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A Microscopic Approach to Quark and Lepton Masses and Mixings

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Abstract

In recent papers a microscopic model for the SM Higgs mechanism has been proposed, and an idea how to determine the 24 quark and lepton masses of all 3 generations has emerged in that framework. This idea is worked out in detail here by accommodating the fermion masses and mixings to microscopic parameters. The top quark mass turns out to be $m_t \approx 170$GeV and can be given in terms of the weak boson masses and of certain exchange couplings of isospin vectors obeying a tetrahedral symmetry. The observed hierarchy in the family spectrum is attributed to a natural hierarchy in the microscopic couplings. The neutrinos will be shown to vibrate within the potential valleys of the system, thus retaining very tiny masses. This is related to a Goldstone effect inside the internal dynamics. A discussion of the quark and lepton mixing matrices is also included. The mixing angles of the PMNS matrix are calculated for an example set of parameters, and a value for the CP violating phase is given.

1 Introduction

The Standard Model of elementary particles (SM) is very successful on the phenomenological level. The outcome of (almost) any particle physics experiment can be predicted accurately within this model, and, where not, by some straightforward extension. For example, one may introduce right handed neutrinos to account for tiny neutrino masses[1].

Nevertheless, it is widely believed that the SM is only an effective low-energy theory valid below a certain energy scale Λ_r, which is supposed to be of the order of 1-10 TeV. This view is based on the fact that the SM has many unknown parameters and one rather mysterious component, the so-called Higgs field, which is needed for the spontaneous symmetry breaking (SSB) taking place in the model. Within the Higgs sector the most challenging part is the set of Yukawa couplings to fermions, which comprises the majority of the unknown parameters of the SM Lagrangian.

In recent papers a microscopic model of the Higgs mechanism has been developed[2, 3, 4], and also an idea how the quark and lepton states arise in that model. This concept will be used in the present article in an attempt to determine the fermion masses and mixings.

At first sight, the spectrum of quarks and leptons seems difficult to explain, because it extends over many orders of magnitude, starting from the neutrinos with their tiny masses below 1 eV, passing over to the 'everyday life' particles e, u and d with masses of order 10^6 eV, proceeding to muon and strange-quark (about 10^8eV), ascending to charm, τ and bottom, which have masses of order 10^9eV, and finishing with the top quark, whose mass of 1.7×10^{11}eV lies suspiciously close to the SSB scale Λ_F, the value of the Higgs mass and to twice the W-mass. For the neutrinos it is reasonable to believe that their mass might be some kind of higher order effect[5], is protected by symmetry or generated by a variant of the popular seesaw mechanism[6]. Other approaches to explain the hierarchy in the particle masses consider textures[7] like 'democratic' mass matrices[8] with identical entries. These show the desirable feature that after diagonalization there is one very heavy particle (the top quark), and the rest have small masses.

Unfortunately, a physical understanding of the underlying dynamics responsible for these effects is still lacking. For example, in (supersymmetric) grand unified theories fermion masses essentially remain free parameters. Furthermore, those models usually introduce many more additional degrees of freedom without much ambition to determine them from first principles. The point is that theories of that kind only extrapolate and extend the symmetries observed at low energies to small distances and that there is a strong amount of arbitrariness in this procedure. In my opinion it is obvious that a physical understanding of the masses and mixings is only possible in a microscopic theory. Superstring theories seem to offer such an understanding. However, although 'in principle' able to determine the masses as energies of string excitations, to my knowledge they have not come up with definite and verifiable predictions.

The present paper is devoted to partially fill this gap. Some of the above mentioned 'textures' will reappear in the sections below. For example, neutrinos are indeed protected by the symmetries of new interactions. Furthermore, a kind of democratic texture will be derived which makes the top quark the heaviest fermion. More precisely, it arises from a symmetry breaking contribution modified by a Dzyaloshinskii-Moriya component[10] in such a way that all entries of the mass matrix effectively get identical contributions. The appearance of this modification turns out to be a reflection of the $SU(2)_L$ gauge symmetry (breaking) on the microscopic level.

A related question is how the mixing between the generations can be understood. Most of the mixing angles are now known with a reasonable accuracy[9, 1]. In particular, there is a hierarchy in the mixing matrix for the quark sector, but not in the lepton sector. Approaching the mixing problem in the present model I will be able to give some preliminary results mainly for the neutrinos and also set up the environment to derive the CKM mixing angles.

2 Quarks and Leptons as Isospin Excitations of a Tetrahedral Shubnikov Group

The Higgs doublet of the Standard Model can be parametrized as

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} i(\pi_x - i\pi_y) \\ \sigma - i\pi_z \end{pmatrix} \tag{1}$$

where $\sigma = \Lambda_F + \phi$ acquires a vacuum expectation value $\langle\sigma\rangle = \Lambda_F = \sqrt{\frac{\mu^2}{\lambda}}$ through the form of the potential

$$\begin{aligned} V(\Phi) &= -\mu^2\Phi^+\Phi + \lambda(\Phi^+\Phi)^2 = -\frac{1}{2}\mu^2(\sigma^2 + \vec{\pi}^2) + \frac{1}{4}\lambda(\sigma^2 + \vec{\pi}^2)^2 \\ &= \frac{1}{4}\lambda[-\Lambda_F^4 + 4\Lambda_F\phi\vec{\pi}^2 + 4\Lambda_F^2\phi^2 + 4\Lambda_F\phi^3 + \phi^4 + \vec{\pi}^4 + 2\phi^2\vec{\pi}^2] \end{aligned} \tag{2}$$

and $\vec{\pi} = (\pi_x, \pi_y, \pi_z)$ gets 'eaten' by the longitudinal modes of the afterwards massive W-bosons. This can be made explicit by a SU(2) gauge transformation of the form[11]

$$U = \exp(\frac{i\vec{\tau}\vec{\pi}}{2\Lambda_F}) \tag{3}$$

which formally removes $\vec{\pi}$ from the Higgs doublet.

The present calculation uses a Nambu-Jona-Lasinio (NJL) type of interpretation of the SSB mechanism. The starting point is an isospin pair $\psi = (U, D)$ of Dirac fermions distantly similar as in technicolor models[12, 13, 14, 15], however without a technicolor quantum number. Rather we shall assume that the pairing mechanism is due to exchange interactions and strong correlations between fermions, effects which in many body physics are known to be responsible for SSB in superconductors and (anti)-ferromagnets. In contrast to solid state physics we do not consider these effects in physical space, but attribute them to arise from an independent dynamics which is active in the internal spaces. It is this dynamics which will allow us to put hand on the values of the fermion masses.

The main idea is that (weak) isospin arises from a nonrelativistic internal R^3 space much like ordinary spin arises from physical space. In other words, the internal space is assumed to possess a rotational SO(3)-symmetry for which the doublet $\psi = (U, D)$

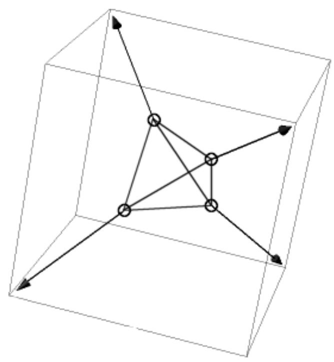

Figure 1: The local ground state of the model, living in a 3-dimensional internal R^3 space. Shown are the corner points (small circles) of the internal tetrahedron, which can be represented by their coordinate vectors $\vec{r}_i$. The origin of coordinates is taken to be the center of the tetrahedron, and is identical to the base point of the fiber in Minkowski space. On each corner point $i = 1, 2, 3, 4$ there is an axial spin vector $\vec{\pi}_i$, pointing in the same radial direction as $\vec{r}_i$.

serves as an (internal) Pauli spinor with an initial internal SU(2) spin symmetry. These internal spins are assumed to undergo interactions in the internal space which can be described by internal Heisenberg spin interactions which are formally similar to those describing spin interactions in solids.

