A Nonabelian Gauge Theory of Gravitation

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Abstract

The aim of the paper is to develop a gauge theory, which shall be on the one hand as similar as possible to the original ansatz of Einstein's theory of general relativity, and on the other hand in agreement with other gauge theories as, for instance, those of the electroweak or of the strong interaction. The result is a nonabelian gauge theory with the general linear group $GL(4, \mathbb{R})$ as its gauge group.

1. What is the problem?

There are three fundamental forces in nature. Each of them has its own theory by which it is described. These three theories are:

(a) the nonabelian gauge theory of strong interaction

(b) the nonabelian gauge theory of electro weak interaction

(c) the 'allgemeine Relativitätstheorie'

If the third theory is compared with the two other ones, there are considerable differences between them. The theory of general relativity is stemming from the year 1916 and hence an elder lady of almost one hundred years, while the other two theories are of recent date. On the one hand the theory of general relativity is not quantized and is treating space as a dynamical background, while on the other hand the two mentioned gauge theories are quantized, but are acting before a rigid background. These differences are a severe problem for the task to develop a quantum theory of gravitation, a program that shall not be discussed here (cf. f.i. [4] [5]).

The aim of the present paper is to construct a gauge theory of gravitation, which is satisfying two conditions: On the one hand it shall be similar to the theory of general relativity as far as possible, and on the other hand fulfil the standards of modern gauge theories, also as far as possible.

In the next section it is investigated, to what extent the theory of general relativity already has the shape of a gauge theory. As a result it can be shown that part of the theory of general relativity can be considered as a nonabelian gauge theory with the general linear group $GL(4,\mathbb{R})$ as its gauge group. The rest of the paper is oriented towards modern gauge theories. The ansatz in section 4 is in direct correspondence with the gauge theories of strong and electroweak interaction, and rather suited for studying gravitation in subatomic regions, while the ansatz of section 5 is still more general and perhaps useful for astrophysical investigations

2. What part of the 'allgemeine Relativitätstheorie' has already the form of a gauge theory?

The element of general relativity most similar to a gauge theory is the formula

$$R^{\varrho}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\varrho}_{\sigma\nu} - \partial_\nu \Gamma^{\varrho}_{\sigma\mu} + \Gamma^{\varrho}_{\tau\mu}\Gamma^{\tau}_{\sigma\nu} - \Gamma^{\varrho}_{\tau\nu}\Gamma^{\tau}_{\sigma\mu} \tag{1}$$

Especially the last two terms are reminding of a commutator, like the last two terms in the formula

$$F^{a}_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} A^b_\mu A^c_\nu \tag{2}$$

being valid in nonabelian gauge theories.

In order to investigate, whether this resemblance is only superficial or more profound, one should write equation (1) as

$$R^{\varrho}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\varrho}_{\sigma\nu} - \partial_\nu \Gamma^{\varrho}_{\sigma\mu} + K^{\varrho}_{\sigma\mu\nu} \tag{3}$$

with the quadratic part

$$K^{\varrho}_{\sigma\mu\nu} = \Gamma^{\varrho}_{\tau\mu}\Gamma^{\tau}_{\sigma\nu} - \Gamma^{\varrho}_{\tau\nu}\Gamma^{\tau}_{\sigma\mu} \tag{4}$$

being separated. By interchanging the two factors on the right hand side of equation (4) one can see that the right hand side of equation (4) is antisymmetric under the interchange of the indices μ and ν.

$$K^{\varrho}_{\sigma\nu\mu} = -\Gamma^{\varrho}_{\tau\mu}\Gamma^{\tau}_{\sigma\nu} + \Gamma^{\tau}_{\sigma\mu}\Gamma^{\varrho}_{\tau\nu} = -K^{\varrho}_{\sigma\mu\nu} \tag{5}$$

Hence, if one is trying to write this expression as

$$K^{\varrho}_{\sigma\mu\nu} = k^{\varrho\alpha\beta}_{\sigma\gamma\delta}\Gamma^{\alpha}_{\gamma\mu}\Gamma^{\beta}_{\delta\nu} \tag{6}$$

with certain constants k, then one has to choose these such that

$$k^{\varrho\varrho\tau}_{\sigma\tau\sigma} = +1 \qquad k^{\varrho\tau\varrho}_{\tau\sigma\sigma} = -1 \qquad k^{\varrho\alpha\beta}_{\sigma\gamma\delta} = 0 \quad \text{else} \tag{7}$$

The next calculation will reveal that the constants k are the same numbers as the structure constants h for the Liealgebra $gl(4,\mathbb{R})$ of the general linear group $GL(4,\mathbb{R})$.

