore, the pattern of prime numbers cannot be of algebraic type because algebra is on a lower level than geometry. It is from the high level of geometry where perhaps it would be posible to find the pattern of prime numbers.

The strategy to follow: the general principles

The multiple attempts along history trying to find the pattern of prime numbers have studied its distribution from the analytical, algebraic, arithmetic and quantitative points of view. But for understanding prime numbers we must situate ourselves at a higher level. It is not a question of finding horizontal relations among prime numbers but ascending as posible and searching for general (or universal) laws or principles that project or manifest as prime numbers. The principles that can be applied are the following ones:

- The downwarding causation principle.
  This universal principle states that every effect comes from a deeper level. That the superficial level is a manifestation of the deep one. And that from the superficial level is not posible express the deep one.

  The set of prime numbers has to be considered at holistic level, as a whole in which all prime numbers must be related from a higher level.

- The simplicity principle.
  Since prime numbers are the foundation of all natural numbers, their structure must be necessarily simple, because the simplicity is asociated with the deep level and with consciousness.

  The simplicity principle has been established in two ways:

  1. The Ockham' razor: among the diverse theories trying to explain a phenomenon, we must to choose the most simple one.

  2. The Einstein principle, also called "Einstein's razor". This principle is based on a phrase attributed to the famous scientist: "Everything should be made as simple as posible, but not simpler".

  This statement has been the subject of controversy, as it has been much discussion about its meaning. For example, it has been said that it is contradictory, because if something has been made as simple as posible, it can not be more simple. Others say that the phrase should be "Everything should be made as simple as posible, but not too simple".

  Surely, Einstein meant that the simplicity has its limits. That we should not go beyond the conceptual essence of something. When we exceed this limit, we enter in a purely superficial, formal, mechanical or syntactical field, with loss of semantics. For example, a hammer can be made more simple taking out its head, but then it is not a hammer, as it loses its esence.

  A paradigmatic example is the binary logic, operatively defined by three concepts: negation ($\neg p$), conjuntion ($p \wedge q$) and disjuntion ($p \vee q$). However, it is posible to define a simpler operation through the Sheffer stroke: “neither p nor q” ($p|q$), from which it is possible to define the logical operations. This leads, paradoxically, to greater complexity:

$$\neg p = p|p \qquad p\vee q = (p|q) \mid (p|q)$$
$$p\wedge q = ((p|p)|(q|q)|(p|p)|(q|q)) \mid ((p|p)|(q|q)|(p|p)|(q|q))$$

- The duality principle.
  This principle states that, in the manifested world, “everything is dual, everything has two poles”. At deep level (not manifested) there is no duality. Duality manifest mainly as simmetry, as complementarity or as opposite concepts.

  Duality is a key principle to understand reality. The key of every theory lies in the identification of dualities. The unification of theories is based on the unification of dualities. This is specially useful in physics, where dualities are unified in concepts as space-time, mass-energy and wave-corpuscle. The key issue is harmonizing dualities searching for higher concepts.

  There exists duality between prime and composite numbers. But prime numbers should also have a dual nature. They should have opposite, symmetric or complementary primes. As a matter of fact, there are symmetric primes: the twin primes (primes that differ in 2). The interest on twin primes lies in that they symbolize the consciousness, the union of opposites.

  This duality must also exists in the composite numbers.

- The fractal paradigm.
  This is an universal principle. It states that there is a set of few principles that manifest themselves in all levels: in the macrocosm, in the microcosm, in the inner (the mind) and in the outer (the nature).

  The definition of prime number is very simple, but the distribution of primes is very complex. But it may occur likewise as with fractals, that they have simple generating rules but applied recursively produce large complexity. The paradigm of this approach is the Mandelbrot set, where a simple equation ($z = z^2+c$) in the complex plane, recursively applied, produce an infinitely complex structure.

Specifically, the strategy to follow for trying to find the pattern of the prime numbers will be based on:

1. The enclosure pattern $4k\pm1$.
   This pattern has the advantage that fulfills the Einstein’s razor –the maximun simplicity would be the pattern $2k+1$–, and also the duality principle: the dual numbers ($4k-1$ and $4k+1$) differ in 2. Besides it includes all odd numbers, supplying an environment that allows relating prime and composite numbers.

2. Pythagoras’ theorem.
   Pythagoras’ theorem is considered the most fundamental theorem of mathematics

because it connects algebra and geometry. It is a deep theorem, a theorem of consciousness. Since prime numbers belong to the realm of the deep, they must have a close relation with this theorem.

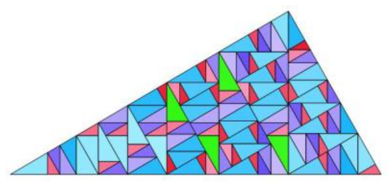
Pythagoras' theorem fulfills the requiered properties of:

- Downwardig causation. Pythagoras' theorem connects geometry and algebra, where algebra is a manifestation of geometry.

- Simplicity. The right triangle is the simplest geometric form. Every 2D geometric shape can be decomposed in triangles. In turn, every triangle can be decomposed in two right triangles.

- Duality. Pythagoras' theorem, expresed as sum of squares ($x^2+y^2 = z^2$) has its dual as difference of squares ($z^2–y^2 = x^2$), in which it is implicit the product: $z^2–y^2 = (z+y)(z–y)$. This duality also manifests in the first and third terms of the Pythagorean triples, which have the form ($a^2–b^2$, $2ab$, a^2+b^2).

- Fractality. Fractality is manifested dividing a right triangle in two similar triangles, and applying this mecanism recursively (see figure).



Fractalidad of a right triangle

Properties of Pythagorean Expressions

Product of two numbers that are sums of squares

The product of two numbers that are sum of two squares, $a_1^2+b_1^2$ and $a_2^2+b_2^2$, is also a sum of two squares, in two forms:

1. $(a_1^2 + b_1^2)(a_2^2 + b_2^2) = (a_1a_2 – b_1b_2)^2 + (a_1b_2 + a_2b_1)^2$
2. $(a_1^2 + b_1^2)(a_2^2 + b_2^2) = (a_1a_2 + b_1b_2)^2 + (a_1b_2 – a_2b_1)^2$

For example:

1. $65 = 5×13 = (1^2 + 2^2)\ (2^2 + 3^2) = (2 – 6)^2 + (3 + 4)^2 = 4^2 + 7^2$
2. $65 = 5×13 = (1^2 + 2^2)\ (2^2 + 3^2) = (2 + 6)^2 + (3 – 4)^2 = 8^2 + 1^2$