The geometrical picture is that the world is a fiber bundle over Minkowski space with fibers given by the R^3 spaces, and that within these fibers physical processes take place, which can be described by a higher dimensional quantum electrodynamics. This idea was worked out in ref. [2] and the interested reader is referred to that paper for more details. It is further assumed that at high temperatures there is a symmetric phase in which the internal spins are distributed randomly in the fibers, giving rise to a local internal SU(2) symmetry of the Lagrangian, local in the sense that on each site in each fiber the spins may be rotated independently.

When the temperature of the universe decreases from big bang energies to TeV values the internal magnetic interactions within the fiber lead to the frustrated[21] tetrahedral structure shown in fig. 1, and when it falls below the Fermi scale all the tetrahedrons over Minkowski space align as in fig. 2, a process which in ref.[2] was claimed to be the microscopic origin of the Higgs mechanism. A pairing process for the formation of the Higgs particle has also been described in that paper.

The configuration fig. 1 is the starting point of the present calculation, because it is considered as the *local* ground state of the system. In other words, it is assumed that in each of the 3-dimensional internal R^3 fibers there is a discrete tetrahedral structure and that the internal dynamics is such that spin vectors arrange themselves according to this internal tetrahedral configuration. The tetrahedron itself has the tetrahedral group S_4 as point group symmetry. However, due to the pseudovector property of the internal spin vectors the whole system loses its reflection symmetries and obtains instead the Shubnikov symmetry group $A_4+S(S_4-A_4)$[27, 16, 28], where S is the internal time reversal operation and A_4 is the subgroup of S_4 which does not contain reflections. Note that S itself does not belong to the Shubnikov group, and also the internal reflections do not. The Shubnikov group is chiral, the configuration with opposite chirality being given when the 4 spin vectors would point inwards instead of outwards. Before the formation of the chiral tetrahedron the internal spins U and D, which according to eq. (6) are the building blocks of the spin vectors $\vec{\pi}_i$, can freely rotate and thus there is an internal spin SU(2) symmetry group, which however is broken to $A_4 + S(S_4 - A_4)$ when the chiral tetrahedron is formed.

With respect to (external) Lorentz symmetry both U and D can appear as lefthanded or righthanded objects, so that one may in fact consider separately a $SU(2)_L$ for the lefthanded and $SU(2)_R$ for the righthanded objects. Before the advent of the gauge bosons the Higgs sector of the SM is symmetric under $SU(2)_L \times SU(2)_R$, and a Nambu-Jona-Lasinio (NJL) model with this symmetry may we formulated.

The NJL philosophy operates as follows: the SSB is induced by formation of bound states and condensates of the fermion doublet $\psi = (U, D)$. Namely, the quadratic part

$$V_2(\Phi) = -\mu^2\Phi^+\Phi = -\frac{1}{2}\mu^2(\sigma^2 + \vec{\pi}^2) \tag{4}$$

of the potential eq. (2) is equivalent to a NJL interaction of the form

$$V_{NJL} = G_{NJL}[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\vec{\tau}\psi)^2] \tag{5}$$

where G_{NJL} denotes the NJL coupling strength, which in the SSB regime, where

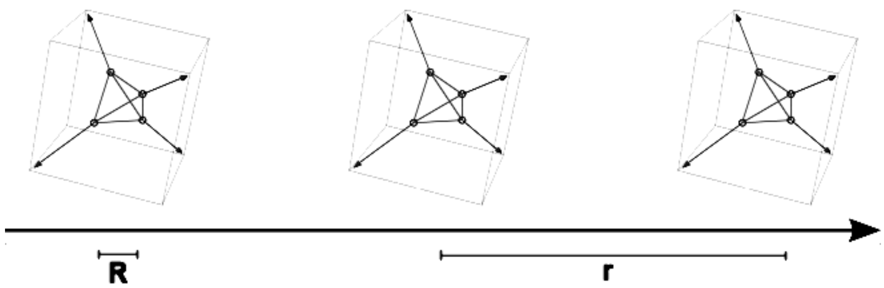

Figure 2: The global ground state of the model after SSB consists of an aligned system of chiral tetrahedrons over Minkowski space (the latter represented by the long arrow). R is the magnitude of one tetrahedron and r the distance between two of them. Note that contrary to what is drawn here, the tetrahedra extend into internal space alone, not into Minkowski space. Before the SSB the chiral tetrahedrons are oriented randomly (not shown) and there is a corresponding local $SO(3)$ symmetry, because each rigid tetrahedron can be rotated freely and independently from the others.

$V_2(\Phi) < 0$, must be negative as well. (Note that $\bar{\psi} i\gamma_5 \vec{\tau}\psi$ is always real.) In fact using the method of auxiliary fields one can show $V_2(\Phi) = V_{NJL}$ provided one chooses

$$\begin{aligned} \sigma &= -2G_{NJL}\bar{\psi}\psi \\ \vec{\pi} &= -2G_{NJL}\bar{\psi} i\gamma_5\vec{\tau}\psi \end{aligned} \tag{6}$$

and thus obtain a sigma model from the original NJL potential (5).

For later use I will rewrite eq. (5) as

$$V_{NJL} = -J_F[(\frac{\bar{\psi}\psi}{\mu^3})^2 + (\frac{\bar{\psi} i\gamma_5\vec{\tau}\psi}{\mu^3})^2] \tag{7}$$

so that in brackets there are dimensionless quantities and $J_F = -G_{NJL}\mu^6$ has the dimension 4 of an energy density.

This equation has been interpreted in ref. [2] to describe the dynamics of interacting chiral internal spin vectors $\vec{\pi}$, normalized to the SSB scale μ. Here 'chiral' refers both to internal and physical space, because γ_5 is a building block for chiral objects in physical space, while the appearance of (internal) Pauli matrices $\vec{\tau}$ signals chiral objects in (internal) space. Moreover, in the framework of the internal Heisenberg spin theory J_F is to be interpreted as the internal exchange energy density corresponding to certain exchange integrals over internal R^3 space to be described later. In the SSB regime one has $J_F > 0$ (because of $G_{NJL} < 0$) corresponding to a ferromagnetic interaction. This interaction accounts for neighbouring tetrahedrons aligning themselves over Minkowski space as shown in fig. 2.

There is a slight complication on these considerations, because at SSB energies after redefinition of $\sigma = \Lambda_F + \phi$, the $\vec{\pi}$-$\vec{\pi}$ interaction eq. (4) seems to disappear, because the sum of terms $\sim \vec{\pi}^2$ vanishes in the potential $V(H)$ as made explicit by the last of eqs. (2). However, when the $\vec{\pi}$ triplet is absorbed as the longitudinal mode of the $\vec{W}$-boson the internal Heisenberg spin interaction reappears as part of the mass term $m_W^2 W_\mu W^\mu$. The alignment of tetrahedrons in fig. 2 will then experience a modification which is dictated by the gauge symmetry of the fiber bundle formed by all tetrahedrons. As a consequence the ferromagnetic Heisenberg interaction has to be modified by a Dzyaloshinskii-Moriya component[10, 25, 26] in this regime. Details will be given in the next section.