The Liealgebra $gl(4,\mathbb{R})$ of $GL(4,\mathbb{R})$ can be spanned by the 16 generators

$$(s^{\varrho}_{\sigma}) = (\delta^{\varrho}_{\sigma}) \qquad \text{row } \varrho \quad \text{column } \sigma \qquad 1 \le \varrho, \sigma \le 4 \tag{8}$$

with the Kronecker symbol $\delta^{\varrho}_{\sigma}$. Determining the structure constants by calculating the commutators

$$[\left(s^{\alpha}_{\gamma}\right), \left(s^{\beta}_{\delta}\right)] = h^{\varrho\alpha\beta}_{\sigma\gamma\delta} \cdot (s^{\varrho}_{\sigma}) \tag{9}$$

one first of all will find that

$$(s^{\varrho}_{\tau}) \cdot (s^{\tau}_{\sigma}) = \delta^{\alpha}_{\varrho} \cdot \delta^{\beta}_{\gamma} \cdot \delta^{\delta}_{\sigma} \cdot (s^{\varrho}_{\sigma}) \qquad (s^{\tau}_{\sigma}) \cdot (s^{\varrho}_{\tau}) = \delta^{\alpha}_{\delta} \cdot \delta^{\beta}_{\varrho} \cdot \delta^{\gamma}_{\sigma} \cdot (s^{\varrho}_{\sigma})$$

This implies

$$h^{\varrho\varrho\tau}_{\sigma\tau\sigma} = +1 \qquad h^{\varrho\tau\varrho}_{\tau\sigma\sigma} = -1 \qquad kh^{\varrho\alpha\beta}_{\sigma\gamma\delta} = 0 \quad \text{else} \tag{10}$$

Plugging this into the formula

$$R^{\varrho}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\varrho}_{\sigma\nu} - \partial_\nu \Gamma^{\varrho}_{\sigma\mu} + h^{\varrho\alpha\beta}_{\sigma\gamma\delta} \Gamma^{\alpha}_{\gamma\mu} \Gamma^{\beta}_{\delta\nu} \tag{11}$$

for the tensor of curvature, one will find that (10) is identical with (7) and hence (11) with (1).

Result of this section:

A part of Einstein's general relativity can be written as a nonabelian gauge theory for the gauge group $GL(4,\mathbb{R})$.

3. The next part of the construction

There are two obstacles standing against the intention to continue the construction of a gauge theory:

(a) In equation (1) there is no coupling constant. But such a constant is needed for the quadratic term in nonabelian gauge theories.

(b) Instead of the two indices ϱ and σ there is only one variable a in a gauge theory.

In order to remove the first fault let be

$$\Gamma^{\varrho}_{\sigma\mu} = gA^{\varrho}_{\sigma\mu} \qquad R^{\varrho}_{\sigma\mu\nu} = gF^{\varrho}_{\sigma\mu\nu} \tag{12}$$

Under the condition that the coupling constant g doesn't vanish, one will have

$$F^{\varrho}_{\sigma\mu\nu} = \partial_\mu A^{\varrho}_{\sigma\nu} - \partial_\nu A^{\varrho}_{\sigma\mu} + gh^{\alpha\beta\varrho}_{\gamma\delta\sigma} A^{\alpha}_{\gamma\mu} A^{\beta}_{\delta\nu}$$

after having divided all parts of the equation by g. After the further substitution

$$A^{\varrho}_{\sigma\mu} = A^{a}_{\mu} \qquad A^{\alpha}_{\gamma\mu} = A^{b}_{\mu} \qquad A^{\beta}_{\delta\nu} = A^{c}_{\nu} \qquad F^{\varrho}_{\sigma\mu\nu} = F^{a}_{\mu\nu} \qquad h^{\varrho\alpha\beta}_{\sigma\gamma\delta} = h_{abc} \tag{13}$$

the second deficiency will be removed, too. The result is

$$F^{a}_{\mu\nu} = \partial_\mu A^{a}_{\nu} - \partial_\nu A^{a}_{\mu} + gh_{abc} A^{b}_{\mu} A^{c}_{\nu} \tag{14}$$

As a sake of simplicity and with the intention to write down some explicit results the rest of the section is reduced to one spatial dimension. In this case the Lie algebra $gl(2,\mathbb{R})$ has four generators

$$s^1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad s^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad s^3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad s^4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{15}$$

The commutators

$$[s^b, s^c] = h_{abc} \cdot s^a \tag{16}$$

are listed in the table

$[s^a, s^b]$	s^1	s^2	s^3	s^4
s^1	0	$+s^2$	$-s^3$	0
s^2	$-s^2$	0	$s^1 - s^4$	$+s^2$
s^3	$+s^3$	$s^4 - s^1$	0	$-s^3$
s^4	0	$-s^2$	$+s^3$	0

The structure constants are

$$h_{122} = h^{111}_{122} = +1 \qquad h_{212} = h^{111}_{212} = -1$$

$$h_{231} = h^{121}_{211} = +1 \qquad h_{234} = h^{122}_{212} = -1$$

$$h_{242} = h^{121}_{222} = +1 \qquad h_{422} = h^{211}_{222} = -1$$

$$h_{313} = h^{212}_{111} = +1 \qquad h_{133} = h^{122}_{111} = -1$$

$$h_{324} = h^{212}_{122} = +1 \qquad h_{321} = h^{211}_{121} = -1$$

$$h_{433} = h^{222}_{211} = +1 \qquad h_{343} = h^{222}_{121} = -1$$

4. A microscopic ansatz

In this section the ansatz has great resemblance with the other nonabelian theories.

