The Quantum Theory of Black Holes

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The present investigation is concerned with the problem of finding a quantum description applicable to all phenomena including black holes. Thus, this paper introduces the universal uncertainty principle which is an extension of Heisenberg uncertainty principle. This extension includes the quantum nature of space-time due to the Planck length while the effects of the zero point momentum/zero point energy are omitted. This formulation predicts the thermodynamics properties -temperature and the entropy- of black holes. The main prediction emerging from this theory is a general equation for the temperature of the black hole. This equation shows a surprising result - the temperature of the black hole depends not only on the mass of the black hole but also on its radius. This theory is applicable to black holes of all sizes; however, the impact of the radius is significant for microscopic black holes only. Finally, this theory predicts that the Berkenstain-Hawking temperature formula is a special case of the more general formulation presented here.

Keywords: *quantum fluctuations, zero point energy, event horizon, entropy, virtual photon, Boltzmann constant, Schwarzchild radius, wave function, wave-packet.*

1. Introduction

The idea that the black hole entropy depends on the area of the event horizon of the black hole was proposed by J. D. Berkenstein in 1972. In 1974 Stephen Hawking discovered an evaporation mechanism [1] by which a black hole can radiate energy into space through the emission of photons originated near the event horizon [2] by the vacuum quantum fluctuations. This along with the Berkenstein-Hawking's formula of the black hole entropy was a great milestone in understanding how these mysterious objects behave. The quantum fluctuations of the vacuum produce pairs of virtual particles and anti-particles around the black hole (in fact these pairs are created everywhere in empty space). If one of these pairs turns out to be two photons (a virtual anti-photon is identical to a virtual photon) and if the pair is close enough to the event horizon of the black hole, one of the virtual photons can be absorbed by the hole and the other one can escape to infinity and become real. This evaporation process is known as *Hawking radiation*. The detection of this radiation is extremely difficult due to the relatively slow rate of emission. This explains why this radiation has not been observed yet. The next section explains the foundation of this theory – *The universal uncertainty principle*.

2. The Universal Uncertainty Principle

The principle I developed to derive the thermodynamic properties of the black hole (temperature and entropy) is an extension of the Heisenberg uncertainty principle [3]
This theory includes two different effects: a) quantum mechanical effects due to the quantum fluctuations of empty space (the energy of empty space is also known as the zero point energy), and b) gravitational effects due to the strong gravitational field surrounding the black hole. To differentiate this principle from other generalized principles I shall call it the *Universal Uncertainty Principle* (UUP). The expression that defines this principle is

$$\Delta p \Delta x \geq \sqrt{\left(\frac{h}{4\pi}\right)^2 - \frac{h}{4\pi}\Delta p L_Z} \qquad (1)$$

(The generalization presented here is justified in **Appendix 1.**)

Where

$\Delta p =$ Uncertainty in the momentum of a particle due to its wave nature (wave-packet representing the particle). This uncertainty does not include the uncertainty P_Z in the momentum due to the quantum fluctuations of space-time.

$\Delta x =$ Uncertainty in the position of the particle due to the wave-packet representing the particle. This uncertainty does not include the uncertainty L_Z in the position due to the quantum fluctuations of space-time.

$L_Z =$ Uncertainty in the position of the particle due to the quantum fluctuations of space-time. This uncertainty does not include the uncertainty Δx due to the wave-packet representing the particle. The minimum value of this uncertainty cannot be measured experimentally with the present technology. Further, it seems logical to assume that this uncertainty is identical to the Planck length L_P. However, these two lengths could be different but the difference should not be significant.

$P_Z =$ Uncertainty in the momentum of a particle due to to the quantum fluctuations of space-time (uncertainty due to the zero point momentum). This uncertainty does not include the uncertainty Δp in the momentum due to the wave nature of the wave-packet representing the particle. We shall neglect the effects of P_Z in this formulation. On way of extending this principle to include the zero point momentum (or zero point energy if the temporal form of the uncertainty principle is used) is to use the Schwinger formulation.