Next I want to extent the view to small distances and high energies. At high energies, there is no SSB and instead of the negative potential term V_2 one has a strictly positive potential, which still can be described by eq. (5), however with a positive coupling G_{NJL}. Rewriting that equation as

$$V_{NJL} = -J_A[(\frac{\bar{\psi}\psi}{\Lambda_r^3})^2 + (\frac{\bar{\psi}i\gamma_5\vec{\tau}\psi}{\Lambda_r^3})^2] \tag{8}$$

one should not take the SSB scale μ as normalization scale any more. Instead another reference scale $\Lambda_r \gg \mu$ has to be introduced which physically corresponds to the distance between two tetrahedrons (alternatively one could utilize the extension of one of them). One thus obtains an antiferromagnetic internal spin interaction with a negative exchange coupling J_A. This repulsion effect leads to the frustrated antiferromagnetic vacuum structure shown in fig. 1. J_F and J_A differ because they correspond to exchange integrals over different regions of space. J_A is dominated by an integral over the volume of one internal tetrahedron (leading to the frustrated internal antiferromagnetic configuration), while J_F is the exchange integral for spin vectors of different tetrahedrons (leading to the aligment of different tehtrahedrons over Minkowski space).

The situation is reminiscent to the theory of ordinary magnets, where the exchange coupling integral J is known to vary with the distance. At large lattice spacings and corresponding large distances between spin vectors (much larger than the extension of the electron wave function) one has $J > 0$ and a ferromagnetic behavior. On the other hand, antiferromagnets like Cr and Mn are characterized by small lattice spacings and corresponding small distances between spin vectors, typically not much larger than the extension of the electron wave function. In these cases $J < 0$, i.e. antiferromagnetic behavior.

According to fig. 1 there is one chiral internal spin vector $\vec{\pi}_i$ for each of the 4 constituents of the internal tetrahedron. In the ground state these vectors point radially away from the origin. Their sum

$$\langle \vec{\pi} \rangle = \sum_{i=1}^{4} \langle \vec{\pi}_i \rangle \tag{9}$$

vanishes corresponding to a vanishing vev $\langle \vec{\pi} \rangle = 0$ in accordance with the SM vacuum structure of the Higgs doublet (1). Excited states arise as vibrations of the vectors $\vec{\pi}_i$ in fig. 1 and will be interpreted as quarks and leptons. They can be classified according to the system's symmetry group, the Shubnikov group $A_4 + S(S_4 - A_4)$. This group, which remains unbroken at low energies, has only 1- and 3- dimensional representations, i.e. singlets (interpreted as leptons) and triplets (interpreted as the 3 colors of quarks).

When studying the dynamics of the internal spin vectors to derive the spectrum of the excited states one notes that $\vec{\pi} \sim \bar{\psi} i \gamma_5 \vec{\tau} \psi$ is not a quantity simple to handle, because it does not fulfill the canonical commutation relations for spin vectors. Secondly, it turns out that the internal Hamiltonian (related to the internal time variable) cannot be written in terms of $\vec{\pi}$. The appropriate internal vector to use is $\psi^\dagger \gamma_5 \vec{\tau} \psi$, a well known charge observable from current algebra[20]. However, due to the factor of γ_5 these vectors still do not fulfill the usual angular momentum commutation relations because their commutator is a scalar, and not a pseudoscalar any more. In order that the algebra of internal spin vectors closes one is forced to consider the following linear combinations of internal spin vectors

$$\vec{S} = \frac{1}{\Lambda^3}\psi^\dagger(1+\gamma_5)\vec{\tau}\psi$$
$$\vec{T} = \frac{1}{\Lambda^3}\psi^\dagger(1-\gamma_5)\vec{\tau}\psi \qquad (10)$$

which fulfill the canonical commutation relations of a system of two decoupled angular momentum operators[20] in the sense that

$$[S_a, S_b] = i\epsilon_{abc}S_c \qquad [T_a, T_b] = i\epsilon_{abc}T_c \qquad [S_a, T_b] = 0 \qquad (11)$$

The point to note here is the impact of the chiral symmetry group $SU(2)_R \times SU(2)_L$, because $\vec{S}$ and $\vec{T}$ can be considered as generators of its $SU(2)_R$ and $SU(2)_L$ factors, respectively. In current algebra they correspond to the conserved charges of the $SU(2)_R \times SU(2)_L$ symmetry, the factor Λ^{-3} arising from the spatial integral of the time component of the left and right handed currents.

On the other hand, within the internal dynamics advocated in this paper $\vec{S}$ and $\vec{T}$ play the role of angular momentum observables corresponding to rotations of the internal R^3 space. The fact that they are dimensionless according to eqs. (10) agrees with this interpretation, because an angular momentum $\vec{r}\times\vec{p}$ is always dimensionless. Physically, the scale Λ can be identified as $\Lambda = \mu$ in the SSB regime and Λ_r at high energies.

Using eq. (10) one is effectively including further dynamical vibrators $\sim \psi^\dagger\vec{\tau}\psi$ in addition to the axial vectors $\vec{\pi}$. This kindly solves another problem not discussed so far, namely the 4×3 d.o.f. of the internal spin vectors in fig. 1 yield only 12 excitation states instead of the necessary 24 quarks and leptons. In order to obtain the remaining 12 (which turn out to be their isospin partners), in ref. [2] it was proposed that internal displacive vibrations should be included in addition to spin wave excitations. In the present context the doubling of the number of excitations is obtained without displacive vibrations by going to the closed algebra eq. (21) of the 8 internal spin vectors $\vec{S}_i$ and $\vec{T}_i$, i=1-4, whose vacuum values are depicted in fig. 3.

In that figure the vectors $\langle\vec{S}_i\rangle$ are shown pointing outwards and $\langle\vec{T}_i\rangle$ pointing inwards fulfilling

$$\langle\vec{S}_i\rangle = -\langle\vec{T}_i\rangle \qquad (12)$$

If $\vec{S}_i$ and $\vec{T}_i$ were identical observables, the configuration would possess an internal time reversal symmetry (with symmetry group the 'grey' group $S_4 \times \{1, S\}$), because the time reversal invariance broken by the set of vectors pointing outwards would be restored by those pointing inwards. However, since $\vec{S}$ and $\vec{T}$ are physically different, the ground state has still the original Shubnikov group $A_4 + S(S_4 - A_4)$ as symmetry.

Eq. (12) implies that the ground state gets contributions only from the γ_5 terms in eq. (10) and $\langle\psi^\dagger \vec{\tau}\psi\rangle_i$ vanishes in the vacuum for each of the constituents $i = 1, 2, 3, 4$. In principle the opposite situation is conceivable as well, namely that $\langle\psi^\dagger \vec{\tau}\psi\rangle_i \neq 0$ while $\langle\psi^\dagger \gamma_5 \vec{\tau}\psi\rangle_i$ vanishes. In that case one would have

$$\langle\vec{S}_i\rangle = +\langle\vec{T}_i\rangle \tag{13}$$

a perfectly reasonable configuration, which maintains the Shubnikov symmetry for the system of 8 spin vectors as long as all internal vectors point e.g. outwards, in radial directions. In the numerical analysis presented in the following sections the configuration (13) will actually be preferred because technically it is easier to handle. In other words, eq. (13) will be used as equilibrium conditions for the concrete calculations of masses and eigenstates carried out in the next section, cf. eq. (25).

The 24 eigenmodes of the system can be arranged in six 1-dimensional and six 3-dimensional representations of the Shubnikov group $A_4 + S(S_4 - A_4)$[27, 28, 16] to yield precisely the multiplet structure of the 24 quark and lepton states of the 3 generations, not less and not more.

$$\begin{aligned} A_\uparrow(\nu_e) + A_\uparrow(\nu_\mu) + A_\uparrow(\nu_\tau) \quad &+ \quad T_\uparrow(d) + T_\uparrow(s) + T_\uparrow(b) + \\ A_\downarrow(e) + A_\downarrow(\mu) + A_\downarrow(\tau) \quad &+ \quad T_\downarrow(u) + T_\downarrow(c) + T_\downarrow(t) \end{aligned} \tag{14}$$

Here $A_{\uparrow,\downarrow}$ and $T_{\uparrow,\downarrow}$ denote singlet and triplet representations of the Shubnikov group. As shown later, the $\uparrow$ excitations can be obtained from the $\downarrow$ excitations by interchanging the roles of $\vec{S}$ and $\vec{T}$. Since interchanging $\vec{S}$ and $\vec{T}$ can be seen to correspond to internal time reversal within the $\uparrow$ and $\downarrow$ states, this is precisely the behavior one expects from weak isospin partners after the SSB.

