Why Exponential Disk?

Jin He

Wuhan FutureSpace Scientific Corporation Limited,
Wuhan, Hubei 430074, China
E-mail: mathnob@yahoo.com

Abstract Galaxies demonstrate spectacular structure and ordinary spiral galaxies are simply an exponential disk. A rational structure has at least one net of orthogonal Darwin curves, and the exponential disk has infinite nets. This paper proves that the nets of Darwin curves of exponential disk define an intrinsic vector field in the disk plane. Finally, a proposition is given that the vector field should connect to the phenomenon of constant rotation curves.

keywords: Spiral Galaxy, Rational Structure, Exponential Disk

PACS: 98.52.Nr, 02.60.Lj

1 Introduction

Our world is beautiful is because of its spectacular display of varied kinds of natural structure. Despite decades of concerned investigation, the origin of natural structure remains mysterious. To study the structure whose formation is solely responsive to gravity, we need look into the heaven. The solar system that displays a discrete distribution of many bodies, is in fact a tiny point of the splendid galaxy Milky Way which demonstrates a globally smooth structure composed of billions of stars. To humans' surprise, galaxy structure is much simpler than that on Earth. Relatively independent galaxies are either 3-dimensional (elliptical galaxies) or disk-shaped (spiral galaxies ignoring their central bulges; see Figures 1, 2, and the reference [1]). There are two types of spiral galaxies. A barred galaxy has additional bar structure. A spiral galaxy without bar structure is called an ordinary spiral. The main structure of ordinary spiral galaxies is an axi-symmetric disk, with the stellar density decreasing exponentially along radial direction. It is the so-called exponential disk. I have studied galaxy structure since 2000 (see [2-8]). In this Section I present my old result on the study of exponential disk. New result is presented in the next Section. The final Section is a simple discussion on the new result.

The stellar density ρ for the exponential disk of ordinary spiral galaxies is

$$\rho = \rho_0 \, e^{-c\,r} \tag{1}$$

where $\rho_0(>0)$, $c(>0)$ are constants, $r = \sqrt{x^2+y^2}$, and (x, y) are the Cartesian coordinates in the disk plane with its origin being the disk center. The corresponding logarithmic density is (in a difference of a constant)

$$f = -c\,r \tag{2}$$

We know that spiral galaxies demonstrate more or less the phenomenon of spiral arms. The French scientist Henri Poincaré called them “the big question mark” in heaven [6]. Astronomic observation shows that the arms of ordinary spiral galaxies are golden spirals (at least locally). Could arms be explained by the simple linear function (2)? The answer is yes [3]. The gradient to the function $f(x, y)$ is a vector which points always to the disk center and whose modulus is the constant c. Therefore, if a curve in the plane whose tangent at any point on the curve always makes a constant angle to the radial line crossing the point then the directional directive to $f(x, y)$ along the curve is constant. Such curve must be a golden spiral. Certainly, the directional directive in the perpendicular direction to the curve is also constant along the curve. My galaxy study is based on the result, and a concept of rational structure was initiated [2-4].

A distribution of stars in a plane is not arbitrary. There exists a net of orthogonal curves in the plane. If the matter density on one side of the curve is in constant ratio to the one on the other side then the curve is called a Darwin curve or a proportion curve. Such a structure of material distribution with a net of orthogonal Darwin curves is called a rational structure. The exponential disk of ordinary spiral galaxies is a rational structure which has infinite nets of orthogonal Darwin curves [3, 5],

$$\begin{cases} x = e^{a\lambda+b\mu}\cos(A\lambda + B\mu) = r\cos\theta, \\ y = e^{a\lambda+b\mu}\sin(A\lambda + B\mu) = r\sin\theta \end{cases} \tag{3}$$

where

$$r = e^{a\lambda+b\mu}, \quad \theta = A\lambda + B\mu \tag{4}$$

are polar coordinates, λ, μ are the parameters of the curves, $a(> 0), b(> 0), A(< 0), B(> 0)$ are constants, and the formula

$$A = -\frac{ab}{B} \tag{5}$$

is the necessary and sufficient condition for the net of curves to be orthogonal. The above curves (3) are all golden spirals.

The paper [7] showed that rational structure must satisfy the Riccati equation with constant coefficients. Therefore, there exist only a few solutions to rational structure. Coincidentally, astronomical observation shows that, ignoring those strongly interacted ones, there exist only a few types of galaxies. Surprisingly, the simplest solution to rational structure is the exponential disk. It is likely that the exponential disk is the only rational structure that has more than one net of orthogonal Darwin curves. In addition, the exponential disk has the trivial net of Darwin curves which is composed of the polar coordinate lines in the disk plane.

