On Existence of Infinitely Many Primes of the Form x^2+1

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Abstract

It is well known that there are infinitely many prime factors of Fermat numbers, because prime factor of a Fermat prime is the Fermat prime itself but a composite Fermat number has at least two prime factors and Fermat numbers are pairwise relatively prime. Hence we conjecture that there is at least one prime factor $(k^{1/2}\,2^{a/2})^2+1$ of Fermat number for $F_n-1 \le a < F_{n+1}-1$ (n = 0,1,2,3,…), where $k^{1/2}$ is odd positive integer , a is even positive integer and F_n is Fermat number. The conjecture holds till $a < F_{4+1} - 1 = 4294967296$ from known evidences. Two corollaries of the conjecture imply existence of infinitely many primes of the form x^2+1, which is one of four basic problems about primes mentioned by Landau at ICM 1912.

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Are there infinitely many primes of the form x^2+1? It has been an unsolved problem in mathematics. Landau listed it as one of four basic problems about primes at ICM 1912[1]. H. Iwaniec showed that there are infinitely many numbers of the form n^2+1 with at most two prime factors in 1987[2]. A theorem proved in 1997 by J. Friedlander and H. Iwaniec shows that there are infinitely many primes of the form x^2+y^4[3]. However, such two results do not imply that there is an infinite number of primes of the form x^2+1 so that above problem mentioned by Landau is still an unsolved problem and we try to consider the problem by the way related to prime factors of Fermat numbers.

Definition. Prime factor of a Fermat prime is the Fermat prime itself.

Lemma. There are infinitely many prime factors of Fermat numbers.

Proof. We see prime factor of a Fermat prime is the Fermat prime itself by above definition but a composite Fermat number has at least two prime factors by definition of composite number. Since Fermat numbers are pairwise relatively prime by Goldbach's theorem (no two Fermat numbers share a common factor)[4] and Fermat numbers are infinite. Hence there are infinitely many prime factors of Fermat numbers.

Basing on above lemma, we have the following conjecture.

Conjecture. There is at least one prime factor $(k^{1/2}\,2^{a/2})^2+1$ of Fermat number for $F_n-1 \le a < F_{n+1}-1$ ($n = 0,1,2,3,\ldots$), where $k^{1/2}$ is odd positive integer , a is even positive integer and F_n is Fermat number.

Some of known evidences support the conjecture to be true till $a < F_{4+1} - 1 = 4294967296$.

Fact 1. There exists one prime factor $(1^{1/2}\,2^{2/2})^2+1=5$ of Fermat number F_1 to be prime of the form $(k^{1/2}\,2^{a/2})^2+1$ for $F_0-1 \le a < F_{0+1}-1$ i.e. $2 \le a < 4$.

Fact 2. There exist two prime factors $(1^{1/2}\,2^{4/2})^2+1=17$ and $(1^{1/2}\,2^{8/2})^2+1=257$ of Fermat numbers F_2, F_3 to be primes of the form $(k^{1/2}\,2^{a/2})^2+1$ for $F_1-1 \le a < F_{1+1}-1$ i.e. $4 \le a < 16$.

Fact 3. There exists one prime factor $(1^{1/2}\,2^{16/2})^2+1=65537$ of Fermat number F_4 to be prime of the form $(k^{1/2}\,2^{a/2})^2+1$ for $F_2-1 \le a < F_{2+1}-1$ i.e. $16 \le a < 256$.

Fact 4. There exists one prime factor $169\,2^{63686}+1=(13\,2^{31843})^2+1$ of Fermat number F_{63679} to be prime of the form $(k^{1/2}\,2^{a/2})^2+1$ for $F_3-1 \le a < F_{3+1}-1$ i.e. $256 \le a < 65536$ (it was discovered by H. Dubner on 19 May 1998[5]).

Fact 5. There exists one prime factor $25\,2^{2141884}+1=(5\,2^{1070942})^2+1$ of Fermat number $F_{2141872}$ to be prime of the form $(k^{1/2}\,2^{a/2})^2+1$ for $F_4-1\leq a<F_{4+1}-1$ i.e. $65536\leq a<4294967296$ (it was discovered by G. Granowski on 9 Sep 2011[5]).

From the conjecture we obtain its two corollaries.

Corollary 1. If the conjecture is true, then there are infinitely many prime factors $(k^{1/2}\,2^{a/2})^2+1$ of Fermat numbers.

Proof. Take $n\to\infty$ in the conjecture then we will get the result.

Corollary 2. If Corollary 1 is true, then there are infinitely many primes of the form x^2+1.

Proof. Since prime factors $(k^{1/2}\,2^{a/2})^2+1$ of Fermat numbers are a special case of primes of the form x^2+1, we will get the result.

From it we see the conjecture and its two corollaries may imply whether there are infinitely many primes of the form x^2+1, but the conjecture is one to be proved. The conjecture was published at the Global Journal of Pure and Applied Mathematics in 2012[6](the GJPAM is abstracted and indexed in the Mathematical Reviews, MathSciNet, Zentralblat MATH and EBSCO databases), so above discussion should be thought a note of the published paper, which emphasizes to give a proof of existence of infinitely many prime factors of Fermat numbers as lemma of the conjecture.

References

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