In the framework of the chiral NJL dynamics the introduction of the second spin vector corresponds to introducing an additional term including $\bar{\psi}\vec{\tau}\psi$ into the potential. Actually it means to consider the most general $SU(2)_L \times SU(2)_R$ invariant

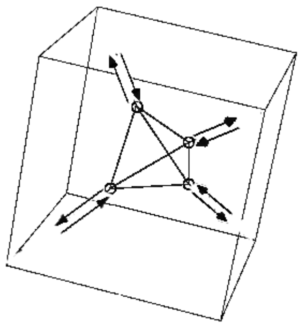

Figure 3: The local ground state of the generalized NJL-model eqs.(15) and (12). The total of 8 internal spin vectors accounts for 3×8 d.o.f. corresponding to 24 spin vibrations which can be classified according to the multiplet structure of the Shubnikov group, eq.(14). The vectors $\vec{S}_i^0$ are assumed to point outwards and $\vec{T}_i^0 = -\vec{S}_i^0$ inwards. According to eq. (10) this corresponds to $\langle\psi^\dagger\gamma_5\vec{\tau}\psi\rangle_i \neq 0$. The alternative vacuum configuration (13) where $\langle\psi^\dagger\vec{\tau}\psi\rangle_i \neq 0$ and the $\vec{T}_i^0$ are parallel to the $\vec{S}_i^0$ instead of anti-parallel is not drawn.

potential based on a fundamental isospin doublet $\psi = (U, D)$, the general 2-flavor NJL model[18]

$$\begin{aligned} V_{2NJL} &= V_+ + V_- \\ V_+ &= -J_+[(\frac{\bar\psi\psi}{\Lambda^3})^2 + (\frac{\bar\psi\vec{\tau}\psi}{\Lambda^3})^2 + (\frac{\bar\psi i\gamma_5\psi}{\Lambda^3})^2 + (\frac{\bar\psi i\gamma_5\vec{\tau}\psi}{\Lambda^3})^2] \\ V_- &= -J_-[(\frac{\bar\psi\psi}{\Lambda^3})^2 - (\frac{\bar\psi\vec{\tau}\psi}{\Lambda^3})^2 - (\frac{\bar\psi i\gamma_5\psi}{\Lambda^3})^2 + (\frac{\bar\psi i\gamma_5\vec{\tau}\psi}{\Lambda^3})^2] \end{aligned} \quad (15)$$

V_+ and V_- are separately chiral $SU(2)_L \times SU(2)_R$ invariant and in addition possess a $U(1)_V$ fermion number symmetry. Furthermore, V_+ is invariant under axial $U(1)_A$ transformations, while V_- explicitly breaks this symmetry and can only be used if an axial anomaly is present (see below). The same scale Λ (= μ oder Λ_r) as in eq. (10) has been introduced to make the fermion operators dimensionless. As before, the NJL couplings can be written in terms of exchange energy densities $J_\pm$, and one needs $J_\pm > 0(< 0)$ to obtain internal (anti)ferromagnetic behavior.

Rewriting V_{2NJL} as

$$V_{2NJL} = -(J_+ + J_-)[(\frac{\bar{\psi}\psi}{\Lambda^3})^2 + (\frac{\bar{\psi}i\gamma_5\vec{\tau}\psi}{\Lambda^3})^2] - (J_+ - J_-)[(\frac{\bar{\psi}i\gamma_5\psi}{\Lambda^3})^2 + (\frac{\bar{\psi}\vec{\tau}\psi}{\Lambda^3})^2] \quad (16)$$

and using the method of auxiliary fields one can transform the theory in the SSB region into a sigma-model, similar to eq. (6), by identifying

$$\begin{aligned} \sigma &= -2(G_+ + G_-)\bar{\psi}\psi \\ \vec{\pi} &= -2(G_+ + G_-)\bar{\psi}i\gamma_5\vec{\tau}\psi \\ \eta &= -2(G_+ - G_-)\bar{\psi}i\gamma_5\psi \\ \vec{v} &= -2(G_+ - G_-)\bar{\psi}\vec{\tau}\psi \end{aligned} \quad (17)$$

i.e. a scalar iso-scalar field (the physical Higgs $\sigma = \Lambda_F + \phi$), a pseudo-scalar iso-vector (the would be Goldstone bosons $\vec{\pi}$ absorbed by the weak bosons), a pseudo-scalar iso-scalar η and a scalar iso-vector triplet $\vec{v}$ consisting of 2 charged fields $v^\pm$ and a neutral v_z.

This field content seems to indicate a second scalar doublet formed by η and $\vec{v}$. However, in order to avoid a chiral (γ_5) vacuum structure of physical space, the η-field should not acquire a vacuum expectation value or, in other words, the second doublet should not be part of the SSB-process, with negative mass terms, a non-trivial minimum of the potential etc.

Although technically possible, one should not put the octet of fields eq. (17) in the adjoint representation

$$\Sigma = \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{1}{\sqrt{2}}(\sigma + v_z) & v^+ \\ v^- & \frac{1}{\sqrt{2}}(\sigma - v_z) \end{pmatrix} + \frac{i}{\sqrt{2}}\begin{pmatrix} \frac{1}{\sqrt{2}}(\eta + \pi_z) & \pi^+ \\ \pi^- & \frac{1}{\sqrt{2}}(\eta - \pi_z) \end{pmatrix} \quad (18)$$

of a model with a larger $U(2)_R \times U(2)_L$ symmetry[22]. The point is that the universal mass term $\mathrm{tr}(\Sigma^\dagger\Sigma)$ of such a model would imply $J_- = 0$ in eq. (16). If any, this would be useful only as long as there are no axial anomalies in the theory which break the $U(1)_A$ subgroup of $U(2)_R \times U(2)_L$.

As explained in ref [2] the underlying theory of the present model exhibits such an anomaly. In general this anomaly not only allows a non-vanishing value for J_- but also makes the mass of the η and $\vec{v}$ much larger than that of the weak gauge bosons and of the Higgs field[22]. Those bound states may appear as heavy resonances in the TeV regime. They could be interesting dark matter candidates[23, 24] and play a role at higher energies or higher temperatures of the universe. Their phenomenology, however, will not be discussed at this point, their d.o.f.s just being used as part of the components of the internal spin vectors whose vibrations give the quark and lepton mass spectrum.

3 Masses and Mixings from Isospin Wave Equations

The model set up in the last section will now be applied to calculate the quark and lepton masses. The idea is that masses can be identified with eigenfrequencies of excitations of the isospin vectors $\vec{S}$ and $\vec{T}$ eqs. (10) and that these eigenfrequencies get contributions both from inner- and from inter-tetrahedral interactions. The *inner*-tetrahedral interactions are antiferromagnetic in nature and responsible for the frustrated tetrahedral configuration figs. 1 and 3, i.e. for the structure of the local vacuum. They are small distance contributions and relatively simple to treat because they can be described by an internal antiferromagnetic Heisenberg Hamiltonian for one tetrahedron alone, with corresponding internal spin vector excitations.

On the other hand there are *inter*-tetrahedral interactions fed by the 'ferromagnetic' SSB interactions between different tetrahedrons. Their leading effect turns out to be a contribution of order $O(\Lambda_F)$ solely to the top quark mass. Physically speaking, this interaction handicaps the specific eigenmode describing the top quark, because this mode disturbs the SSB alignment in the strongest possible way. Mathematically, the effect will be described by adding an effective universal term to the inner-tetrahedral Heisenberg interaction with a normal ferromagnetic plus a Dzyaloshinskii-Moriya component[10]. The sum of the 2 components will yield a quasi-democratic mass matrix which in leading order only contributes a term of order Λ_F to the top-quark mass and nothing to the masses of the other quarks and leptons.