The next Section is dedicated to the study on the curvature vectors of Darwin curves. We know that a circle has everywhere its curvature equal to the reciprocal of its radius. A smaller circle bends sharply and has larger curvature. The curvature of a straight line is zero. The curvature of a smooth curve is defined as the curvature of its osculating circle at each point. A curvature vector takes into account of its direction of the bend as well as its sharpness. At any point in a rational structure, there cross two Darwin curves at a right angle. Hence there exist a pair of curvature vectors at the point. We calculate their vectorial sum. Since exponential disk has more than one net of orthogonal Darwin

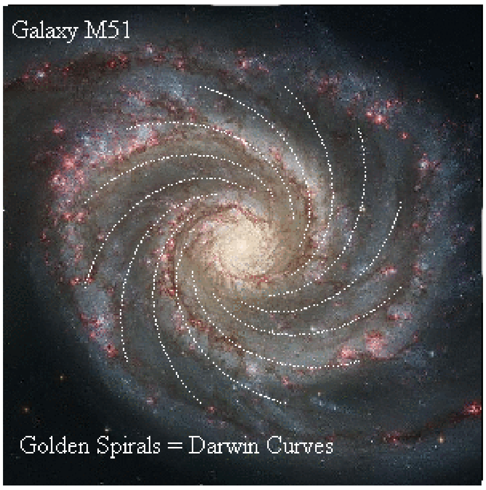

Figure 1: Image of ordinary spiral galaxy M51 with optical light (image credit: Nasa/ESA). Its arms follow the Darwin curves of the rational structure (the exponential disk).

curves, there exist more than one pair of curvature vectors at each point. Now a question comes to us. Does the sum of one pair of curvature vectors equal to the sum of the other pair? A positive answer is shown in the next Section. The sum is called the intrinsic curvature vectorial field of rational structure, or called intrinsic curvature in short. The final Section is dedicated to the discussion that the intrinsic curvature should connect to the phenomenon of constant rotation curves.

2 Summation of Curvature Vectors

The components of the tangent vector to the Darwin curve of parameter λ are

$$\begin{cases} x'_\lambda = e^{a\lambda+b\mu}(a\cos(A\lambda+B\mu) - A\sin(A\lambda+B\mu), \\ y'_\lambda = e^{a\lambda+b\mu}(a\sin(A\lambda+B\mu) + A\cos(A\lambda+B\mu) \end{cases} \tag{6}$$

The squared modulus of the vector is

$$x'^2_\lambda + y'^2_\lambda = (a^2+A^2)e^{2(a\lambda+b\mu)} = (a^2+A^2)r^2 \tag{7}$$

The second derivatives are

$$\begin{cases} x''_{\lambda\lambda} = e^{a\lambda+b\mu}((a^2-A^2)\cos(A\lambda+B\mu) - 2aA\sin(A\lambda+B\mu), \\ y''_{\lambda\lambda} = e^{a\lambda+b\mu}(2aA\cos(A\lambda+B\mu) + (a^2-A^2)\sin(A\lambda+B\mu) \end{cases} \tag{8}$$

To obtain the curvature k_1 of the λ-curve, we calculate

$$\begin{aligned} & |x'_\lambda y''_{\lambda\lambda} - y'_\lambda x''_{\lambda\lambda}| \\ = \; & r^2|(a\cos\theta - A\sin\theta)(2aA\cos\theta + (a^2-A^2)\sin\theta) \\ & -(a\sin\theta + A\cos\theta)((a^2-A^2)\cos\theta - 2aA\sin\theta)| \\ = \; & r^2|2a^2A\cos^2\theta + (A^3-a^2A)\sin^2\theta + (-2aA^2+a^3-aA^2)\cos\theta\sin\theta \\ & +(A^3-a^2A)\cos^2\theta + 2a^2A\sin^2\theta + (aA^2-a^3+2aA^2)\cos\theta\sin\theta| \\ = \; & r^2|(a^2A+A^3)\cos^2\theta + (a^2A+A^3)\sin^2\theta| \\ = \; & (a^2+A^2)|A|r^2 \end{aligned} \tag{9}$$

Finally we have the curvature

$$k_1 = \frac{1}{R_1} = \frac{|x'_\lambda y''_{\lambda\lambda} - y'_\lambda x''_{\lambda\lambda}|}{(x'^2_\lambda + y'^2_\lambda)^{3/2}} = \frac{(a^2 + A^2)|A|}{(a^2 + A^2)^{3/2} r} = \frac{|A|}{\sqrt{a^2 + A^2} r} \tag{10}$$

Similarly we have

$$\begin{array}{l} x'_\mu = e^{a\lambda + b\mu}(b\cos(A\lambda + B\mu) - B\sin(A\lambda + B\mu), \\ y'_\mu = e^{a\lambda + b\mu}(b\sin(A\lambda + B\mu) + B\cos(A\lambda + B\mu), \\ x'^2_\mu + y'^2_\mu = (b^2 + B^2)r^2, \\ |x'_\mu y''_{\mu\mu} - y'_\mu x''_{\mu\mu}| = (b^2 + B^2)Br^2 \end{array} \tag{11}$$

Finally we have the curvature k_2 of the μ-curve

$$k_2 = \frac{1}{R_2} = \frac{|x'_\mu y''_{\mu\mu} - y'_\mu x''_{\mu\mu}|}{(x'^2_\mu + y'^2_\mu)^{3/2}} = \frac{B}{\sqrt{b^2 + B^2} r} \tag{12}$$