3. Black Hole Temperature

Let's consider a black hole of radius R and mass M. Let's assume that a pair of virtual photons is created near the event horizon due to the quantum fluctuations of empty space. Let's also assume that one of the photons of this pair is absorbed by the black hole while the other one escapes to space and thus becomes real (see Fig 1). The Schwarzchild radius of this black hole is

$$R_S = \frac{2GM}{c^2} \tag{2}$$

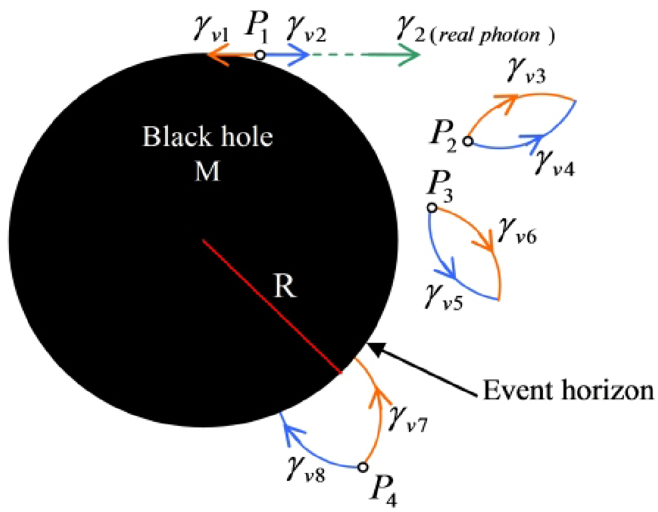

Fig 1: Four pairs of virtual photons are created near a black hole's event horizon due to the quantum fluctuations of vacuum. The points of creation of these pairs are labeled as P_1, P_2, P_3 and P_4. One of the photons (γ_{v1}) of the pair (γ_{v1}, γ_{v2}) is absorbed by the black hole while the other one (γ_{v2}) escapes to space and becomes real ($\gamma_{v2} \to \gamma_2$). This evaporation mechanism was discovered by Hawking and is known as Hawking radiation.

Equation (1) leads to

$$c^2 = \frac{2GM}{R} \tag{3}$$

In order to simplify the equations I shall use R instead of R_S.

Let's multiply both sides of equation (2) by the equivalent mass m_v of the virtual photon that becomes real (from now on I shall call it "the escaping photon")

$$m_v c^2 = \frac{2GMm_v}{R} \tag{4}$$

But according to Einstein

$$E_v = m_v c^2 \tag{5}$$

$$E_v = hf \tag{6}$$

Where E_v is the total relativistic energy of the escaping photon.
From equations (4) , (5) and (6) we have

$$\frac{2GMm_v}{R} = hf \tag{7}$$

Solving this equation for m_v we obtain

$$m_v = \left(\frac{hR}{2GM}\right) f \tag{8}$$

The condition that the black holes imposes on the photon to absorb it is that the dominant wavelength of the photon must be roughly the size of the black hole. This is because if the dominant wavelength of the photon is too short the photon will be absorbed by the quantum fluctuation of the vacuum surrounding the photon before it gets the chance to go through the event horizon of the black hole, and if, on the other hand, the wavelength of the photon is too long, the photon will miss out the black hole entirely. Thus the uncertainty in the position of the absorbed photon, due to its wave nature, is

$$\Delta x = 2\pi R \tag{9}$$

In order to simplify the equations we shall substitute Δx with a , thus

$$a = \Delta x \tag{10}$$

Because the photon's dominant wavelength must be exactly the size of the circumference that results from slicing the black hole along any great circle, the momentum will be related to the photon's dominant wavelength through the universal uncertainty principle and not through the De Broglie relationship (you can develop an identical theory from the Heisenberg uncertainty principle $\Delta p \Delta x \geq h/4\pi$ to see that the De Broglie relationship $p_v = h/2\pi R$ will not produce the correct expression of the black hole temperature. The correct relationship is $p_v = h/8\pi^2 R$ which is derived from the Heisenberg uncertainty principle. In this formulation we use a more general expression - equation 12 - instead). Therefore we shall substitute Δp with p_v , thus

$$\Delta p = p_v \tag{11}$$

Substituting into equation (1) then gives

$$p_v a \geq \sqrt{\left(\frac{h}{4\pi}\right)^2 - \frac{h}{4\pi} p_v L_Z} \tag{12}$$

If we square both sides we have

$$p_v^2 a^2 \geq \left(\frac{h}{4\pi}\right)^2 - \frac{h}{4\pi} p_v L_Z \tag{13}$$