Let me start with the high energy / small distance contributions. The antiferromagnetic inner-tetrahedral Heisenberg Hamiltonian density for the spin vectors $\vec{S}$ and

$\vec{T}$ reads

$$V_H = -J_{SS}\sum_{i,j=1}^{4}\vec{S}_i\vec{S}_j - J_{ST}\sum_{i,j=1}^{4}[\vec{S}_i\vec{T}_j + \vec{T}_i\vec{S}_j] - J_{TT}\sum_{i,j=1}^{4}\vec{T}_i\vec{T}_j \tag{19}$$

where i and j run over the corners of the tetrahedron fig. 1 and the exchange energy densities J can be identified with the couplings introduced in eq. (16)

$$J_{SS} = J_{TT} = -\frac{1}{2}J_- \qquad\qquad J_{ST} = \frac{1}{2}J_+ \tag{20}$$

Using the commutation relation for the internal spin operators

$$[\vec{S}_i^a, \vec{S}_j^b] = i\epsilon_{abc}\delta_{ij}S_i^c \qquad [\vec{T}_i^a, \vec{T}_j^b] = i\epsilon_{abc}\delta_{ij}T_i^c \qquad [\vec{S}_i^a, \vec{T}_j^b] = 0 \tag{21}$$

one can derive their (internal) time evolution in the Heisenberg picture

$$\Lambda^3\frac{d\vec{S}_i}{dt} = i[V_H, \vec{S}_i] \qquad\qquad \Lambda^3\frac{d\vec{T}_i}{dt} = i[V_H, \vec{T}_i] \tag{22}$$

to obtain

$$\begin{aligned}\frac{d\vec{S}_i}{dt} &= \vec{S}_i^0 \times \sum_{i,j=1}^{4}[j_{SS}\vec{S}_j + j_{ST}\vec{T}_j] + k_{ST}\vec{S}_i^0 \times \vec{T}_i \\ \frac{d\vec{T}_i}{dt} &= \vec{T}_i^0 \times \sum_{i,j=1}^{4}[j_{TT}\vec{T}_j + j_{ST}\vec{S}_j] + k_{ST}\vec{T}_i^0 \times \vec{S}_i\end{aligned} \tag{23}$$

These equations have been linearized for small displacements $\delta\vec{S}_i = \vec{S}_i - \vec{S}_i^0$ and $\delta\vec{T}_i = \vec{T}_i - \vec{T}_i^0$ of the spin vectors from their ground state positions in fig. 3, and the letter δ has then been left out. Furthermore, we have switched from exchange energy densities J to exchange energies $j = J\Lambda^{-3}$. Finally, eq. (23) includes a contribution, which accounts for the possibility that $\vec{S}_i\vec{T}_i$ interact with a different strength than $\vec{S}_i\vec{T}_j$ for $j \neq i$, because the internal distance between $\vec{S}_i$ and $\vec{T}_i$ is different from the distance between $\vec{S}_i$ and $\vec{T}_j$ for $j \neq i$. This corresponds to the fact that in principle the Heisenberg couplings J can be different for each term ij in the sums in eq. (19), as long as the point symmetry is respected.

The ground state positions are given by

$$\vec{S}_1^0 = \frac{1}{\sqrt{3}}(-1,-1,-1) \qquad \vec{S}_2^0 = \frac{1}{\sqrt{3}}(-1,+1,+1) \tag{24}$$

$$\vec{S}_3^0 = \frac{1}{\sqrt{3}}(+1,-1,+1) \qquad \vec{S}_4^0 = \frac{1}{\sqrt{3}}(+1,+1,-1) \tag{25}$$

and $\vec{T}_i^0 = \pm\vec{S}_i^0$ depending on whether one is analyzing the parallel or anti-parallel configuration fig. 3. An overall normalization factor of the vacuum spin vectors was put to 1, because it does not influence the eigenfrequencies and mixing angles. So everything fixed now to solve the differential equations (23)? Not quite, because for a frustrated, i.e. non-collinear ground state configuration like fig. 1 it is necessary to go beyond the collinear spin wave analysis and transform to a rotating frame with the z-axis pointing along the local spin direction[25, 26]. Applied to the present case this procedure modifies eqs. (23) in such a way that

$$
\begin{aligned}
\frac{d\overrightarrow{U_i S_i}}{dt} &= \overrightarrow{U_i S_i}^0 \times \sum_{i,j=1}^{4} [j_{SS}\overrightarrow{U_j S_j} + j_{ST}\overrightarrow{U_j T_j}] + k_{ST}\overrightarrow{U_i S_i}^0 \times \overrightarrow{U_i T_i} \\
\frac{d\overrightarrow{U_i T_i}}{dt} &= \overrightarrow{U_i T_i}^0 \times \sum_{i,j=1}^{4} [j_{TT}\overrightarrow{U_j T_j} + j_{ST}\overrightarrow{U_j S_j}] + k_{ST}\overrightarrow{U_i T_i}^0 \times \overrightarrow{U_i S_i} \qquad (26)
\end{aligned}
$$

where U_i are diagonal 3×3 matrices which act on the vector components of the spin vectors

$$U_1 = D(1,1,1) \quad U_2 = D(1,-1,-1) \quad U_3 = D(-1,1,-1) \quad U_4 = D(-1,-1,1) \qquad (27)$$

The resulting 24×24 matrix is given in table 1 and has to be diagonalized in order to obtain the 24 eigenfrequencies ω. Due to the Shubnikov symmetry of the system the corresponding eigenstates can be arranged into 6 singlets and 6 triplets as in eq. (14), i.e. as leptons and quarks. Each triplet consists of 3 states with degenerate eigenvalues, because the Shubnikov symmetry $A_4 + S(S_4 - A_4)$ is unbroken.

The result of the diagonalization procedure gives the following non-vanishing masses / eigenfrequencies consisting of 2 singlets

$$\omega(\mu) = -\omega(\tau) \;=\; 2(4j_{ST} + k_{ST}) = 4j_+ + 2k_{ST} \qquad (28)$$

and 4 triplets

$$
\begin{aligned}
\omega(t) = -\omega(c) &= 4(j_{ST} + j_{SS}) = 2(j_+ - j_-) \qquad (29) \\
\omega(s) = -\omega(b) &= 2(2j_{ST} + 2j_{TT} + k_{ST}) = 2(j_+ - j_- + k_{ST}) \qquad (30)
\end{aligned}
$$

The corresponding eigenvectors are not given because the formulas are too cumbersome to be presented here. Apart from these 14 modes at this stage there are 10 zero modes (4 singlets and 2 triplets), which can be attributed to the 3 neutrinos, the electron and the up- and down-quark.

One concludes that considering only the antiferromagnetic Heisenberg contributions eqs. (19) and (23) leads to a stronger degeneracy than dictated by the Shubnikov symmetry alone. As (28) and (30) show this does not only concern the zero modes. What will be done next, is to include effects of inter-tetrahedral interactions to partially lift the degeneracies and in particular to shift the masses of the third family to larger values. Most prominently, the top quark mass will be equipped with a contribution of order Λ_F. Afterwards torsion and anisotropic corrections will be included. They are tiny effects and cannot be attributed to a SM piece of the interactions like the contributions discussed so far. However, they are needed, because they are responsible for the light quark and lepton masses, and in particular for the neutrino masses and mixings. Hence they will remove the 'accidental' degeneracies which one obtains if one only considers the inner-tetrahedral Heisenberg exchange contributions to the eigenfrequencies.