In the case of two equal numbers, it only makes sense applying the first formula, since the second one leads to an identity:

1. $(a^2 + b^2)(a^2 + b^2) = (a^2 – b^2)^2 + (ab + ab)^2 = (a^2 – b^2)^2 + (2ab)^2$
2. $(a^2 + b^2)(a^2 + b^2) = (a^2 + b^2)^2 + (ab – ab)^2 = (a^2 + b^2)^2$

For example, $25 = 5×5 = (1^2+2^2)(1^2+2^2) = (1^2–2^2)^2 + (2×2)^2 = 3^2 + 4^2$

Every composite number is a difference of squares

Every composite number n of type $x \times y$, where $x<y$, can be expressed as a difference between two squares ($a^2–b^2$). Indeed, doing $x = (a–b)$ and $y = (a+b)$, we have:

$$a = (x+y)/2 \qquad b = (y–x)/2 \qquad n = x \times y = (a+b)\times(a–b) = a^2 – b^2$$

Therefore, $n = x \times y = ((x+y)/2)^2 – ((y–x)/2)^2$

In order a and b to be integers, x and y must have the same parity. For example:

$n = 5\times37$, $a = (37+5)/2 = 21$, $b = (37–5)/2 = 16$, $n = 21^2 – 16^2$
$n = 15\times9$, $a = (15+9)/2 = 12$, $b = (15–9)/2 = 3$, $n = 12^2 – 3^2$
$n = 4\times14$, $a = (14+4)/2 = 18/2 = 9$, $b = (14–4)/2 = 10/2 = 5$, $n = 9^2 – 5^2$

If they have not the same parity, then we obtain for example,

$n = 4\times13$, $a = (13+4)/2 = 17/2$, $b = (13–4)/2 = 9/2$, $n = (17/2)^2 – (9/2)^2$

Conversely, if a natural number n can be expressed as a difference between two squares, $n = a^2–b^2$, then n is a composite number: $n = x \times y$, where $x = (a–b)$ and $y = (a+b)$, i.e., $n = (a+b)\times(a–b)$.

In the case where $a–b = 1$, i.e., if a and b are consecutive numbers ($a = b+1$), we have

$$n = a^2–b^2 = (a+b)\times(a–b) = a+b$$

which is always an odd number. For example, $7^2–6^2 = 6+7 = 13$, $8^2–7^2 = 8+7 = 15$. From this property follows that every odd number can be expressed as a difference between the squares of two consecutive numbers.

Product of two numbers that are difference of squares

Given two natural numbers that can be expressed as difference of two squares, $a_1^2–b_1^2$ and $a_2^2–b_2^2$, then their product can be expressed as a difference of two squares, also (as in the case of the sum of squares) of two forms:

1. $(a_1^2–b_1^2)(a_2^2–b_2^2) = (a_1a_2 + b_1b_2)^2 – (a_1b_2 + a_2b_1)^2$
2. $(a_1^2–b_1^2)(a_2^2–b_2^2) = (a_1a_2 – b_1b_2)^2 – (a_1b_2 – a_2b_1)^2$

For example:

$21 = 3\times7 = (2^2–1^2)\,(4^2–3^2) = (8+3)^2 – (6+4)^2 = 11^2–10^2$
$21 = 3\times7 = (2^2–1^2)\,(4^2–3^2) = (8–3)^2 – (6–4)^2 = 5^2–2^2$

In the case of two equal numbers, only makes sense applying the first formula, since the second one leads to an identity:

1. $(a^2 - b^2)(a^2 - b^2) = (a^2 + b^2)^2 - (ab + ab)^2 = (a^2 + b^2)^2 - (2ab)^2$
2. $(a^2 - b^2)(a^2 - b^2) = (a^2 - b^2)^2 - (ab - ab)^2 = (a^2 - b^2)^2$

For example, $49 = 7\times7 = (4^2-3^2)\,(4^2-3^2) = (4^2+3^2)^2 - 24^2 = 25^2 - 24^2$

Product of sum of squares by difference of squares

Given two natural numbers that can be expressed, one as sum of two squares ($a_1^2+b_1^2$), and the other as a difference of two squares ($a_2^2-b_2^2$), then their product can also be expressed as a difference of two squares:

$$(a_1^2+b_1^2)(a_2^2-b_2^2) = ((a_1^2 + b_1^2 + a_2^2 - b_2^2)/2)^2 - ((a_1^2 + b_1^2 - a_2^2 + b_2^2)/2)^2$$

This formula results from the equation $(a_1^2+b_1^2)(a_2^2-b_2^2) = x\times y$, making $x = a_1^2+b_1^2$ e and $y = a_2^2-b_2^2$.

If n_1 and n_2 are both odd, then a_1 and b_1, and also a_2 and b_2, have different parity. Therefore, it results a difference of squares between integers. For example: $7\times29 = (4^2-3^2)\times(2^2+5^2) = 18^2-11^2$.

Even numbers vs. Pythagorean expressions

Number 2 can be expressed as the positive Pythagorean expression 1^2+1^2. Powers 2^n can also be expressed as positive Pythagorean expressions. In general:

If n is even, $2^n = (2^{n/2})^2$.
If n is odd, $2^n = (2^{(n-1)/2})^2 + (2^{(n-1)/2})^2$

An odd number is of the form 2^nk, where $n\geq1$ and $k\geq1$ is odd. Number k can be of type a^2, a^2+b^2 or a^2-b^2. Calling $c = 2^{(n-1)/2}$, we have:

k	Value of 2^nk if n is even	Value of 2^nk if n is odd
a^2	$(2^{n/2}a)^2$	$(ac)^2 + (ac)^2$
a^2+b^2	$(2^{n/2}a)^2 + (2^{n/2}b)^2$	$((ac)^2-(bc)^2)^2 + ((bc)^2+(ac)^2)^2$
a^2-b^2	$(2^{n/2}a)^2 - (2^{n/2}b)^2$	$(2c^2+a^2-b^2)/2)^2 - (2c^2-a^2+b^2)/2)^2$

Since a y b have different parity, the last expression is a difference of squares of non-integer numbers.