Rearranging the terms and substituting the inequality with an equality sign lead us to a second degree equation

$$a^2 p_v{}^2 + \frac{hL_Z}{4\pi} p_v - \left(\frac{h}{4\pi}\right)^2 = 0 \tag{14}$$

The coefficients of this equation are

$$A = a^2 \tag{15}$$

$$B = \frac{hL_Z}{4\pi} \tag{16}$$

$$C = -\left(\frac{h}{4\pi}\right)^2 \tag{17}$$

And the solution to this equation is

$$p_v = \frac{-\frac{hL_Z}{4\pi} \pm \sqrt{\left(\frac{hL_Z}{4\pi}\right)^2 + 4a^2\left(\frac{h}{4\pi}\right)^2}}{2a^2} \tag{18}$$

We shall neglect the negative square root because negative temperatures have no physical meaning

$$p_v = \frac{1}{2a^2}\left[\sqrt{\left(\frac{hL_Z}{4\pi}\right)^2 + 4a^2\left(\frac{h}{4\pi}\right)^2} - \frac{hL_Z}{4\pi}\right] \tag{19}$$

$$p_v = \frac{1}{2(2\pi R)^2}\left[\sqrt{\left(\frac{hL_Z}{4\pi}\right)^2 + 4(2\pi R)^2\left(\frac{h}{4\pi}\right)^2} - \frac{hL_Z}{4\pi}\right] \tag{20}$$

$$p_v = \frac{1}{8\pi^2 R^2}\left[\sqrt{\frac{h^2 L_Z^2}{16\pi^2} + h^2 R^2} - \frac{hL_Z}{4\pi}\right] \tag{21}$$

$$p_v = \frac{1}{8\pi^2 R^2}\left[\sqrt{\frac{h^2R^2}{16\pi}\left(\frac{L_Z^2}{\pi R^2}+16\pi\right)}-\frac{hL_Z}{4\pi}\right] \tag{22}$$

$$p_v = \frac{1}{8\pi^2 R^2}\left(\frac{hR}{4\sqrt{\pi}}\sqrt{\frac{L_Z^2}{\pi R^2}+16\pi}-\frac{hL_Z}{4\pi}\right) \tag{23}$$

$$p_v = \frac{h}{32\pi^2\sqrt{\pi}R}\left(\sqrt{\frac{L_Z^2}{\pi R^2}+16\pi}-\frac{L_Z}{\sqrt{\pi}R}\right) \tag{24}$$

According to Einstein the energy of a photon is

$$p_v c = hf \tag{25}$$

Hence

$$f = \frac{c}{h}p_v \tag{26}$$

Substituting *f* in equation (8) with the value obtained from equation (26) gives

$$m_v = \left(\frac{hR}{2GM}\right)\left(\frac{c}{h}p_v\right) = \left(\frac{Rc}{2GM}\right)p_v \tag{27}$$

Multiplying both sides by c^2

$$m_v c^2 = \left(\frac{Rc^3}{2GM}\right)p_v \tag{28}$$

Considering that the energy E_v of the escaping photon is proportional to the temperature T of the black hole we can write

$$E_v = k_B T \tag{29}$$

Hence

$$T = \frac{E_v}{k_B} \tag{30}$$

Substituting E_v in equation (30) with the right hand side of equation (28) gives

$$T = \left(\frac{Rc^3}{2k_B GM} \right) p_v \tag{31}$$

Substituting p_v in equation (31) with the right hand side of equation (24) we have

$$T = \left(\frac{Rc^3}{2k_B GM} \right) \frac{h}{32\pi^2 \sqrt{\pi} R} \left(\sqrt{\frac{L_Z^2}{\pi R^2} + 16\pi} - \frac{L_Z}{\sqrt{\pi} R} \right) \tag{32}$$

$$T = \left(\frac{1}{4\sqrt{\pi}} \right) \left(\frac{hc^3}{16\pi^2 k_B GM} \right) \left(\sqrt{\frac{L_Z^2}{\pi R^2} + 16\pi} - \frac{L_Z}{\sqrt{\pi} R} \right) \tag{33}$$

