

On the Composite Terms in Sequence Generated from Mersenne-type Recurrence Relations

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Abstract

We conjecture that there is at least one composite term in sequence generated from Mersenne-type recurrence relations. Hence we may expect that all terms are composite besides the first few continuous prime terms in Catalan-Mersenne number sequence and composite Mersenne numbers with exponents restricted to prime values are infinite.

Keywords : Mersenne-type recurrence relations , generalized Catalan-Mersenne number sequence , Catalan-Mersenne number sequence, composite Mersenne number

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1. Introduction

After Lucas had discovered the primality of $M_{127} = 2^{127} - 1$ in 1876[1], Catalan came up with an infinite sequence $2, M_2, MM_2, MMM_2, MMMM_2, MMMMM_2, \dots$, which are generated from recurrence relations and called Catalan-Mersenne numbers. Are these numbers all prime? It has been an unsolved problem[2,3,4,5], because in Catalan-Mersenne number sequence the first five terms $2, M_2, M_3, M_7, M_{127}$ are all prime but the sixth term MM_{127} is much too large for any currently known primality test to be practical so that it has not been known if the number is prime, and it is also very difficult to prove the number to be prime or to find any factor of the number. If a composite term appears in Catalan-Mersenne number sequence then all of the following terms are composite so that infinitely many composite terms will appear in the sequence, which means there are no any prime terms behind a composite term in Catalan-Mersenne number sequence. Although we have not known which term is a composite term among terms $MM_{127}, MMM_{127}, MMMM_{127}, \dots$ in Catalan-Mersenne number sequence, if it is able to be presented that there exists a composite term in Catalan-Mersenne number sequence then Catalan-Mersenne number sequence will become a new example of Guy's strong law of small numbers[5,6]. A general and easy accepted conjecture may be helpful for getting an elementary result on the problem and we will study the problem by the way so that we may consider another unsolved problem on existence of infinitely many composite Mersenne numbers with exponents restricted to prime values.

2. Mersenne-type recurrence relations

In this paper we only consider Mersenne numbers whose exponents are prime numbers.

Definition 2.1 If x is a prime number then 2^x-1 is called a Mersenne number.

Definition 2.2 Let x be integer greater than 1 and n be natural number then the recurrence relations to be able to be written $X(n+1) = 2^{X(n)}-1$ with $X(0) = x$ for $n \geq 0$ are called Mersenne-type recurrence relations.

Case 2.1 Mersenne-type recurrence relations for x being a composite number.

Lemma 2.1 If x is a composite number then 2^x-1 is a composite number.

Proof It has been proved that if 2^x-1 is prime then x is prime[1]. Then if x is composite then 2^x-1 is composite.

Theorem 2.1 If x is a composite number, then all terms are composite in the infinite sequence generated from Mersenne-type recurrence relations $X(n+1) = 2^{X(n)}-1$ with $X(0) = x$ for $n \geq 0$.

Proof Since every term $X(n)$ is exponent of next term $X(n+1)$ and the first term x

is a composite number in the infinite sequence. Then by Lemma 2.1 all terms are composite in the sequence.

Case 2.2 Mersenne-type recurrence relations for x being exponent of composite Mersenne number.

Theorem 2.2 If x is exponent of composite Mersenne number, then all terms are composite besides the first term in the infinite sequence generated from Mersenne-type recurrence relations $X(n+1) = 2^{X(n)} - 1$ with $X(0) = x$ for $n \geq 0$.

Proof Since the first term x is prime but the second term $2^x - 1$ is composite in any such infinite sequence. Then by Lemma 2.1 the second term and all of the following terms are composite in any such infinite sequence.

Case 2.3 Mersenne-type recurrence relations for x being exponent of Mersenne prime.

Example 2.1 For $x = 5$, Mersenne-type recurrence relations $X(n+1) = 2^{X(n)} - 1$ with $X(0) = 5$ for $n \geq 0$ generate an infinite sequence: $5, M_5, MM_5, MMM_5, \dots$

Proposition 2.1 All terms are composite besides the first three terms in the infinite sequence generated from Mersenne-type recurrence relations $X(n+1) = 2^{X(n)} - 1$ with $X(0) = 5$ for $n \geq 0$.

