# THE GENERALIZED CONTINUUM HYPOTHESIS

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In this paper we prove the Generalized Continuum Hypothesis, which is, as Universal Algebra and Category Theory, a complete theory, proving that the infinite initial ordinals and the transfinite cardinals are isomorphic universal algebras.

For isomorphic universal algebras are isomorphic categories, isomorphic categories are isomorphic theories, by the Fundamental Theorem of Categorical Logic, and so, isomorphic universal algebras are isomorphic theories.

**Theorem "Generalized Continuum Hypothesis"** *For every transfinite cardinal $\alpha$, there is no cardinal between $\alpha$ and $2^\alpha$.*

**Proof.** Let **Card** be the class of transfinite cardinals and let **Ord** be class of infinite initial ordinals. Since **Card** and **Ord** are structures of second-order logic, both are categories, namely, well-ordered semirings, as every structure of a formal language is a category, which is the foundation of Categorical Logic.

We prove that **Card** and **Ord** are isomorphic universal algebras, acted upon by the exponential functor, proving there exists a full and faithful functor $T\colon$ **Card**$\to$**Ord** such that each infinite initial ordinal $\beta$ is isomorphic to an infinite initial ordinal $T\alpha$ for some transfinite cardinal $\alpha$.

Let $T\colon$ **Card**$\to$**Ord** be the function of categories that assigns to each transfinite cardinal $\alpha$ the infinite initial ordinal $T\alpha$ of its equipotence class and to every arrow $f\colon \alpha \to \alpha'$ in **Card** the arrow $Tf\colon T\alpha \to T\alpha'$ in **Ord** for every dyad of transfinite cardinals $\alpha$ and $\alpha'$. The function of categories $T$ is well-defined, because is well-defined on objects, for each transfinite cardinal $\alpha$ lies in a unique equipotence class, and because is well-defined on arrows, for each arrow $f$ in **Card** is a unique dyad of objects $\alpha$ and $\alpha'$ in **Card**, by the axioms of Category Theory, to each dyad of transfinite cardinals $\alpha$ and $\alpha'$ there is a unique dyad of infinite initial ordinals $T\alpha$ and $T\alpha'$, for $T$ is a well-defined function of categories on objects, and each dyad of infinite initial ordinals $T\alpha$ and $T\alpha'$ is a unique arrow $Tf\colon T\alpha \to T\alpha'$, by the axioms of Category Theory and for **Ord** is a preorder.

The function of categories $T$ is a functor because preserves preorders, that is, preserves identities and composable dyads of arrows, or, $T1_\alpha = 1_{T\alpha}$ and $T(f \circ g) = Tf \circ Tg$ for every identity $1_\alpha$ and for every composable dyad of arrows $f$ and $g$ in **Card**. For each identity $1_\alpha$ in **Card** is a transfinite cardinal $\alpha$ and every infinite initial ordinal $T\alpha$ is an identity $1_{T\alpha}$ in **Ord**, by the axioms of Category Theory. And for $T(f \circ g)\colon T\alpha \to T\alpha''$ is an arrow in **Ord** for each arrow $f \circ g\colon \alpha \to \alpha''$ in **Card**, because $T$ is a function of categories, each arrow $f \circ g\colon \alpha \to \alpha''$ in **Card** is a dyad of composable arrows $f\colon \alpha \to \alpha'$ and $g\colon \alpha' \to \alpha''$ in **Card**, by the axioms of Category Theory, each dyad of composable arrows $f\colon \alpha \to \alpha'$ and $g\colon \alpha' \to \alpha''$ in **Card** is a triad of transfinite cardinals $\alpha$, $\alpha'$ and $\alpha''$, by the axioms of Category Theory, to each triad of transfinite cardinals $\alpha$, $\alpha'$ and $\alpha''$ there is a triad of infinite initial ordinals $T\alpha$, $T\alpha'$ and $T\alpha''$, for $T$ is a function of categories, each triad of infinite initial ordinals $T\alpha$, $T\alpha'$ and $T\alpha''$ is a dyad of composable arrows $Tf\colon T\alpha \to T\alpha'$ and $Tg\colon T\alpha' \to T\alpha''$, by the axioms of Category Theory, and so, $Tf \circ Tg\colon T\alpha \to T\alpha''$ is a unique arrow in **Ord** for $T\alpha$ and $T\alpha''$, that is, $T(f \circ g) = Tf \circ Tg$, by the axioms of Category Theory and for **Ord** is a preorder.

The functor $T$ is full, because to each dyad of transfinite cardinals $\alpha$ and $\alpha'$ in **Card** and to each arrow $g\colon T\alpha \to T\alpha'$ in **Ord** there is an arrow $f\colon \alpha \to \alpha'$ in **Card** such that $g = Tf$, by definition of $T$. The functor $T$ is faithful, because for each dyad of transfinite cardinals $\alpha$ and $\alpha'$ and for each dyad of arrows $f_1$, $f_2\colon \alpha \to \alpha'$ in **Card** the equality $Tf_1 = Tf_2$ implies $f_1 = f_2$, for $T$ is a function of categories on preorders. And each infinite initial ordinal $\beta$ is isomorphic to the infinite initial ordinal $T|\beta|$, because is isomorphic to the transfinite cardinal $|\beta|$ and every transfinite cardinal $|\beta|$ is isomorphic to the infinite initial ordinal $T|\beta|$, by definition of $T$. Hence **Card** $\cong$ **Ord**.

Therefore, since the theories of isomorphic categories are isomorphic, by the Fundamental Theorem of Categorical Logic, the theories of **Card** and **Ord** are isomorphic, and so, since there is no infinite initial ordinal between $\omega$ and $\omega^\omega$, by the Theorem on the Cardinality of Ordinals, and $\aleph_0^{\aleph_0} = 2^{\aleph_0}$, by the Fundamental Theorem of Transfinite Cardinal Arithmetic, there exists no transfinite cardinal between the transfinite cardinals $\aleph_0$ and $2^{\aleph_0}$, henceforth there exists no transfinite cardinal between any transfinite cardinals $\alpha$ and $2^\alpha$. Consequently, there exist no inaccessible cardinals. In fact, the category **Card** of transfinite cardinals is isomorphic to $\omega$, because the function $f$ of $\omega$ to **Card** which assigns to each ordinal $\alpha \in \omega$ the $\alpha$-th transfinite cardinal $f\alpha$ is an order-preserving isomorphism, which is unique by transfinite construction. ■

## In Categorical Logic

In Categorical Logic, the theorem not only proves that the theories of the universal algebras of the transfinite cardinals and the infinite initial ordinals are isomorphic, but also that higuer-order theories have cardinal greater than or equal to the cardinal of the continuum.

## In Universal Algebra

In Universal Algebra, the theorem not only proves that the large universal algebra of transfinite cardinals is nondiscrete closed complete and cocomplete, with arrows the polynomial maps and the exponential maps, but also that is acted upon by the covariant exponential functor universal algebra.

This paper is dedicated to my brothers Sergio Cordero Grau and Pablo Cordero Grau