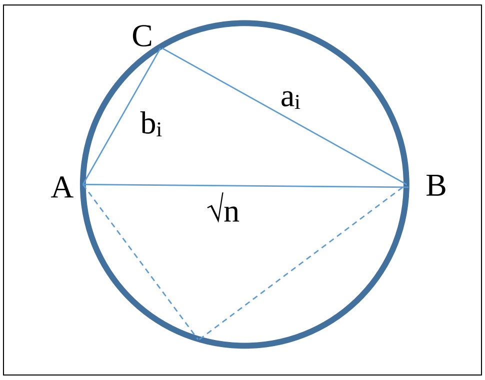
Geometry of equivalent Pythagorean expressions

When we have several sums of squares, of the type $4k$+1, which correspond to a same composite number,

$$n = a_1^2+b_1^2 = a_2^2+b_2^2 = . . . = a_m^2+b_m^2$$

for example,

$$1105 = 4^2 + 33^2 = 9^2 + 32^2 = 12^2 + 31^2 = 23^2 + 24^2$$



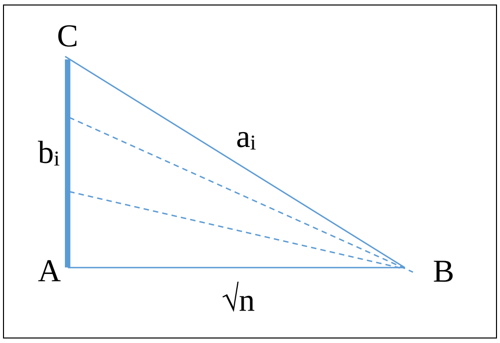
then the vértices C of the right triangles of legs a_i and b_i belong to a circunference of diameter $\sqrt{n}$ (see figure).

When we have several difference of squares, which correspond to a same composite number,

$$n = a_1^2–b_1^2 = a_2^2–b_2^2 = . . . = a_m^2–b_m^2$$

for example,

$$885 = 37^2 – 22^2 = 91^2 – 86^2 = 149^2 – 146^2$$



If we consider the right triangles of base AB = $\sqrt{n}$, then all vértices C belong to a perpendicular straight line to AB passing through A (see figure).

Fermat's Theorem of Sum of Squares and its Dual

Fermat's theorem of sum of squares

This theorem [9] [10] [11] states that every prime number p of type $4k$+1, i.e., $p \equiv 1$ (module 4), can be expressed as a unique sum of two squares: $p = a^2+b^2$. And vice versa: if a prime number can be expressed as sum of squares, then it is of type $4k$+1. For example,

$$13 = 4×3 + 1 = 2^2 + 3^2 \qquad 17 = 4×4 + 1 = 1^2 + 4^2 \qquad 29 = 4×7 + 1 = 2^2 + 5^2$$

The first prime numbers of this type are: 5, 13, 17, 29, 37, 41, 53, …

In this theorem it is necessary to take into account the following restrictions:

1. Numbers a and b must have different parity, because if they would have the same, $p = a^2+b^2$ would be even and it would not be prime.
2. Numbers a and b cannot be equal because a^2+b^2 would be also even. Therefore, we can assume that $a<b$.
3. Numbers a and b must be coprimes, because if they would have a common factor, then a^2+b^2 would not be prime because it would be multiple of that factor.

Fermat enunciated this theorem in a letter to Mersenne on 25th December 1640. For this reason, this theorem is also known as "Fermat's Christmas theorem". The theorem appeared in the publication of 1670 of Fermat'son of the notes of his father in his copy of Diophantus' Arithmetic.

Euler said in a letter to Christian Goldbach, on 12th Abril 1749, that he had proved it. He published in 1783 a complete proof, using the infinitely descending method. Lagrange published a proof in 1775, based on quadratic forms. This last proof was simplified by Gauss in his *Disquisitiones Arithmeticae*. Dedekind provided other two proofs based on the gaussian integers.

This theorem was qualified by G.H. Hardy as "one of the most beautiful of arithmetic" [12]. Mathematicians are still interested in it for two reasons: 1) For its role in the conection between algebra and geometry; 2) For attempting to find the most simple posible proof in order to help understanding the true nature of prime numbers.

Other numbers that are sums of squares

Besides the primes of type $4k+1$, there are other numbers that can be expressed as sum of two squares a^2+b^2, but they are composites, in two cases:

1. When this decomposition is not unique. For example:

   $85 = 5\times 17 = 2^2 + 9^2 = 6^2 + 7^2 \qquad 221 = 5^2 + 14^2 = 10^2 + 11^2$

2. When a and b are the two first elements of a Pythagorean triple (a, b, c), i.e., when exists a number c such $a^2+b^2 = c^2$. For example,

   $3^2+4^2 = 5^2 \quad 5^2+12^2 = 13^2 \quad 8^2+15^2 = 17^2 \quad 20^2+21^2 = 29^2 \quad 12^2+35^2 = 37^2$

These two types of composite numbers are also type $4k+1$. In general, every odd number n that can be expressed as sum of squares is type $4k+1$. This can be easily proved:

> In effect, every square number ($2^2, 3^2, 4^2, 5^2, \ldots$) fulfills that their remainders after dividing by 4 are 0 or 1. Therefore,
>
> $a^2 \equiv 0$ or 1 (module 4) $\qquad b^2 \equiv 0$ or 1 (module 4)
> $n = a^2+b^2 \equiv 0$ or 1 or 2 (module 4)
>
> But $n \equiv 0$ or 2 (module 4) must be discarded because n would be even. Therefore, it only remains $n \equiv 1$ (module 4).

Dual theorem of the Fermat's Christmas theorem

Fermat's Christmas theorem is truly remarkable because it relates prime numbers of type $4k+1$ (objets whose definition implies multiplication and division) with the additive structure of square numbers. When considering square numbers we are "raising" to geometry, since it is at a higher level than algebra. Thus the search of relations among natural numbers (primes and composite) are notably simplified.

Therefore, Fermat's Christmas theorem is not any other theorem of number theory, but a fundamental theorem, since it connects algebra and geometry, revealing the close relationship between prime numbers and Pythagoras' theorem. This theorem is comparable in importance to the fundamental theory of arithmetic.

However, Fermat's Christmas theorem needs to be complemented, since it is natural to consider the dual operation: difference of squares.

We know, according the Fermat's Christmas theorem, that if n is prime and of type $4k$+1, then it is expressible in a unique form as a sum of two squares. This is the fundamental property that characterizes the primes of type $4k$+1 of the odd numbers.

If n is prime and is of type $4k$–1, then:

- It is not expressible as sum of squares, because if it were, then it would be of type $4k$+1.
- It is not expressible as difference of squares, because if it were, then it would be a composite number.