It is starting with the homogeneous Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \tag{17}$$

In order to be in accordance with the usual notation the generators s^a are altered into $t^a = is^a$. Then

$$[t^b, t^c] = ih_{abc}t^a$$

The covariant derivative is

$$D_\mu = \partial_\mu + igA^a_\mu \tag{18}$$

Equation (17) then changes into

$$(i\gamma^\mu D_\mu - m)\psi = 0 \tag{19}$$

The tensor of the field strength is

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gh_{abc}A^b_\mu A^c_\nu \tag{20}$$

Variation of the Lagrangian density

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F^a_{\mu\nu}F^{a,\mu\nu} \tag{21}$$

with respect to the three fields inherent in it will give a system

$$\begin{aligned} i\gamma^\mu \partial_\mu \psi(x) - m\psi(x) &= g\gamma^\mu A^a_\mu(x) t^a \psi(x) \\ -i\partial_\mu \bar{\psi}(x)\gamma^\mu - m\bar{\psi}(x) &= g\bar{\psi}(x) t^a \gamma^\mu A^a_\mu(x) \\ \partial^\mu F^a_{\mu\nu}(x) &= g\bar{\psi}(x)\gamma_\nu t^a \psi(x) + g h_{abc} A^{b,\mu}(x) F^c_{\mu\nu}(x) \end{aligned} \tag{22}$$

of three coupled equations. If additionally the Lorentz-t'Hooft convention

$$\partial^\mu A^a_\mu = 0 \tag{23}$$

is taken as a gauge fixing condition, then one has

$$\partial^\mu(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) = \partial^\mu \partial_\mu A^a_\nu - \partial_\nu \partial^\mu A^a_\mu = \Box A^a_\nu$$

and

$$\partial^\mu(A^b_\mu A^c_\nu) = (\partial^\mu A^b_\mu) A^c_\nu + A^b_\mu \partial^\mu A^c_\nu = A^b_\mu \partial^\mu A^c_\nu$$

and hence the counterpart

$$\Box A^a_\nu(x) = g j^a_\nu(x) + g h_{abc}(A^b_\mu \partial^\mu A^c_\nu + A^{b,\mu}(x) F^c_{\mu\nu}(x)) \qquad j^a_\nu(x) = \bar{\psi}(x)\gamma_\nu s^a \psi(x) \tag{24}$$

of the inhomogeneous wave equation.

The three equations (22) in connection with (20) and (24) form a system of coupled field equations. It might be considered as a useful description of gravity in microscopic dimensions, especially because there are well known propagator functions for the left hand side of the equations. Hence perturbation theory is possible.

5. A macroscopic ansatz

The result of the last section is not suited to describe gravitation in the magnitude of cosmic dimensions, because the objects of the universe surely are no solutions of the Dirac equation. But one can take the last equation of (22) without the restriction that the material current density j^a_ν shall be of as the special form in (22). From

$$\partial^\nu \partial^\mu F^a_{\mu\nu} = -\partial^\mu \partial^\nu F^a_{\nu\mu} = -\partial^\nu \partial^\mu F^a_{\mu\nu} \tag{25}$$

one can conclude that

$$\partial^\nu \partial^\mu F^a_{\mu\nu} = 0 \tag{26}$$

By a similar argumentation, that is to say, by interchanging the variables μ und ν, making use of the symmetry of $F^a_{\mu\nu}$, and finally a reverse change of the variables, one will get

$$\partial^\nu (A^{b,\mu} F^a_{\mu\nu}) = 0 \tag{27}$$

and hence

$$\partial^\nu j^a_\nu = 0 \tag{28}$$

Thus the material current density is also conserved, even if it is not of the special form in (22). If again the Lorentz-t'Hooft-konvention is chosen as a gauge fixing condition, one will get a system

$$\begin{aligned} \Box A^a_\nu(x) &= g j^a_\nu(x) + g h_{abc}(A^b_\mu \partial^\mu A^c_\nu + A^b_\mu(x) F^c_{\mu\nu}(x)) \\ F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g h_{abc} A^b_\mu A^c_\nu \end{aligned} \tag{29}$$

of two coupled equations. In case of lacking matter the system still has its legitimation. Then it is describing the self interaction of the gravitational field.

The transformation of the matter density under gauge transformations shall be determined such, that the first equation of (29) is invariant under it.

That is the first part of the interaction between matter and the gravitational field. The other one is given by the rule that a particle, may it be consistent of matter or may it be a photon, is moving in the gravitational field along a geodesic line

$$\ddot{y}^\varrho = A^\varrho_{\mu\nu}(x) \dot{y}^\mu \dot{y}^\nu \tag{30}$$

In this point the ansatz is following Einstein in that it is making use of the concept of geodesic line and also in treating matter and light on the same footing. The result of the construction can be considered as a nonabelian extension of classical electrodynamics, and simultaneously as a refinement of the Newtonian law of gravity. It might be suited for the description of gravity in macroscopic and especially in cosmic dimensions.

References

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