We recognize the second factor of equation (33) as the Berkenstein-Hawking temperature T_{BH}, thus we write

$$T_{BH} = \frac{hc^3}{16\pi^2 k_B GM} \tag{34}$$

Now we shall assume that the Planck length L_P is the minimum length with physical meaning. Thus we shall make two changes to equation (33) a) we shall substitute the zero point energy's length L_Z with the Planck length L_P, and b) we shall substitute the second factor with T_{BH}. With these two changes equation (33) transforms into

$$T = \frac{T_{BH}}{4\sqrt{\pi}} \left(\sqrt{\frac{L_P^2}{\pi R^2} + 16\pi} - \frac{L_P}{\sqrt{\pi} R} \right) \tag{35}$$

This is the equation of the black hole temperature in terms of the Berkenstein-Hawking temperature.

Finding 1
The black hole temperature depends on both the mass M and the radius R of the black hole.

The values of the T/T_{BH} ratio are tabulated on Table 1 for different values of R/L_P ratios. The relative error is shown on the last column as a percentage

Radius to Planck length ratio $\frac{R}{L_P}$	Temperature to BH temperature ratio $\frac{T}{T_{BH}}$	Relative error (percentage) e(%) $100 \times \left(\frac{T_{BH} - T}{T_{BH}} \right)$
1	0.923 584	7.64
2	0.961 003	3.90
3	0.973 826	2.62
4	0.980 304	1.97
5	0.984 211	1.58
6	0.986 825	1.32
7	0.988 696	1.13
8	0.990 102	0.99
9	0.991 197	0.88
10	0.992 074	0.79
20	0.996 029	0.40
30	0.997 351	0.27
40	0.998 013	0.20
50	0.998 410	0.16
60	0.998 675	0.13
70	0.998 864	0.11
80	0.999 006	0.10
90	0.999 116	0.09
100	0.999 205	0.08
1 000	0.999 920	0.008
10 000	0.999 992	0.000 8
100 000	0.999 999	0.000 1
1 000 000	0.999 999 92	0.000 008

TABLE 1: *This table shows that the actual temperature T predicted by equation (35) is smaller than the corresponding temperatures* (T_{BH}) *predicted by the Berkenstein-Hawking equation.*

From the above table we see that the temperature T predicted by equation (35) is smaller than the corresponding temperatures predicted by the Berkenstein-Hawking's equation (34). We also see that the maximum relative error is about 8 % and occurs for black holes of the size of the Planck length ($R = L_P$). For black holes whose radii R are greater than $7L_P$ the relative error is less than 1 %. For black holes of radii R greater or equal than $90L_P$ the relative error is less than 0.1 %. Thus we conclude that the impact of the black hole radius on the temperature is significant in the case of microscopic black holes only. I shall quote the following paragraph from Aurélien Barrau and Julien Grain published by the CERN Currier [4]

"*Microscopic black holes are thus a paradigm for convergence. At the intersection of astrophysics and particle physics, cosmology and field theory, quantum mechanics and general relativity, they open up new fields of investigation and could constitute an invaluable pathway towards the joint study of gravitation and high-energy physics. Their possible absence already provides much information about the early universe; their detection would* *constitute a major advance. The potential existence of extra dimensions opens up new avenues for the production of black holes in colliders, which would become, de facto, even more fascinating tools for penetrating the mysteries of the fundamental structure of nature."*

Now let's return to equation (35) taking the limit of T when L_P tends to zero, which gives

$$\lim_{L_P \to 0} T = \frac{T_{BH}}{4\sqrt{\pi}} \sqrt{16\pi} \tag{36}$$

$$\lim_{L_P \to 0} T = T_{BH} \tag{37}$$

Then we find that

Finding 2
The Berkenstein-Hawking temperature is a special case of the general formula of the black hole temperature when the Planck length is zero

If we take the limit when the radius tends to infinity we obtain

$$\lim_{R \to \infty} T = \frac{T_{BH}}{4\sqrt{\pi}} \sqrt{16\pi} = T_{BH} \tag{38}$$

We find that

Finding 3
The Berkenstein-Hawking temperature is a special case of the general formula of the black hole temperature when the radius of the black hole is infinite.