Therefore, every prime number of type $4k$–1 is not expressible as sum or difference of squares, obviating the trivial relation that every odd number is the difference between the squares of two consecutive numbers. Another form of saying this is that a prime number of type $4k$–1 is only expressible in a trivial way, i.e., as difference between the squares of two consecutive numbers. This is the fundamental property that characterizes the primes of type $4k$–1 of the odd numbers. The first prime numbers of this type are: 3, 7, 11, 19, 23, 31, 43, …

Table of Pythagorean Expressions of Odd Numbers. Properties

Making use of the experimental aspect of number theory, it has been generated (through a computer program) the table of Pythagorean expressions of the odd numbers (up to 501), with the two branches ($4k$–1 and $4k$+1). (See Appendix.)

- Prime numbers appear shaded.
- Every Pythagorean expression is accompanied by the number asociated to its dual Pythagorean expression. The letters A and B, mean type $4k$–1 and $4k$+1, respectively.
- Trivial Pythagorean expressions are not included, i.e., the difference between the difference of squares of two consecutive numbers, except in the dual numbers. For example, the expression dual of 3^2+4^2 is the trivial 4^2-3^2.

In the table, dual Pythagorean expressions appear (a^2-b^2 and a^2+b^2), i.e., they form the dual extremes of a Pythagorean triple. The number a^2-b^2 is always composite because it is equivalent to the product $(a+b)(a-b)$, unless $a-b=1$, in which case can be prime or composite. The number a^2+b^2 can be prime or composite. These expressions reflect the Fermat's Christmas theorem and its dual, and also the properties previously mentioned, The first thing that we can observe in the table is that the branch $4k+1$ is more complex because is more "populated" by Pythagorean expressions than the branch $4k-1$, which is simpler, although the number of primes in both branches is more or less the same.

Types of odd numbers

In the branch $4k-1$ there are two types of numbers:

1. Those expressed by one or several negative Pythagorean expressions. They are all composite numbers of type $\text{even}^2-\text{odd}^2$. For example,

   $$15 = 4^2-1^2 \qquad 27 = 6^2-3^2 \qquad 135 = 12^2-3^2 = 16^2-11^2 = 24^2-21^2$$

2. Those not expressed by any Pythagorean expression (positive or negative). They can only be expressible as difference between the squares of two consecutive numbers. They are all prime numbers also of type $\text{even}^2-\text{odd}^2$. For example, $3 = 2^2-1^2$ and $7 = 4^2-3^2$.

In the branch $4k+1$ there are 5 types of numbers:

1. Those expressed by only one positive Pythagorean expression and are not perfect squares. They are all prime numbers of type $\text{odd}^2+\text{even}^2$. For example, $5 = 1^2+2^2$ and $13 = 2^2+3^2$.

2. Those expressed by only one positive Pythagorean expression and are perfect squares. They are all composite numbers of type $\text{odd}^2+\text{even}^2$. For example: $5^2 = 3^2+4^2$ and $13^2 = 5^2+12^2$.

3. Those being perfect squares but not expressible as a sum or a difference of squares. They are all composite numbers of type odd^2. For example, 3^2 and 31^2.

4. Those expressed by one o several positive Pythagorean expressions, accompanied by one or several negative Pythagorean expressions. They are all composite numbers of type $\text{odd}^2\pm\text{even}^2$. For example: $45 = 3^2+6^2 = 7^2-2^2 = 9^2-6^2$ and $65 = 1^2+8^2 = 4^2+7^2 = 9^2-4^2$.

5. Those expressed by one or several negative Pythagorean expressions. They are all composite numbers of type $\text{odd}^2-\text{even}^2$. For example: $33 = 7^2-4^2$ and $105 = 11^2-4^2 = 13^2-8^2 = 19^2-16^2$.

Pythagorean triples

All Pythagorean expressions appearing in the table are connected by Pythagorean triples of type $(a^2-b^2, 2ab, a^2+b^2)$.

Since a and b have different parity, ab is even and $2ab$ is of type $4k$ (multiple of 4). That is to say, the middle term of a Pythagorean triple is between $4k$–1 and $4k$+1, the two branches of odd numbers. For example: in (3, 4, 5) it is 4×1; in (15, 8, 17) it is 4×2; in (5, 12, 13) it is 4×3; etc.

Pythagorean triples of the table are of two types:

1. Horizontal.
   In the branch $4k$–1, the negative Pythagorean expressions are all of type $n_1 = a^2–b^2$, with a even and b odd, and the dual of n_1 is $n_2 = a^2+b^2$ in the branch $4k$+1. The values of a are: 2, 4, 6, 8, 10, 12, etc. The values of b are a–3, a–5, a–7, a–9, etc., i.e., $b = a–k$ (or $k = a–b$), with k = 3, 5, 7, 9, …

   There are two ways of looking at these infinite Pythagorian expressions: setting k or setting b. If k is set, we have the Pythagorean triples

   $(a^2–b^2, 2ab, a^2+b^2)$ with $b = a–k$, with a = 4, 6, 8, 10,…

   And if b is set, we have the expressions

   $(a^2–b^2, 2ab, a^2+b^2)$ with a = 4, 6, 8, 10,…

   Example with $b = 1$ ($k = a–b$):

n_1	n_2
$3 = 2^2–1^2$ (prime)	$5 = 2^2+1^2$ (prime)
$15 = 4^2–1^2$ (composite)	$17 = 4^2+1^2$ (prime)
$35 = 6^2–1^2$ (composite)	$37 = 6^2+1^2$ (prime)
$63 = 8^2–1^2$ (composite)	$65 = 8^2+1^2$ (composite)
$99 = 10^2–1^2$ (composite)	$101 = 10^2+1^2$ (prime)

   This example corresponds precisely to the case of twin numbers (those differing in 2).

   Example with $k = 3$ ($b = a–k$):

n_1	n_2
$15 = 4^2–1^2$ (composite)	$17 = 4^2+1^2$ (prime)
$27 = 6^2–3^2$ (composite)	$45 = 6^2+3^2$ (composite)
$39 = 8^2–5^2$ (composite)	$89 = 8^2+5^2$ (prime)
$51 = 10^2–7^2$ (composite)	$149 = 10^2+7^2$ (prime)
$63 = 12^2–9^2$ (composite)	$223 = 12^2+9^2$ (prime)

2. Vertical.
   In the branch $4k$+1, the dual Pythagorean expressions are $n_1 = a^2–b^2$, with a odd and b even, and $n_2 = a^2+b^2$ in the own branch $4k$+1. The posible values of a are: 3, 5, 7, 9, 11, 13, etc. The values of b are a–3, a–5, a–7, a–9, etc., i.e., $b = a–k$ (or $k = a–b$), with k = 3, 5, 7, 9, …