Proof In the infinite sequence, the first three terms $5, M_5, M_{31}$ are all prime but the fourth term MM_{31} is known composite number (its four factors have been found[7]). Then by Lemma 2.1 MM_{31} and all of the following terms are composite in the sequence.

Example 2.2 For $x = 13$, Mersenne-type recurrence relations $X(n+1) = 2^{X(n)} - 1$ with $X(0) = 13$ for $n \geq 0$ generate an infinite sequence: $13, M_{13}, MM_{13}, MMM_{13}, \dots$

Proposition 2.2 All terms are composite besides the first two terms in the infinite sequence generated from Mersenne-type recurrence relations $X(n+1) = 2^{X(n)} - 1$ with $X(0) = 13$ for $n \geq 0$.

Proof In the infinite sequence, the first two terms $13, M_{13}$ are all prime but the third term MM_{13} is known composite number (its two factors have been found[7]). Then by Lemma 2.1 MM_{13} and all of the following terms are composite in the sequence.

Example 2.3 For $x = 17$, Mersenne-type recurrence relations $X(n+1) = 2^{X(n)} - 1$ with $X(0) = 17$ for $n \geq 0$ generate an infinite sequence: $17, M_{17}, MM_{17}, MMM_{17}, \dots$

Proposition 2.3 All terms are composite besides the first two terms in the infinite sequence generated from Mersenne-type recurrence relations $X(n+1) = 2^{X(n)} - 1$ with

$X(0) = 17$ for $n \geq 0$.

Proof In the infinite sequence, the first two terms 17, M_{17} are all prime but the third term MM_{17} is known composite number (its one factor has been found[7]). Then by Lemma 2.1 MM_{17} and all of the following terms are composite in the sequence.

Example 2.4 For $x = 19$, Mersenne-type recurrence relations $X(n+1) = 2^{X(n)}-1$ with $X(0) = 19$ for $n \geq 0$ generate an infinite sequence: 19, M_{19} , MM_{19} , MMM_{19}, \dots

Proposition 2.4 All terms are composite besides the first two terms in the infinite sequence generated from Mersenne-type recurrence relations $X(n+1) = 2^{X(n)}-1$ with $X(0) = 19$ for $n \geq 0$.

Proof In the infinite sequence, the first two terms 19, M_{19} are all prime but the third term MM_{19} is known composite number (its one factor has been found[7]). Then by Lemma 2.1 MM_{19} and all of the following terms are composite in the sequence.

Example 2.5 For $x = 31$, Mersenne-type recurrence relations $X(n+1) = 2^{X(n)}-1$ with $X(0) = 31$ for $n \geq 0$ generate an infinite sequence: 31, M_{31} , MM_{31} , MMM_{31}, \dots

Proposition 2.5 All terms are composite besides the first two terms in the infinite sequence generated from Mersenne-type recurrence relations $X(n+1) = 2^{X(n)}-1$ with

$X(0) = 31$ for $n \geq 0$.

Proof In the infinite sequence, the first two terms 31, M_{31} are all prime but the third term MM_{31} is known composite number (its four factors have been found[7]). Then by Lemma 2.1 MM_{31} and all of the following terms are composite in the sequence.

Example 2.6 For $x = 2$, Mersenne-type recurrence relations $X(n+1) = 2^{X(n)} - 1$ with $X(0) = 2$ for $n \geq 0$ generate an infinite sequence: 2, M_2 , MM_2 , MMM_2 , $MMMM_2$, $MMMMM_2, \dots$. In the sequence, we have known the first five terms 2, M_2 , M_3 , M_7 , M_{127} are all prime but we do not know whether the sixth term MM_{127} is prime.

Example 2.7 For $x = 3$, Mersenne-type recurrence relations $X(n+1) = 2^{X(n)} - 1$ with $X(0) = 3$ for $n \geq 0$ generate an infinite sequence: 3, M_3 , MM_3 , MMM_3 , $MMMM_3$, $MMMMM_3, \dots$. In the sequence, we have known the first four terms 3, M_3 , M_7 , M_{127} are all prime but we do not know whether the fifth term MM_{127} is prime.