4. Black Hole Entropy

Let's analyze equation (35). We notice that the ratio $\frac{L_P^2}{\pi R^2}$ is dimensionless, thus we define the parameter ρ as

$$\rho = \frac{L_P^2}{\pi R^2} \tag{39}$$

Now we substitute $\frac{L_P^2}{\pi R^2}$ with ρ in equation (35) which gives

$$T = \frac{T_{BH}}{4\sqrt{\pi}} \left(\sqrt{\rho + 16\pi} - \sqrt{\rho} \right) \tag{40}$$

Now let's consider the definition of the Planck length L_P

$$L_P = \sqrt{\frac{hG}{2\pi c^3}} \tag{41}$$

Hence

$$L_P^2 = \frac{hG}{2\pi c^3} = \frac{\hbar G}{c^3} \tag{42}$$

Now we substitute L_P^2 in equation (40) with the value obtained in equation (42). This gives

$$\rho = \frac{\hbar G}{c^3} \frac{1}{\pi R^2} \tag{43}$$

Multiplying by 4/4 yields

$$\rho = \frac{4\hbar G}{c^3} \frac{1}{4\pi R^2} \tag{44}$$

We recognize the denominator $4\pi R^2$ of the second factor as the area of a sphere of radius R. This sphere is the event horizon of the black hole.

$$A_H = 4\pi R^2 \tag{45}$$

Where
$A_H =$ area of the event horizon (Area of the sphere of radius R)

Then we write the parameter ρ in terms of the area of the event horizon

$$\rho = \frac{4\hbar G}{c^3 A_H} \tag{46}$$

Now we consider the following two limits

$$\lim_{R \to \infty} \left(\frac{1}{\rho} \right) = \lim_{R \to \infty} \left(\frac{\pi R^2}{L_P^2} \right) = \infty \tag{47}$$

$$\lim_{R \to L_P} \left(\frac{1}{\rho} \right) = \lim_{R \to L_P} \left(\frac{\pi R^2}{L_P^2} \right) = \pi \tag{48}$$

Because the possible values of $1/\rho$ are between π (minimum) and ∞ (maximum), we can define this thermodynamic property as a quantity proportional to the entropy S of the black hole (it cannot be the black hole temperature because we have already found the relationship). Since the entropy has units of $J/{}^0K$ while $1/\rho$ is dimensionless, the proportionality constant must be the inverse of the Boltzmann's constant k_B. Then we can write

$$\frac{1}{\rho} = \frac{S}{k_B} \tag{49}$$

Where

S = entropy of the black hole
k_B = Boltzmann's constant

Hence

$$S = \frac{k_B}{\rho} \tag{50}$$

$$S = \left(\frac{k_B c^3}{4\hbar G} \right) A_H \tag{51}$$

We recognize this formula as the Berkenstein-Hawking black hole entropy S_{BH}, then we write

$$S_{BH} = \frac{k_B c^3 A_H}{4\hbar G} \tag{52}$$

This result confirms that the second order uncertainty principle we have adopted in this theory provides the correct description of nature.

Finding 4
The Berkenstein-Hawking entropy formula of the Black hole is a direct consequence of the universal uncertainty principle used in this theory.

Finally we express the equation of the black hole temperature (either equation 35 or equation 40) as a function of the entropy S_{BH}

$$T = \frac{T_{BH}}{4\sqrt{\pi}} \left\{ \sqrt{\frac{k_B}{S_{BH}} + 16\pi} - \sqrt{\frac{k_B}{S_{BH}}} \right\} \tag{53}$$

5. Conclusions

In summary, the present theory predicts that the black hole temperature depends on the mass M and the radius R of the black hole. The impact of the black hole radius on the temperature is significant in the case of microscopic black holes only. The theory also shows that the Berkenstein-Hawking temperature is a special case of the more general formulation shown here. Furthermore, this theory agrees with the Berkenstein-Hawking's formula of the black hole entropy. More importantly, the black hole entropy emerges naturally from the present formulation without making any additional assumptions. This is an indication of the predicting potential and correctness of the present theory.

Appendix 1

The universal uncertainty principle we want has to satisfy the following conditions

1) The principle will be quadratic in $\Delta p \Delta x$
2) When $L_Z = 0$ the principle will reduce to $\Delta p \Delta x \geq h/4\pi$
3) When $\Delta x = 0$ the principle will reduce to $\Delta p L_Z \geq h/4\pi$

Let's consider these three conditions separately

1) The uncertainty principle has to be quadratic because of the following reasons
   a) the theory should predict the black hole entropy without any additional assumptions, and
   b) the theory should predict the size of the electron (this is not included in this paper but I shall publish the results shortly).

