On Existence of Infinitely Many Primes of the Form $x^2+1$

Pingyuan Zhou

E-mail: zhoupingyuan49@hotmail.com

Abstract

It is well known that there are infinitely many prime factors of Fermat numbers, because prime factor of a Fermat prime is the Fermat prime itself but a composite Fermat number has at least two prime factors and Fermat numbers are pairwise relatively prime. Hence we conjecture that there is at least one prime factor $(k^{1/2}2^{a/2})^2+1$ of Fermat number for $F_n−1 ≤ a < F_{n+1}−1$ ($n = 0,1,2,3,...$), where $k^{1/2}$ is odd positive integer , $a$ is even positive integer and $F_n$ is Fermat number. The conjecture holds till $a < F_{4+1}−1 = 4294967296$ from known evidences. Two corollaries of the conjecture imply existence of infinitely many primes of the form $x^2+1$, which is one of four basic problems about primes mentioned by Landau at ICM 1912.

Keywords: Fermat number; prime factor $(k^{1/2}2^{a/2})^2+1$ of Fermat number; primes of the form $x^2+1$.

2010 Mathematics Subject Classification: 11A41
Are there infinitely many primes of the form $x^2+1$? It has been an unsolved problem in mathematics. Landau listed it as one of four basic problems about primes at ICM 1912[1]. H. Iwaniec showed that there are infinitely many numbers of the form $n^2+1$ with at most two prime factors in 1987[2]. A theorem proved in 1997 by J. Friedlander and H. Iwaniec shows that there are infinitely many primes of the form $x^2+y^4$[3]. However, such two results do not imply that there is an infinite number of primes of the form $x^2+1$ so that above problem mentioned by Landau is still an unsolved problem and we try to consider the problem by the way related to prime factors of Fermat numbers.

**Definition.** Prime factor of a Fermat prime is the Fermat prime itself.

**Lemma.** There are infinitely many prime factors of Fermat numbers.

**Proof.** We see prime factor of a Fermat prime is the Fermat prime itself by above definition but a composite Fermat number has at least two prime factors by definition of composite number. Since Fermat numbers are pairwise relatively prime by Goldbach’s theorem ( no two Fermat numbers share a common factor )[4] and Fermat numbers are infinite. Hence there are infinitely many prime factors of Fermat numbers.

Basing on above lemma, we have the following conjecture.
**Conjecture.** There is at least one prime factor \((k^{1/2} \cdot 2^{a/2})^2 + 1\) of Fermat number for \(F_n - 1 \leq a < F_{n+1} - 1\) ( \(n = 0, 1, 2, 3, \ldots\) ), where \(k^{1/2}\) is odd positive integer, \(a\) is even positive integer and \(F_n\) is Fermat number.

Some of known evidences support the conjecture to be true till \(a < F_{4+1} - 1 = 4294967296\).

**Fact 1.** There exists one prime factor \((1^{1/2} \cdot 2^{2/2})^2 + 1 = 5\) of Fermat number \(F_1\) to be prime of the form \((k^{1/2} \cdot 2^{a/2})^2 + 1\) for \(F_0 - 1 \leq a < F_0 + 1 - 1\) i.e. \(2 \leq a < 4\).

**Fact 2.** There exist two prime factors \((1^{1/2} \cdot 2^{4/2})^2 + 1 = 17\) and \((1^{1/2} \cdot 2^{8/2})^2 + 1 = 257\) of Fermat numbers \(F_2, F_3\) to be primes of the form \((k^{1/2} \cdot 2^{a/2})^2 + 1\) for \(F_1 - 1 \leq a < F_1 + 1 - 1\) i.e. \(4 \leq a < 16\).

**Fact 3.** There exists one prime factor \((1^{1/2} \cdot 2^{16/2})^2 + 1 = 65537\) of Fermat number \(F_4\) to be prime of the form \((k^{1/2} \cdot 2^{a/2})^2 + 1\) for \(F_2 - 1 \leq a < F_2 + 1 - 1\) i.e. \(16 \leq a < 256\).

**Fact 4.** There exists one prime factor \(169 \cdot 2^{63686} + 1 = (13 \cdot 2^{31843})^2 + 1\) of Fermat number \(F_{63679}\) to be prime of the form \((k^{1/2} \cdot 2^{a/2})^2 + 1\) for \(F_3 - 1 \leq a < F_3 + 1 - 1\) i.e. \(256 \leq a < 65536\) (it was discovered by H. Dubner on 19 May 1998[5]).

**Fact 5.** There exists one prime factor \(25 \cdot 2^{2141884} + 1 = (5 \cdot 2^{1070942})^2 + 1\) of Fermat
number $F_{2141872}$ to be prime of the form $(k^{1/2} \cdot 2^{a/2})^2 + 1$ for $F_4 - 1 \leq a < F_{4+1} - 1$ i.e. $65536 \leq a < 4294967296$ (it was discovered by G. Granowski on 9 Sep 2011[5]).

From the conjecture we obtain its two corollaries.

**Corollary 1.** If the conjecture is true, then there are infinitely many prime factors $(k^{1/2} \cdot 2^{a/2})^2 + 1$ of Fermat numbers.

**Proof.** Take $n \to \infty$ in the conjecture then we will get the result.

**Corollary 2.** If Corollary 1 is true, then there are infinitely many primes of the form $x^2 + 1$.

**Proof.** Since prime factors $(k^{1/2} \cdot 2^{a/2})^2 + 1$ of Fermat numbers are a special case of primes of the form $x^2 + 1$, we will get the result.

From it we see the conjecture and its two corollaries may imply whether there are infinitely many primes of the form $x^2 + 1$, but the conjecture is one to be proved. The conjecture was published at the Global Journal of Pure and Applied Mathematics in 2012[6](the GJPAM is abstracted and indexed in the Mathematical Reviews, MathSciNet, Zentralblat MATH and EBSCO databases), so above discussion should be thought a note of the published paper, which emphasizes to give a proof of existence of infinitely many prime factors of Fermat numbers as lemma of the conjecture.
References

   http://en.wikipedia.org/wiki/Landau%27s_problems


   http://en.wikipedia.org/wiki/Fermat_number

   http://www.prothsearch.net/fermat.html

[6] Pingyuan Zhou, On the Existence of Infinitely Many Primes of the Form $x^2+1$, 