What can be done already at this point is to take the formulas (28) and (30) for the second family and to adopt it to the known mass values of the muon, the charmed and the strange quark. It is advisable to take the running values of the masses at order 1 TeV[29] and try to determine from that the values for the internal exchange couplings. One obtains

$$j_{ST} \approx -0.14 \text{ GeV} \qquad j_{TT} = j_{SS} \approx -0.12 \text{ GeV} \qquad k_{ST} \approx 0.51 \text{ GeV} \tag{31}$$

and concludes that the coupling strengths are ≤ 1 GeV and are dominated by the coupling k_{ST} of adjacent spin vectors $\vec{S}_i$ and $\vec{T}_i$, while the interactions with $i \neq j$ are somewhat smaller in magnitude.

As discussed before, the inter-tetrahedral interactions yield the SSB contributions to the fermion masses. To account for these contributions one should include a sum over neighboring tetrahedrons in eq. (23) and add the corresponding interactions. Since I do not have the resources to treat these many terms properly the following trick will be used to approximately solve the problem: the effects of all other tetrahedrons on a given tetrahedron fig. 3 will be subsumed as an effective contribution generated by the internal spins. The idea is that this effective contribution can be attributed to the gauge transformation eq. (3) which transfers the $\vec{\pi}$-field from the Higgs sector to the W-boson mass term. As well known such a transformation modifies the W-field by

$$\vec{W}_\mu \to \vec{W}_\mu - \frac{1}{g\Lambda_F}\partial_\mu \vec{\pi} - \frac{1}{\Lambda_F}\vec{\pi} \times \vec{W}_\mu \tag{32}$$

Thus, while the bilinear terms in $\vec{\pi}$ disappear from the Higgs potential eq. (2), they re-appear in the W-mass term of the Lagrangian. Furthermore, their sign is such that the antiferromagnetic $\vec{\pi} - \vec{\pi}$ coupling at high energies gets transformed into a ferromagnetic interaction plus an additional term which is due to the non-abelian nature of the gauge transformation. All in all

$$m_W^2 \vec{W}_\mu \vec{W}^\mu \to m_W \sum_{i,j=1}^{4} [\vec{A}_i \vec{A}_j + i(\vec{A}_i \times \vec{A}_j)(\vec{A}_k - \vec{A}_l)] \tag{33}$$

has to be added to the Heisenberg Hamiltonian (19). Here the notation $\vec{A}_i = \vec{S}_i - \vec{T}_i$ has been used to denote the axial component of the spin vectors which is related to the $\vec{\pi}$-transformation (3). The additional term with the cross product in it can be interpreted as a Dzyaloshinskii-Moriya component[10], a contribution which in solid state physics is sometimes used to describe leading anisotropic corrections to the ordinary Heisenberg equations of motion. Quite in general such a component stands for a tendency to form a rotational structure (instead of the ordinary ferromagnetic alignment of neighboring tetrahedrons depicted in fig. 2) simply because the DM-term tends to rotate the spin vectors instead of aligning them. In the present case it was deduced as a consequence of the gauge transformation eq. (3). Therefore the DM-term can be interpreted quite naturally, namely by that the $SU(2)_L$ gauge fields induce a curvature of the fiber bundle formed by the system of all tetrahedrons, and the DM-term simply takes care of this curvature effect to effectively maintain the aligned structure.

It may be noted that in general a DM-component to the Heisenberg Hamiltonian has a more complicated coupling structure than eq. (33), of the form $\sum \vec{D}_{ij}(\vec{A}_i \times \vec{A}_j)$[10]. However, in the present case the couplings $\vec{D}_{ij}$ are fixed by 2 symmetry requirements, namely that the term must be $SU(2)_L$ gauge invariant and that it must respect the $A_4 + S(S_4 - A_4)$ Shubnikov symmetry. While the former fixes the modulus of the DM coupling strength relatively to the Heisenberg term in eq. (33), the latter forces the direction of $\vec{D}_{ij}$ to be $\vec{A}_k - \vec{A}_l$[10], where $kl \neq ij$ is chosen such that the sign of the permutation (ijkl) of (1234) is positive[10].

Since $\vec{A}$ is a linear combination of the internal spin vectors $\vec{S}$ and $\vec{T}$, new terms for the equations of motion of $\vec{S}$ and $\vec{T}$ can be derived from the SSB contribution (33):

$$\begin{aligned}
\frac{d\vec{S}_i}{dt} &= \frac{m_W}{4}\{\vec{S}_i^0 \times [1 + i\vec{S}_i^0 \times] \sum_{j=1}^{4} \vec{S}_j + \vec{T}_i^0 \times [1 + i\vec{T}_i^0 \times] \sum_{j=1}^{4} \vec{T}_j\} \\
\frac{d\vec{T}_i}{dt} &= \frac{m_W}{4}\{\vec{T}_i^0 \times [1 + i\vec{T}_i^0 \times] \sum_{j=1}^{4} \vec{T}_j + \vec{S}_i^0 \times [1 + i\vec{S}_i^0 \times] \sum_{j=1}^{4} \vec{S}_j\}
\end{aligned} \tag{34}$$

Eqs. (34) have to be combined with (23) and give a quasi-democratic type of contribution $\sim m_W$ to the mass matrix, as made explicit in table 1, where each entry is seen to have a contribution $\sim u = 2m_W$. Evaluating the eigenvalues, this equips the top quark triplet with a mass term $2m_W$ while leaving the other quark and lepton masses unchanged.

Combining this with eq. (29) one obtains the sum rule $m_t = 2m_W + m_c$. Physically, the factor of 2 in this result can be interpreted as stemming from a double contribution of the vibration of $\vec{A}$ which is contained in both the internal spin vectors $\vec{S}$ and $\vec{T}$. Unfortunately, this is a rather crude result which holds only in the limit of vanishing Weinberg angle θ, i.e. neglecting the mixing with the photon. It is possible, however, to include the mixing effect by considering the Z-mass term of the SM Lagrangian $m_Z^2 Z_\mu Z^\mu$, where $m_Z \cos\theta = m_W$ and $Z = W_z \cos\theta - B \sin\theta$. As compared to (33) this induces additional terms in the e.o.m. for the internal spin vectors $\vec{S}$ and $\vec{T}$ which not only increases the prediction for the top mass but also leads to a lifting of some of the degeneracies in eqs. (28), (29) and (30).

The situation is further complicated by other possibilities of mixings among the various fields (17). In fact, the quantum numbers of the Higgs allow for a mixing with the z-component of the iso-vector $\vec{v}$. Similarly, the pseudo-scalar combination $\eta \sim \bar{\psi} i\gamma_5 \psi$ can mix with π_z according to

$$\pi_z \to \pi_z \cos\alpha + \eta \sin\alpha \tag{35}$$

A small but nontrivial mixing angle α will influence the mass eigenvalues in the following way: although $\bar{\psi} i\gamma_5 \psi$ is an internal (pseudo)scalar density and per se does not contribute directly to the spin wave spectrum, there is an indirect effect, because it modifies the equations of motion by furnishing the terms $\sim S_{zi} - T_{zi}$ with a factor $\cos\alpha$, i.e. with an anisotropy $\sim 1 - \cos\alpha \sim \alpha^2$. This modification leads to another lifting of degeneracies in eqs. (28), (29) and (30). A complete analysis of these effects, however, is postponed to a future publication.

The calculations presented so far are based on a certain interpretation of the Standard Model Higgs mechanism - a rather intuitive interpretation, if one is willing to accept that the quark and lepton spectrum is due to a discrete internal tetrahedral structure. As was shown it is possible to identify the terms in the SM Lagrangian responsible for the internal spin interactions, namely the quadratic part of the Higgs potential and the W-mass term. However, at this point 10 of the 24 possible excitations (the neutrinos, the electron and up and down quarks) are left massless.