The corresponding curvature vectors are calculated as follows

$$\mathbf{k}_1 = \frac{1}{R_1} \frac{(-x'_\mu,\ -y'_\mu)}{\sqrt{x'^2_\mu + y'^2_\mu}} = -|A| \frac{(b\cos\theta - B\sin\theta,\ b\sin\theta + B\cos\theta)}{\sqrt{(a^2 + A^2)(b^2 + B^2)}\, r} \tag{13}$$

$$\mathbf{k}_2 = \frac{1}{R_2} \frac{(-x'_\lambda,\ -y'_\lambda)}{\sqrt{x'^2_\lambda + y'^2_\lambda}} = -B \frac{(a\cos\theta - A\sin\theta,\ a\sin\theta + A\cos\theta)}{\sqrt{(a^2 + A^2)(b^2 + B^2)}\, r} \tag{14}$$

Finally we calculate the vectorial sum of the two vectors

$$\mathbf{k}_1 + \mathbf{k}_2 = \frac{(Ab - Ba)\cos\theta,\ (Ab - Ba)\sin\theta)}{\sqrt{(a^2 + A^2)(b^2 + B^2)}\, r} = -\frac{1}{r}(\cos\theta,\ \sin\theta) \tag{15}$$

Surprisingly we see that the vectorial sum of the pair of vectors are independent of the constants a, b, A, B. This means that the different nets of orthogonal Darwin curves share the same vectorial sum. That is, we have an intrinsic curvature vector field which belongs to the corresponding rational structure itself, independent of the choice of the net of Darwin curves.

3 Discussion

Currently the accepted theory applied to galaxies is Newton's universal gravity which, in actuality, is a theory of two bodies. Furthermore, it is a theory of action at a distance. To study the gravitational field in a galaxy, people employ Poisson's equation which is an differential form of the two-body interaction and the law of action at a distance. This theory of galactic dynamics has been used to predict kinematical phenomena [9] . These predictions do not match galaxy observation generally. The first example was presented by Zwichy [10]. Another well known example is the problem of constant rotation curves. To maintain the status of Newton's theory, people introduced the concept of dark matter which, however, has never been observed directly. Our study is directly on galaxy

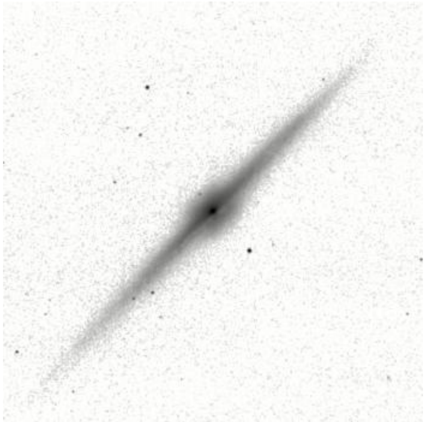

Figure 2: A longer-wavelength image of edge-on spiral galaxy NGC 4565 (Courtesy of Jarrett *et al* [1]). The central bulge is easily recognized.

images, and rational structure was proposed and fitted to galaxy images satisfactorily [2-8]. Rational gravity was proposed which is a generalization to Newton's theory and resulted from the local curvature of Darwin curves [11].

Rational Gravity: Since an intrinsic curvature is proved in the last Section, the new gravitational field, called rational gravity, is assumed which is proportional to the intrinsic curvature vector field.

Although spiral galaxies are flat and considered to be two-dimensional, they still have a certain thickness. We use z to describe the vertical direction to the disk and r to describe the horizontal direction as seen on an edge-on spiral galaxy image (see Figure 2). Astronomical observation shows that the stellar density distribution on an edge-on spiral galaxy image can be described by a formula whose variables z and r can be separated [12],

$$\rho(r, z) = \sigma(r)\,\tau(z) \tag{16}$$

This means that the ratio of stellar density from two sides of each vertical straight line is constant along the line. That is, spiral galaxies considered to be 3-dimensional, are still rational structure. Accordingly, Darwin curves become Darwin surfaces, and curvature vector becomes Gaussian curvature vector. The intrinsic curvature field in 3-dimensional space still exists. The formula (16) tells us that the Darwin surfaces of spiral galaxies are either parallel or perpendicular to the disk plane. Accordingly, the intrinsic curvature vector is 2-dimensional, that is, parallel to the disk plane. It is the parallel gravity that requires constant rotation curves if we ignore the weak contribution from galaxy bulges and halos [8, 11]. We repeat the proof as follows. The rational gravity at the radius r is,

$$F \propto \frac{1}{r} \tag{17}$$

Now we suppose that a star rotates circularly at the same radius. Its acceleration is v^2/r. Therefore,

$$\frac{v^2}{r} \propto \frac{1}{r} \tag{18}$$

Finally, we have proved the constant rotational curves of ordinary spiral galaxies,

$$v = \text{constant} \tag{19}$$

Our proposition needs further investigation and new observational evidences are expected. These are left for the future work.

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