   Like the former case, there are two ways of looking at these infinite Pythagorean expressions: setting k or setting b. If k is set, we have the expressions

$$(a^2–b^2, 2ab, a^2+b^2) \text{ with } b = a–k, \text{ with } a = 5, 7, 9, 11,\ldots$$

And if b is set, we have the expressions

$$(a^2–b^2, 2ab, a^2+b^2) \text{ with } a = 5, 7, 9, 11,\ldots$$

Example with $b = 2$ ($k = a–b$):

n_1	n_2
$5 = 3^2–2^2$ (prime)	$13 = 3^2+2^2$ (prime)
$21 = 5^2–2^2$ (composite)	$29 = 5^2+2^2$ (prime)
$45 = 7^2–2^2$ (composite)	$53 = 7^2+2^2$ (prime)
$77 = 9^2–2^2$ (composite)	$85 = 9^2+2^2$ (composite)
$117 = 11^2–2^2$ (composite)	$125 = 11^2+2^2$ (composite)
$165 = 13^2–2^2$ (composite)	$173 = 13^2+2^2$ (prime)

Example with k = 3 (b = a–k):

n_1	n_2
$21 = 5^2–2^2$ (composite)	$29 = 5^2+2^2$ (prime)
$33 = 7^2–4^2$ (composite)	$65 = 7^2+4^2$ (composite)
$48 = 9^2–6^2$ (composite)	$117 = 9^2+6^2$ (composite)
$117 = 11^2–8^2$ (composite)	$185 =11^2+8^2$ (composite)
$69 = 13^2–10^2$ (composite)	$269 = 13^2+10^2$ (prime)

Traditional System vs. Pythagorean System

With the help of the sum of squares and its dual (the difference of squares), i.e., considering the positive and negative Pythagorean expressions, we have obtained a conceptual frame based on the principles of simplicity, duality and fractality. If we compare the traditional system (T), based on the product of prime numbers, with the Pythagorean system (P), based on the sum and difference of squares, we have the following differences and analogies:

- Structure of fundamental or primary numbers.
  T: They are the prime numbers, in a disordered structure, without known pattern: (2, 3, 5, 7, 11, …).
  P: They are the squares of natural numbers, in an ordered structure, with a simple pattern: ($1^2, 2^2, 3^2, 4^2, 5^2,\ldots$).

- Operations.
  T: Multiplication and division.
  P: Sum and difference of squares. They are more simple operations than multiplication and division. They supply a richer and flexible frame in which it is possible to discover

more easily the properties of numbers (prime and composite ones).

- Fundamental theorem.
  T: Fundamental theorem of arithmetic: every natural number has a unique decomposition in prime factors.
  P: The fundamental theorem is the Fermat´s Christmas theorem and its dual.

- Description of prime numbers.
  T: Prime numbers are the numbers that can only be divided by 1 or itself.
  P: The description is Pythagorean. A prime number of type $4k+1$ is only expressible as a unique sum of squares. A prime number of type $4k-1$ is inexpressible as sum or as difference of squares. It is only expressible as a difference between the squares of two consecutive numbers, like all odd numbers.

- Composite numbers.
  T: A composite number is a unique expression as product of prime numbers.
  P: A composite number may have multiple expressions, as several sums of squares and/or as one or several differences of squares.

  In general, the greater the composite number, the greater is the number of Pythagorean expressions for representing it. This implies a much more flexible frame than the traditional system.

- Number 1.
  T: The number 1 is not considered prime. It does not play any role. It is the neutral element of multiplication: $n\times 1 = n$.
  P: It is an essential number as a component of Pythagorean expressions of type a^2-1^2 and a^2+1^2. It is the only number whose square is equal to itself: $1^2 = 1$.

- The singularity of 2.
  T: Number 2 is the only even prime number.
  P: Number 2 is the only number that is the sum of two equal squares: $2 = 1^2+1^2$. It is not a gaussian prime because $2 = (1+i)(1-i)$.

- Trivial elements.
  T: There are two trivial divisors: the number itself (n) and the unit (1).
  P: The difference between the squares of two consecutive numbers.

- Elements of the primality test.
  T: The non-trivial divisors.
  P: The Pythagorean expressions of sum and difference of squares.

- Primality test
  T: The descriptive level implies the operative one. The traditional description of prime numbers –a prime number is only divisible by 1 and itself– provides a method to test if a number n is prime. The proper divisors of n are searched. If they are not found, then n is prime. Otherwise, it is composite.

P: The Pythagorian description also provides a method to test if a number n (odd) is prime:

1. If n is of type $4k+1$ and it is not a perfect square, the equivalent expressions of n that are sum of squares are searched. If there is only one, n is prime. Otherwise, n is composite.
2. If n is of type $4k–1$, the equivalent expressions of n that are difference of squares are searched. If there is none, n is prime. Otherwise, n is composite.

- Factorization.
  T: It is of linear type, a method of LH consciousness. To know if a number is prime, it suffices to divide it by the prime numbers less than its square root. The reason for this is because the factors operate in pairs. If a number has a factor $>\sqrt{n}$, it has also another factor $<\sqrt{n}$. For example, $n = 35 = 5\times 7$, $5<\sqrt{35}$ and $7>\sqrt{35}$.

  P: It is of fractal descendent type, a method of RH consciosness. There are two cases:

  1. If the number n is odd, two possible prime factors are searched, that correspond to a difference of squares: $n = a^2–b^2 = (a+b)(a–b)$, where a and b have different parity. The same procedure is applied to each of these factors (which are also odd), and so on. It is dual search, in the sense of that two numbers are searched at the same time in every step.

  2. If the number n is even, we convert it to the form $2^m n_1$, where $m\geq 1$ and n_1 is odd. Then, we apply the same former procedure to n_1.

Summary and conclusions

There are two categories of odd prime numbers:

1) Those that can be expressed of a unique form as sum of squares and they are no perfect squares. They are of type $4k+1$. This property corresponds to the Fermat´s Christmas theorem.

2) Those that cannot be expressed as a sum or as a difference of squares (excluding the trivial difference of squares of two consecutive numbers). They are of type $4k–1$. This property corresponds to dual of the Fermat´s Christmas theorem.

Prime numbers do not have an algebraic pattern, but a qualitative pattern based on these two fundamental properties.

There are several reasons for not considering primes of type $4k+1$ as real prime numbers:

1. They are not gaussian primes.

2. They are expressible as sum of two composite numbers (a^2 and b^2), which is conceptually a contradiction. Prime numbers should not be expressible, because the rest

of numbers (the composite numbers) are built from them.