As long as these 4 singlets and 2 triplets remain massless i.e. constant non-vibrating modes, it is no use trying to calculate the CKM matrix elements or even the mixing angles in the neutrino sector. To get rid of those degeneracies one has to relax a condition inherent in the classical Heisenberg model namely that the internal magnetic moments may be treated as classical 3-dimensional vector spins of fixed length. This condition is destroyed by quantum fluctuations in the quantum Heisenberg model and on the classical level by allowing for torsional vibrations. Although these (tiny) torsional effects have no counterpart in the SM Lagrangian it can be shown that the leading up-quark, down-quark and electron mass contributions are provided by isotropic torsional interactions of the internal spin vectors while the neutrino masses can be attributed to anisotropic effects within the torsional couplings.

To start with I am now going to write down the most general form of these torsional interactions. As argued above, torsion is not strictly forbidden in the system under consideration, and for the case of only one spin vector is simply induced by a contribution of the form $d\vec{S}/dt \sim \vec{S}$ to its time variation. In the case at hand with 8 spin vectors $\vec{S}_i$ and $\vec{T}_i$ its main effect is to allow vibrations along the local z-directions and thus lift the degeneracies of all zero modes. The terms supplementing the Heisenberg e.o.m. are

$$
\begin{aligned}
\frac{d\vec{S}_i}{dt} &= ie_{SS}\vec{S}_i + if_{SS}\sum_{j\neq i}\vec{S}_j + ie_{ST}\vec{T}_i + if_{ST}\sum_{j\neq i}\vec{T}_j \\
\frac{d\vec{T}_i}{dt} &= ie_{TT}\vec{T}_i + if_{TT}\sum_{j\neq i}\vec{T}_j + ie_{ST}\vec{S}_i + if_{ST}\sum_{j\neq i}\vec{S}_j
\end{aligned}
\tag{36}
$$

where e and f are the torsion coupling strengths, whose values are assumed to be

	S_{1x}	S_{2x}	S_{3x}	S_{4x}
S_{1x}	$i(\omega+8u)$	$-2iu$	$2iu$	$2iu$
S_{2x}	$-2iu$	$i(\omega+8u)$	$2iu$	$2iu$
-S_{3x}	$2iu$	$2iu$	$i(\omega+8u)$	$-2iu$
-S_{4x}	$2iu$	$2iu$	$-2iu$	$i(\omega+8u)$
T_{1x}	$-2iu$	$-2iu$	$2iu$	$2iu$
T_{2x}	$-2iu$	$-2iu$	$2iu$	$2iu$
-T_{3x}	$2iu$	$2iu$	$-2iu$	$-2iu$
-T_{4x}	$2iu$	$2iu$	$-2iu$	$-2iu$
S_{1y}	$4j+k_s+7u$	$-s-u$	$s+u$	$s+u$
-S_{2y}	$s+u$	$-4j-k_s-7u$	$-s-u$	$-s-u$
S_{3y}	$-s-u$	$-s-u$	$-4j-k_s-7u$	$s+u$
-S_{4y}	$s+u$	$s+u$	$-s-u$	$4j+k_s+7u$
T_{1y}	$-j+k-u$	$-j-u$	$j+u$	$j+u$
-T_{2y}	$j+u$	$j+k+u$	$-j-u$	$-j-u$
T_{3y}	$-j-u$	$-j-u$	$j+k+u$	$j+u$
-T_{4y}	$j+u$	$j+u$	$-j-u$	$-j-k-u$
S_{1z}	$-4j-k_s-7u$	$s+u$	$-s-u$	$-s-u$
-S_{2z}	$-s-u$	$4j+k_s+7u$	$s+u$	$s+u$
-S_{3z}	$-s-u$	$-s-u$	$-4j-k_s-7u$	$s+u$
S_{4z}	$s+u$	$s+u$	$-s-u$	$4j+k_s+7u$
T_{1z}	$j+k+u$	$j+u$	$-j-u$	$-j-u$
-T_{2z}	$-j-u$	$-j-k-u$	$j+u$	$j+u$
-T_{3z}	$-j-u$	$-j-u$	$j+k+u$	$j+u$
T_{4z}	$j+u$	$j+u$	$-j-u$	$-j-k-u$

Table 1: The 24×24 matrix that determines the eigenmodes of the system, including SSB as well as inner-tetrahedral Heisenberg effects. The minus signs in the first column of the table account for the effect of the matrices U on the l.h.s. of (26).

	T_{1x}	T_{2x}	T_{3x}	T_{4x}
S_{1x}	$-2iu$	$-2iu$	$2iu$	$2iu$
S_{2x}	$-2iu$	$-2iu$	$2iu$	$2iu$
-S_{3x}	$2iu$	$2iu$	$-2iu$	$-2iu$
-S_{4x}	$2iu$	$2iu$	$-2iu$	$-2iu$
T_{1x}	$i(\omega+8u)$	$-2iu$	$2iu$	$2iu$
T_{2x}	$-2iu$	$i(\omega+8u)$	$2iu$	$2iu$
-T_{3x}	$2iu$	$2iu$	$i(\omega+8u)$	$-2iu$
-T_{4x}	$2iu$	$2iu$	$-2iu$	$i(\omega+8u)$
S_{1y}	$-j-k-u$	$-j-u$	$j+u$	$j+u$
-S_{2y}	$j+u$	$j+k+u$	$-j-u$	$-j-u$
S_{3y}	$-j-u$	$-j-u$	$j+k+u$	$j+u$
-S_{4y}	$j+u$	$j+u$	$-j-u$	$-j-k-u$
T_{1y}	$4j+k_t+7u$	$-t-u$	$t+u$	$t+u$
-T_{2y}	$t+u$	$-4j-k_t-u$	$-t-u$	$-t-u$
T_{3y}	$-t-u$	$-t-u$	$-4j-k_t-7u$	$t+u$
-T_{4y}	$t+u$	$t+u$	$-t-u$	$4j+k_t+7u$
S_{1z}	$j+k+u$	$j+u$	$-j-u$	$-j-u$
-S_{2z}	$-j-u$	$-j-k-u$	$j+u$	$j+u$
-S_{3z}	$-j-u$	$-j-u$	$j+k+u$	$j+u$
S_{4z}	$j+u$	$j+u$	$-j-u$	$-j-k-u$
T_{1z}	$-4j-k_t+7u$	$t+u$	$-t-u$	$-t-u$
-T_{2z}	$-t-u$	$4j+k_t+7u$	$t+u$	$t+u$
-T_{3z}	$-t-u$	$-t-u$	$-4j-k_t-7u$	$t+u$
T_{4z}	$t+u$	$t+u$	$-t-u$	$4j+k_t+7u$

Table 2: Table 1 continued; SSB terms are given in terms of $u = 2m_W$. Note the u terms are distributed 'democratically' over all entries. Further abbreviations used: $j_{SS} = s$, $j_{TT} = t$, $j_{ST} = j$ and $k_{ST} = k$, $k_s = 3j_{SS} + k_{ST}$ and $k_t = 3j_{TT} + k_{ST}$.