Example 2.8 For $x = 7$, Mersenne-type recurrence relations $X(n+1) = 2^{X(n)} - 1$ with $X(0) = 7$ for $n \geq 0$ generate an infinite sequence: 7, M_7 , MM_7 , MMM_7 , $MMMM_7$, $MMMMM_7, \dots$. In the sequence, we have known the first three terms 7, M_7 , M_{127} are all prime but we do not know whether the fourth term MM_{127} is prime.

Example 2.9 For $x = 61$, Mersenne-type recurrence relations $X(n+1) = 2^{X(n)} - 1$ with

$X(0) = 61$ for $n \geq 0$ generate an infinite sequence: $61, M_{61}, MM_{61}, MMM_{61}, MMMM_{61},$
 $MMMMM_{61}, \dots$ In the sequence, we have known the first two terms $61, M_{61}$ are all
prime but we do not know whether the third term MM_{61} is prime.

Example 2.10 As Example 2.9, for x being exponents of Mersenne primes to be
larger than M_{61} , Mersenne-type recurrence relations $X(n+1) = 2^{X(n)} - 1$ with $X(0) = x$ for
 $n \geq 0$ generate an infinite sequence: $x, M_x, MM_x, MMM_x, MMMM_x, MMMMM_x, \dots$ In
any such sequence, we always know the first two terms x, M_x are all prime but we do
not know whether the third term MM_x is prime.

3. A conjecture and corresponding corollaries

From Theorem 2.1, Theorem 2.2, Proposition 2.1, Proposition 2.2, Proposition 2.3,
Proposition 2.4, Proposition 2.5, Example 2.6, Example 2.7, Example 2.8, Example
2.9 and Example 2.10 (all integers x greater than 1 have been considered in these
studies) we see there are no any Mersenne-type recurrence relations to be able to
generate an infinite sequence in which all terms are prime, therefore, the following
conjecture seems to be acceptable.

Conjecture 3.1 There is at least one composite term in sequence generated from
Mersenne-type recurrence relations.

We see Theorem 2.1, Theorem 2.2, Proposition 2.1, Proposition 2.2, Proposition
2.3, Proposition 2.4 and Proposition 2.5 to support the conjecture to be true.

Definition 3.1 Let x be exponent p of Mersenne prime M_p and n be natural number then corresponding infinite sequence generated from Mersenne-type recurrence relations $X(n+1) = 2^{X(n)} - 1$ with $X(0) = p$ for $n \geq 0$ is called generalized Catalan-Mersenne number sequence.

Because it has been generally believed that Mersenne primes are infinite, there may be infinitely many generalized Catalan-Mersenne number sequences. For $p = 2$, corresponding generalized Catalan-Mersenne number sequence is just Catalan-Mersenne number sequence.

Corollary 3.1 All terms are composite besides the first few continuous prime terms in any generalized Catalan-Mersenne number sequence.

Proof Since any generalized Catalan-Mersenne number sequence is an infinite sequence generated from Mersenne-type recurrence relations. By Conjecture 3.1 there is at least one composite term in the sequence and the first composite term will appear behind the first few continuous prime terms in the sequence. Then by Lemma 2.1 the term p and all of the following terms are composite in the sequence.

By Proposition 2.1, Proposition 2.2, Proposition 2.3, Proposition 2.4 and Proposition 2.5 we have known Corollary 3.1 holds for $p = 5, 13, 17, 19, 31$.

Corollary 3.2 All terms are composite besides the first few continuous prime terms

in Catalan-Mersenne number sequence.

Proof Since Catalan-Mersenne number sequence is a generalized Catalan-Mersenne number sequence generated from Mersenne-type recurrence relations $X(n+1) = 2^{X(n)} - 1$ with $X(0) = 2$ for $n \geq 0$. Then by Corollary 3.1 all terms are composite besides the first few continuous prime terms in Catalan-Mersenne number sequence.

4. Connections with previous results

Definition 4.1 If exponent of a Mersenne number is a Mersenne prime M_p i.e. $MM_p = 2^{M_p} - 1$ then the Mersenne number MM_p is called double Mersenne number.