3. They have the same form than the composite numbers where sums of squares appear.

4. There is an evident asymmetry between the branches $4k{-}1$ and $4k{+}1$, since the branch $4k{+}1$ is more "populated" and is more complex than the $4k{-}1$ one:

   a) From of the point of view of the Pythagorean expressions, there are 2 types of numbers in the branch $4k{-}1$, and 5 ones in the branch $4k{+}1$.
   b) There are only positive Pythagorean expressions in the branch $4k{+}1$.
   c) There are more negative Pythagorean expressions in the branch $4k{+}1$ than in the branch $4k{-}1$.
   d) There are Pythagorean triples in the branch $4k{+}1$, while there are no one in the branch $4k{-}1$.

   Since there are more manifestations in the branch $4k{+}1$, this does suggest that it is in a lower level than the branch $4k{-}1$, i,e., the branch $4k{-}1$ is more fundamental than the $4k{+}1$ one.

Therefore, if we consider the prime numbers of type $4k{+}1$ as composite numbers, only the primes of type $4k{-}1$ are left, whose property is to be inexpressible. They trascend the opposites (sum and difference of squares), standing in a deeper level. This is in line with the philosophy that from the surface level it is not possible to access the deep one.

In conclusion, there is nothing mysterious or strange or complex in the subject of prime numbers. On the contrary, it is something fundamentally simple due to the close relation with the Pythagoras' theorem. The key to understanding prime numbers lies in the sum/difference of squares, i.e., the dual forms of the Pythagoras' theorem. Pythagoras's theorem is the most fundamental theorem of mathematics. Pythagoras's theorem is a theorem of consciousness, the Holy Grail of mathematics.

Appendix

Table of Pythagorean expressions of the odd numbers

Nº A	Type	Expression and its dual	Nº B	Type	Expression and its dual
3	Prime		**5**	Prime	1^2+2^2 **3** (A)
7	Prime		**9**	3^2	
11	Prime		**13**	Prime	2^2+3^2 **5** (B)
15	3*5	4^2-1^2 **17** (B)	**17**	Prime	1^2+4^2 **15** (A)
19	Prime		**21**	3*7	5^2-2^2 **29** (B)

23	Prime		**25**	5^2	3^2+4^2 **7** (A)
27	3^3	$6^2–3^2$ **45** (B)	**29**	Prime	2^2+5^2 **21** (B)
31	Prime		**33**	3*11	$7^2–4^2$ **65** (B)
35	5*7	$6^2–1^2$ **37** (B)	**37**	Prime	1^2+6^2 **35** (A)
39	3*13	$8^2–5^2$ **89** (B)	**41**	Prime	4^2+5^2 **9** (B)
43	Prime		**45**	3^2*5	3^2+6^2 **27** (A) $7^2–2^2$ **53** (B) $9^2–6^2$ **117** (B)
47	Prime		**49**	7^2	
51	3*17	$10^2–7^2$ **149** (B)	**53**	Prime	2^2+7^2 **45** (B)
55	5*11	$8^2–3^2$ **73** (B)	**57**	3*19	$11^2–8^2$ **185** (B)
59	Prime		**61**	Prime	5^2+6^2 **11** (A)
63	3^2*7	$8^2–1^2$ **65** (B) $12^2–9^2$ **225** (B)	**65**	5*13	1^2+8^2 **63** (A) 4^2+7^2 **33** (B) $9^2–4^2$ **97** (B)
67	Prime		**69**	3*23	$13^2–10^2$ **269** (B)
71	Prime		**73**	Prime	3^2+8^2 **55** (A)
75	3*5^2	$10^2–5^2$ **125** (B) $14^2–11^2$ **317** (B)	**77**	7*11	$9^2–2^2$ **85** (B)
79	Prime		**81**	3^4	$15^2–12^2$ **369** (B)
83	Prime		**85**	5*17	2^2+9^2 **77** (B) 6^2+7^2 **13** (B) $11^2–6^2$ **157** (B)
87	3*29	$16^2–13^2$ **425** (B)	**89**	Prime	5^2+8^2 **39** (A)
91	7*13	$10^2–3^2$ **109** (B)	**93**	3*31	$17^2–14^2$ **485** (B)
95	5*19	$12^2–7^2$ **193** (B)	**97**	Prime	4^2+9^2 **65** (B)
99	3^2*11	$10^2–1^2$ **101** (B) $18^2–15^2$ **549** (B)	**101**	Prime	1^2+10^2 **99** (A)

103	Prime		**105**	3*5*7	$11^2–4^2$ **137** (B) $13^2–8^2$ **233** (B) $19^2–16^2$ **617** (B)
107	Prime		**109**	Prime	3^2+10^2 **91** (A)
111	3*37	$20^2–17^2$ **689** (B)	**113**	Prime	7^2+8^2 **15** (A)
115	5*23	$14^2–9^2$ **277** (B)	**117**	3^2*13	6^2+9^2 **45** (B) $11^2–2^2$ **125** (B) $21^2–18^2$ **765** (B)
119	7*17	$12^2–5^2$ **169** (B)	**121**	11^2	
123	3*41	$22^2–19^2$ **845** (B)	**125**	5^3	2^2+11^2 **117** (B) 5^2+10^2 **75** (A) $15^2–10^2$ **325** (B)
127	Prime		**129**	3*43	$23^2–20^2$ **929** (B)
131	Prime		**133**	7*19	$13^2–6^2$ **205** (B)
135	3^3*5	$12^2–3^2$ **153** (B) $16^2–11^2$ **377** (B) $24^2–21^2$ **1017** (B)	**137**	Prime	4^2+11^2 **105** (B)
139	Prime		**141**	3*47	$25^2–22^2$ **1109** (B)
143	11*13	$12^2–1^2$ **145** (B)	**145**	5*29	1^2+12^2 **143** (A) 8^2+9^2 **17** (B) $17^2–12^2$ **433** (B)
147	$3*7^2$	$14^2–7^2$ **245** (B) $26^2–23^2$ **1205** (B)	**149**	Prime	7^2+10^2 **51** (A)
151	Prime		**153**	3^2*17	3^2+12^2 **135** (A) $13^2–4^2$ **185** (B) $27^2–24^2$ **1305** (B)
155	5*31	$18^2–13^2$ **493** (B)	**157**	Prime	6^2+11^2 **85** (B)
159	3*53	$28^2–25^2$ **1409** (B)	**161**	7*23	$15^2–8^2$ **289** (B)
163	Prime		**165**	3*5*11	$13^2–2^2$ **173** (B) $19^2–14^2$ **557** (B)