Therefore, shall adopt a second order uncertainty principle (a first order principle cannot produce the correct results). Thus, the principle will have the following form

$$(\Delta p \Delta x)^2 \geq \left(\frac{h}{4\pi}\right)^2 + another\ term$$

Hence

$$\Delta p \Delta x \geq \sqrt{\left(\frac{h}{4\pi}\right)^2 + another\ term}$$

2) When the effects of quantum fluctuations of space-time are neglected. (mathematically means $L_Z = 0$), the principle will be identical to the Heisenberg uncertainty principle. Thus under these conditions the principle will reduce to

$$\Delta p \Delta x \geq \frac{h}{4\pi}$$

The reason of this is that a wave-packet representing the wave function *ψ(x,y,z,t)* of the particle is formed by the addition of a number of different wavelengths that produce interference (the superposition principle in quantum mechanics gives rise to interference). The more wavelengths we add the more localized the wave function will be and therefore the probability of finding the particle in a cubic box of volume $dV = dxdydz$ will be higher. This is so because the square of the wave function $|\psi(x,y,z,t)|^2$ is the probability density of a measurement of the finding the particle in the cubic volume dV. Thus the probability $P_{x1,x2,y1,y2,z1,z2}(t)$ of finding the particle in a cubic volume defined as

$$x \in [x1, x2] \; and \; y \in [y1, y2] \; and \; z \in [z1, z2] \,)$$

$$\text{where } x1 < x2; \; y1 < y2; \; z1 < z2$$

at time t will be

$$P_{x1,x2,y1,y2,z1,z2}(t) = \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} |\psi(x,y,z,t)|^2 \, dxdydz$$

This integral shows that the more localized the wave function the higher the probability of finding the particle in a given volume. However, this mechanism will make the momentum of the particle more uncertain. The reason is that, according to De Broglie, each individual wavelength has a momentum associated with it which is given by

$$p = \frac{h}{\lambda}$$

Because the wave function of the particle is composed of a large number of different wavelengths of different amplitudes (only the De Broglie relationships are shown here):

$$p_1 = \frac{h}{\lambda_1}; \quad p_2 = \frac{h}{\lambda_2}; \quad p_3 = \frac{h}{\lambda_3}; \quad p_4 = \frac{h}{\lambda_4}; \quad \ldots \quad ; \quad p_n = \frac{h}{\lambda_n}$$

the momentum of the particle becomes more uncertain (which is the momentum of the particle $p_1, p_2, p_3, p_4, \ldots \, or \; p_n$?)

From this analysis we see that the Heisenberg uncertainty principle relates to the wave nature of the wave-packet and not to the quantum fluctuation of the vacuum.

3) When the effects of the uncertainties due to the wave nature of the wave-packet representing the escaping photon are neglected (mathematically means $\Delta x = 0$), the principle will reduce to

$$\Delta p L_Z \geq \frac{h}{4\pi}$$

Solution

The following inequality satisfies all three conditions simultaneously

$$\Delta p \Delta x \geq \sqrt{\left(\frac{h}{4\pi}\right)^2 - \frac{h}{4\pi}\Delta p L_Z}$$

This is the *universal uncertainty principle.*

Verification

(a) $L_Z = 0$

$$\Delta p \Delta x \geq \sqrt{\left(\frac{h}{4\pi}\right)^2 - 0}$$

$$\Delta p \Delta x \geq \frac{h}{4\pi}$$ (which satisfies condition 2)

(b) $\Delta x = 0$

$$0 \geq \sqrt{\left(\frac{h}{4\pi}\right)^2 - \frac{h}{4\pi}\Delta p L_Z}$$

$$0 \geq \left(\frac{h}{4\pi}\right)^2 - \frac{h}{4\pi}\Delta p L_Z$$

$$0 \geq \frac{h}{4\pi} - \Delta p L_Z$$

$$\Delta p L_Z \geq \frac{h}{4\pi}$$ (which satisfies condition 3)

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