In our previous work[8] Corollary 1 (Conjecture 1) implies Mersenne primes M_p are infinite and Corollary 2 (Conjecture 1) implies double Mersenne numbers $MM_p = 2^{M_p} - 1$ for $p = 2,3,5,7$ are all prime but every double Mersenne number MM_p is composite for $p > 7$. By Corollary 2 (Conjecture 1) we expected Catalan-Mersenne number $MM_{127} = MMMMM_2$ to be composite since $127 > 7$. If the result is true then the first composite term will be MM_{127} in Catalan-Mersenne number sequence and by Corollary 3.2 we may expect MM_{127} and all of the following terms will be composite but the first few continuous prime terms are $2, M_2, MM_2, MMM_2, MMMM_2$ in Catalan-Mersenne number sequence.

Now we consider another unsolved problem: Are composite Mersenne numbers infinite? Euler showed a theorem: If $k > 1$ and $p = 4k+3$ is prime, then $2p+1$ is prime if and only if $2^p \equiv 1 \pmod{2p+1}$ [5]. It implies that if both $p = 4k+3$ and $2p+1$ are

prime then Mersenne number 2^p-1 is composite because $2p+1$ is a prime factor of 2^p-1 from the theorem, which means if there are infinitely many prime pairs $(p, 2p+1)$ for $p = 4k+3$ then there are infinitely many composite Mersenne numbers. But it has not been a solved problem because the existence of infinitely many prime pairs $(p, 2p+1)$ for $p = 4k+3$ has not been proved. By Corollary 1 (Conjecture 1)[8], Definition 3.1, Corollary 3.1 and Definition 4.1 we may get an elementary result on the problem.

Corollary 4.1 There are infinitely many composite Mersenne numbers.

Proof By Corollary 1 (Conjecture 1)[8] there are infinitely many Mersenne primes M_p . Then by Definition 3.1 there are infinitely many generalized Catalan-Mersenne number sequences and by Definition 4.1 there are infinitely many double Mersenne numbers. By Corollary 3.1 there are infinitely many composite terms and the first composite term is always a composite double Mersenne number but all of the following composite terms will be composite numbers but will not be composite double Mersenne numbers by Definition 4.1 because of their exponents being all composite in any generalized Catalan-Mersenne number sequence. Since every generalized Catalan-Mersenne number sequence has one composite double Mersenne number and there are infinitely many generalized Catalan-Mersenne number sequences. Suppose there is a finite number of generalized Catalan-Mersenne number sequences whose composite double Mersenne numbers are certain same composite

double Mersenne number then there are infinitely many composite double Mersenne numbers arising from generalized Catalan-Mersenne number sequences. Since every double Mersenne number is also a Mersenne number so that every composite double Mersenne number is also a composite Mersenne number. Then there are infinitely many composite Mersenne numbers.

As all generalized Catalan-Mersenne number sequences, it seems to be another such example that Fermat numbers are generated from the recurrence relations $F_{n+1} = (F_n - 1)^2 + 1$ with $F_0 = 3$ for $n \geq 0$ and the first five Fermat numbers are all prime but F_n are all composite for $5 \leq n \leq 32$ and there are no any found new Fermat primes for $n > 32$ [9]. Is F_n composite for all $n > 4$? In our previous work[8] Corollary 1 (Conjecture 2) implies Fermat numbers F_n for $n = 0, 1, 2, 3, 4$ i.e. 3, 5, 17, 257, 65537 are all prime but every Fermat number is composite for $n > 4$ and Corollary 2 (Conjecture 2) implies double Fermat numbers $F_{2^n} = 2^{F_n - 1} + 1$ for $n = 0, 1, 2$ i.e. 5, 17, 65537 are all prime but every double Fermat number is composite for $n > 2$. We also discovered existence of the recurrence relations $F_{2^{n+1}} = (F_{2^n} - 1)^{F_n - 1} + 1$ with $F_{2^0} = 5$ for $n \geq 0$ to be suitable to double Fermat numbers[8], so it seems to be reasonable that Fermat and double Fermat number sequences generated from Fermat-type recurrence relations are similar to generalized Catalan-Mersenne number sequences generated from Mersenne-type recurrence relations in existing of infinitely many continuous composite terms behind the first few continuous prime terms.

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