					$29^2–26^2$ **1517** (B)
167	Prime		**169**	13^2	5^2+12^2 **119** (A)
171	3^2*19	$14^2–5^2$ **221** (B) $30^2–27^2$ **1629** (B)	**173**	Prime	2^2+13^2 **165** (B)
175	5^2*7	$16^2–9^2$ **337** (B) $20^2–15^2$ **625** (B)	**177**	3*59	$31^2–28^2$ **1745** (B)
179	Prime		**181**	Prime	9^2+10^2 **19** (A)
183	3*61	$32^2–29^2$ **1865** (B)	**185**	5*37	4^2+13^2 **153** (B) 8^2+11^2 **57** (B) $21^2–16^2$ **697** (B)
187	11*17	$14^2–3^2$ **205** (B)	**189**	3^3*7	$15^2–6^2$ **261** (B) $17^2–10^2$ **389** (B) $33^2–30^2$ **1989** (B)
191	Prime		**193**	Prime	7^2+12^2 **95** (A)
195	3*5*13	$14^2–1^2$ **197** (B) $22^2–17^2$ **773** (B) $34^2–31^2$ **2117** (B)	**197**	Prime	1^2+14^2 **195** (A)
199	Prime		**201**	3*67	$35^2–32^2$ **2249** (B)
203	7*29	$18^2–11^2$ **445** (B)	**205**	5*41	3^2+14^2 **187** (A) 6^2+13^2 **133** (B) $23^2–18^2$ **853** (B)
207	3^2*23	$16^2–7^2$ **305** (B) $36^2–33^2$ **2385** (B)	**209**	11*19	$15^2–4^2$ **241** (B)
211	Prime		**213**	3*71	$37^2–34^2$ **2525** (B)
215	5*43	$24^2–19^2$ **937** (B)	**217**	7*31	$19^2–12^2$ **505** (B)
219	3*73	$38^2–35^2$ **2669** (B)	**221**	13*17	5^2+14^2 **171** (A) 10^2+11^2 **21** (B) $15^2–2^2$ **229** (B)
223	Prime		**225**	3^2*5^2	9^2+12^2 **63** (A) $17^2–8^2$ **353** (B)

					$25^2–20^2$ **1025** (B) $39^2–36^2$ **2817** (B)
227	Prime		**229**	Prime	2^2+15^2 **221** (B)
231	3*7*11	$16^2–5^2$ **281** (B) $20^2–13^2$ **569** (B) $40^2–37^2$ **2969** (B)	**233**	Prime	8^2+13^2 **105** (B)
235	5*47	$26^2–21^2$ **1117** (B)	**237**	3*79	$41^2–38^2$ **3125** (B)
239	Prime		**241**	Prime	4^2+15^2 **209** (B)
243	3^5	$18^2–9^2$ **405** (B) $42^2–39^2$ **3285** (B)	**245**	$5*7^2$	7^2+14^2 **147** (A) $21^2–14^2$ **637** (B) $27^2–22^2$ **1213** (B)
247	13*19	$16^2–3^2$ **265** (B)	**249**	3*83	$43^2–40^2$ **3449** (B)
251	Prime		**253**	11*23	$17^2–6^2$ **325** (B)
255	3*5*17	$16^2–1^2$ **257** (B) $28^2–23^2$ **1313** (B) $44^2–41^2$ **3617** (B)	**257**	Prime	1^2+16^2 **255** (A)
259	7*37	$22^2–15^2$ **709** (B)	**261**	3^2*29	6^2+15^2 **189** (B) $19^2–10^2$ **461** (B) $45^2–42^2$ **3789** (B)
263	Prime		**265**	5*53	3^2+16^2 **247** (A) 11^2+12^2 **23** (A) $29^2–24^2$ **1417** (B)
267	3*89	$46^2–43^2$ **3965** (B)	**269**	Prime	10^2+13^2 **69** (B)
271	Prime		**273**	3*7*13	$17^2–4^2$ **305** (B) $23^2–16^2$ **785** (B) $47^2–44^2$ **4145** (B)
275	5^2*11	$18^2–7^2$ **373** (B) $30^2–25^2$ **1525** (B)	**277**	Prime	9^2+14^2 **115** (A)
279	3^2*31	$20^2–11^2$ **521** (B) $48^2–45^2$ **4329** (B)	**281**	Prime	5^2+16^2 **231** (A)

283	Prime		**285**	3*5*19	$17^2–2^2$ **293** (B); $31^2–26^2$ **1637** (B); $49^2–46^2$ **4517** (B)
287	7*41	$24^2–17^2$ **865** (B)	**289**	17^2	8^2+15^2 **161** (B)
291	3*97	$50^2–47^2$ **4709** (B)	**293**	Prime	2^2+17^2 **285** (B)
295	5*59	$32^2–27^2$ **1753** (B)	**297**	3^3*11	$19^2–8^2$ **425** (B); $21^2–12^2$ **585** (B); $51^2–48^2$ **4905** (B)
299	13*23	$18^2–5^2$ **349** (B)	**301**	7*43	$25^2–18^2$ **949** (B)
303	3*101	$52^2–49^2$ **5105** (B)	**305**	5*61	4^2+17^2 **273** (B); 7^2+16^2 **207** (A); $33^2–28^2$ **1873** (B)
307	Prime		**309**	3*103	$53^2–50^2$ **5309** (B)
311	Prime		**313**	Prime	12^2+13^2 **25** (B)
315	3^2*5*7	$18^2–3^2$ **333** (B); $22^2–13^2$ **653** (B); $26^2–19^2$ **1037** (B); $34^2–29^2$ **1997** (B); $54^2–51^2$ **5517** (B)	**317**	Prime	11^2+14^2 **75** (A)
319	11*29	$20^2–9^2$ **481** (B)	**321**	3*107	$55^2–52^2$ **5729** (B)
323	17*19	$18^2–1^2$ **325** (B)	**325**	5^2*13	1^2+18^2 **323** (A); 6^2+17^2 **253** (B); 10^2+15^2 **125** (B); $19^2–6^2$ **397** (B); $35^2–30^2$ **2125** (B)
327	3*109	$56^2–53^2$ **5945** (B)	**329**	7*47	$27^2–20^2$ **1129** (B)
331	Prime		**333**	3^2*37	3^2+18^2 **315** (A); $23^2–14^2$ **725** (B); $57^2–54^2$ **6165** (B)
335	5*67	$36^2–31^2$ **2257** (B)	**337**	Prime	9^2+16^2 **175** (A)

339	3*113	$58^2–55^2$ **6389** (B)	**341**	11*31	$21^2–10^2$ **541** (B)
343	7^3	$28^2–21^2$ **1225** (B)	**345**	3*5*23	$19^2–4^2$ **377** (B) $37^2–32^2$ **2393** (B) $59^2–56^2$ **6617** (B)
347	Prime		**349**	Prime	5^2+18^2 **299** (A)
351	3^3*13	$20^2–7^2$ **449** (B) $24^2–15^2$ **801** (B) $60^2–57^2$ **6849** (B)	**353**	Prime	8^2+17^2 **225** (B)
355	5*71	$38^2–33^2$ **2533** (B)	**357**	3*7*17	$19^2–2^2$ **365** (B) $29^2–22^2$ **1325** (B) $61^2–58^2$ **7085** (B)
359	Prime		**361**	19^2	
363	3*11^2	$22^2–11^2$ **605** (B) $62^2–59^2$ **7325** (B)	**365**	5*73	2^2+19^2 **357** (B) 13^2+14^2 **27** (A) $39^2–34^2$ **2677** (B)
367	Prime		**369**	3^2*41	12^2+15^2 **81** (B) $25^2–16^2$ **881** (B) $63^2–60^2$ **7569** (B)
371	7*53	$30^2–23^2$ **1429** (B)	**373**	Prime	7^2+18^2 **275** (A)
375	3*5^3	$20^2–5^2$ **425** (B) $40^2–35^2$ **2825** (B) $64^2–61^2$ **7817** (B)	**377**	13*29	4^2+19^2 **345** (B) 11^2+16^2 **135** (A) $21^2–8^2$ **505** (B)
379	Prime		**381**	3*127	$65^2–62^2$ **8069** (B)
383	Prime		**385**	5*7*11	$23^2–12^2$ **673** (B) $31^2–24^2$ **1537** (B) $41^2–36^2$ **2977** (B)
387	3^2*43	$26^2–17^2$ **965** (B) $66^2–63^2$ **8325** (B)	**389**	Prime	10^2+17^2 **189** (B)
391	17*23	$20^2–3^2$ **409** (B)	**393**	3*131	$67^2–64^2$ **8585** (B)
395	5*79	$42^2–37^2$ **3133** (B)	**397**	Prime	6^2+19^2 **325** (B)

399	3*7*19	$20^2–1^2$ **401** (B) $32^2–25^2$ **1649** (B) $68^2–65^2$ **8849** (B)	**401**	Prime	1^2+20^2 **399** (A)
403	13*31	$22^2–9^2$ **565** (B)	**405**	3^4*5	9^2+18^2 **243** (A) $21^2–6^2$ **477** (B) $27^2–18^2$ **1053** (B) $43^2–38^2$ **3293** (B) $69^2–66^2$ **9117** (B)
407	11*37	$24^2–13^2$ **745** (B)	**409**	Prime	3^2+20^2 **391** (A)
411	3*137	$70^2–67^2$ **9389** (B)	**413**	7*59	$33^2–26^2$ **1765** (B)
415	5*83	$44^2–39^2$ **3457** (B)	**417**	3*139	$71^2–68^2$ **9665** (B)
419	Prime		**421**	Prime	14^2+15^2 **29** (B)
423	3^2*47	$28^2–19^2$ **1145** (B) $72^2–69^2$ **9945** (B)	**425**	5^2*17	5^2+20^2 **375** (A) 8^2+19^2 **297** (B) 13^2+16^2 **87** (A) $21^2–4^2$ **457** (B) $45^2–40^2$ **3625** (B)
427	7*61	$34^2–27^2$ **1885** (B)	**429**	3*11*13	$23^2–10^2$ **629** (B) $25^2–14^2$ **821** (B) $73^2–70^2$ **10229** (B)
431	Prime		**433**	Prime	12^2+17^2 **145** (B)
435	3*5*29	$22^2–7^2$ **533** (B) $46^2–41^2$ **3797** (B) $74^2–71^2$ **10517** (B)	**437**	19*23	$21^2–2^2$ **445** (B)
439	Prime		**441**	3^2*7^2	$29^2–20^2$ **1241** (B) $35^2–28^2$ **2009** (B) $75^2–72^2$ **10809** (B)
443	Prime		**445**	5*89	2^2+21^2 **437** (B) 11^2+18^2 **203** (A) $47^2–42^2$ **3973** (B)
447	3*149	$76^2–73^2$ **11105** (B)	**449**	Prime	7^2+20^2 **351** (A)

451	11*41	$26^2–15^2$ **901** (B)	**453**	3*151	$77^2–74^2$ **11405** (B)
455	5*7*13	$24^2–11^2$ **697** (B) $36^2–29^2$ **2137** (B) $48^2–43^2$ **4153** (B)	**457**	Prime	4^2+21^2 **425** (B)
459	3^3*17	$22^2–5^2$ **509** (B) $30^2–21^2$ **1341** (B) $78^2–75^2$ **11709** (B)	**461**	Prime	10^2+19^2 **261** (B)
463	Prime		**465**	3*5*31	$23^2–8^2$ **593** (B) $49^2–44^2$ **4337** (B) $79^2–76^2$ **12017** (B)
467	Prime		**469**	7*67	$37^2–30^2$ **2269** (B)
471	3*157	$80^2–77^2$ **12329** (B)	**473**	11*43	$27^2–16^2$ **985** (B)
475	5^2*19	$22^2–3^2$ **493** (B) $50^2–45^2$ **4525** (B)	**477**	3^2*53	6^2+21^2 **405** (B) $31^2–22^2$ **1445** (B) $81^2–78^2$ **12645** (B)
479	Prime		**481**	13*37	9^2+20^2 **319** (A) 15^2+16^2 **31** (A) $25^2–12^2$ **769** (B)
483	3*7*23	$22^2–1^2$ **485** (B) $38^2–31^2$ **2405** (B) $82^2–79^2$ **12965** (B)	**485**	5*97	1^2+22^2 **483** (A) 14^2+17^2 **93** (B) $51^2–46^2$ **4717** (B)
487	Prime		**489**	3*163	$83^2–80^2$ **13289** (B)
491	Prime		**493**	17*29	3^2+22^2 **475** (A) 13^2+18^2 **155** (A) $23^2–6^2$ **565** (B)
495	3^2*5*11	$24^2–9^2$ **657** (B) $28^2–17^2$ **1073** (B) $32^2–23^2$ **1553** (B) $52^2–47^2$ **4913** (B) $84^2–81^2$ **13617** (B)	**497**	7*71	$39^2–32^2$ **2545** (B)
499	Prime		**501**	3*167	$85^2–82^2$ **13949** (B)

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