	S_{1y}	S_{2y}	S_{3y}	S_{4y}
S_{1x}	$-4j-k_s-7u$	$-s-u$	$s+u$	$-s-u$
S_{2x}	$s+u$	$4j+k_s+7u$	$s+u$	$-s-u$
-S_{3x}	$-s-u$	$s+u$	$4j+k_s+7u$	$-s-u$
-S_{4x}	$-s-u$	$s+u$	$-s-u$	$-4j-k_s-7u$
T_{1x}	$j+k+u$	$-j-u$	$j+u$	$-j-u$
T_{2x}	$j+u$	$-j-k-u$	$j+u$	$-j-u$
-T_{3x}	$-j-u$	$-j-u$	$-j-k-u$	$j+u$
-T_{4x}	$-j-u$	$j+u$	$-j-u$	$j+k+u$
S_{1y}	$i(\omega+8u)$	$2iu$	$-2iu$	$2iu$
-S_{2y}	$2iu$	$i(\omega+8u)$	$2iu$	$-2iu$
S_{3y}	$-2iu$	$2iu$	$i(\omega+8u)$	$2iu$
-S_{4y}	$2iu$	$-2iu$	$2iu$	$i(\omega+8u)$
T_{1y}	$-2iu$	$2iu$	$-2iu$	$2iu$
-T_{2y}	$2iu$	$-2iu$	$2iu$	$-2iu$
T_{3y}	$-2iu$	$2iu$	$-2iu$	$2iu$
-T_{4y}	$2iu$	$-2iu$	$2iu$	$-2iu$
S_{1z}	$+4j+k_s+7u$	$s+u$	$-s-u$	$s+u$
-S_{2z}	$s+u$	$4j+k_s+7u$	$s+u$	$-s-u$
-S_{3z}	$s+u$	$-s-u$	$-4j-k_s-7u$	$-s-u$
S_{4z}	$-s-u$	$s+u$	$-s-u$	$-4j-k_s-7u$
T_{1z}	$-j-k-u$	$j+u$	$-j-u$	$j+u$
-T_{2z}	$j+u$	$-j-k-u-$	$j+u$	$-j-u$
-T_{3z}	$j+u$	$-j-u$	$j+k+u$	$-j-u$
T_{4z}	$-j-u$	$j+u$	$-j-u$	$j+k+u$

Table 3: Table 1 continued

	T_{1y}	T_{2y}	T_{3y}	T_{4y}
S_{1x}	$j+k+u$	$-j-u$	$j+u$	$-j-u$
S_{2x}	$j+u$	$-j-k-u$	$j+u$	$-j-u$
-S_{3x}	$-j-u$	$j+u$	$-j-k-u$	$j+u$
-S_{4x}	$-j-u$	$j+u$	$-j-u$	$j+k+u$
T_{1x}	$-4j-k_t-7u$	$-t-u$	$t+u$	$-t-u$
T_{2x}	$t+u$	$4j+k_t+7u$	$t+u$	$-t-u$
-T_{3x}	$-t-u$	$t+u$	$4j+k_t+7u$	$t+u$
-T_{4x}	$-t-u$	$t+u$	$-t-u$	$-4j-k_t-7u$
S_{1y}	$-2iu$	$2iu$	$-2iu$	$2iu$
-S_{2y}	$2iu$	$-2iu$	$2iu$	$-2iu$
S_{3y}	$-2iu$	$2iu$	$-2iu$	$2iu$
-S_{4y}	$2iu$	$-2iu$	$2iu$	$-2iu$
T_{1y}	$i(\omega+8u)$	$2iu$	$-2iu$	$2iu$
-T_{2y}	$2iu$	$i(\omega+8u)$	$2iu$	$-2iu$
T_{3y}	$-2iu$	$2iu$	$i(\omega+8u)$	$2iu$
-T_{4y}	$2iu$	$-2iu$	$2iu$	$i(\omega+8u)$
S_{1z}	$-j-k-u$	$j+u$	$-j-u$	$j+u$
-S_{2z}	$j+u$	$-j-k-u$	$j+u$	$-j-u$
-S_{3z}	$j+u$	$-j-u$	$j+k+u$	$-j-u$
S_{4z}	$-j-u$	$j+u$	$-j-u$	$j+k+u$
T_{1z}	$4j+k_t+7u$	$t+u$	$-t-u$	$t+u$
-T_{2z}	$t+u$	$4j+k_t+7u$	$t+u$	$-t-u$
-T_{3z}	$t+u$	$-t-u$	$-4j-k_t-7u$	$-t-u$
T_{4z}	$-t-u$	$t+u$	$-t-u$	$-4j-k_t-7u$

Table 4: Table 1 continued

	S_{1z}	S_{2z}	S_{3z}	S_{4z}
S_{1x}	$4j+k_s+7u$	$s+u$	$s+u$	$-s-u$
S_{2x}	$-s-u$	$-4j-k_s-7u$	$s+u$	$-s-u$
-S_{3x}	$s+u$	$-s-u$	$4j+k_s+7u$	$s+u$
-S_{4x}	$s+u$	$-s-u$	$-s-u$	$-4j-k_s-7u$
T_{1x}	$-j-k-u$	$j+u$	$j+u$	$-j-u$
T_{2x}	$-j-u$	$j+k+u$	$j+u$	$-j-u$
-T_{3x}	$j+u$	$-j-u$	$-j-k-u$	$j+u$
-T_{4x}	$j+u$	$-j-u$	$-j-u$	$j+k+u$
S_{1y}	$-4j-k_s-7u$	$-s-u$	$-s-u$	$s+u$
-S_{2y}	$-s-u$	$-4j-k_s-7u$	$s+u$	$-s-u$
S_{3y}	$s+u$	$-s-u$	$4j+k_s+7u$	$s+u$
-S_{4y}	$-s-u$	$s+u$	$s+u$	$4j+k_s+7u$
T_{1y}	$j+k+u$	$-j-u$	$-j-u$	$j+u$
-T_{2y}	$-j-u$	$j+k+u$	$j+u$	$-j-u$
T_{3y}	$j+u$	$-j-u$	$-j-k-u$	$j+u$
-T_{4y}	$-j-u$	$j+u$	$j+u$	$-j-k-u$
S_{1z}	$i(\omega+8u)$	$2iu$	$2iu$	$-2iu$
-S_{2z}	$2iu$	$i(\omega+8u)$	$-2iu$	$-2iu$
-S_{3z}	$2iu$	$-2iu$	$i(\omega+8u)$	$2iu$
S_{4z}	$-2iu$	$2iu$	$2iu$	$i(\omega+8u)$
T_{1z}	$-2iu$	$2iu$	$2iu$	$-2iu$
-T_{2z}	$2iu$	$-2iu$	$-2iu$	$2iu$
-T_{3z}	$2iu$	$-2iu$	$-2iu$	$2iu$
T_{4z}	$-2iu$	$2iu$	$2iu$	$-2iu$

Table 5: Table 1 continued

	T_{1z}	T_{2z}	T_{3z}	T_{4z}
S_{1x}	$-j-k-u$	$j+u$	$j+u$	$-j-u$
S_{2x}	$-j-u$	$j+k+u$	$j+u$	$-j-u$
-S_{3x}	$j+u$	$-j-u$	$-j-k-u$	$j+u$
-S_{4x}	$j+u$	$-j-u$	$-j-u$	$j+k+u$
T_{1x}	$4j+k_t+7u$	$t+u$	$t+u$	$-t-u$
T_{2x}	$-t-u$	$-4j-k_t-7u$	$t+u$	$-t-u$
-T_{3x}	$t+u$	$-t-u$	$4j+k_t+7u$	$t+u$
-T_{4x}	$t+u$	$-t-u$	$-t-u$	$-4j-k_t-7u$
S_{1y}	$j+k+u$	$-j-u$	$-j-u$	$j+u$
-S_{2y}	$-j-u$	$j+k+u$	$j+u$	$-j-u$
S_{3y}	$j+u$	$-j-u$	$-j-k-u$	$j+u$
-S_{4y}	$-j-u$	$j+u$	$j+u$	$-j-k-u$
T_{1y}	$-4j-k_t-7u$	$t+u$	$-t-u$	$t+u$
-T_{2y}	$-t-u$	$-4j-k_t-7u$	$+ut$	$-t-u$
T_{3y}	$t+u$	$-t-u$	$4j+k_t+7u$	$t+u$
-T_{4y}	$-t-u$	$t+u$	$t+u$	$4j+k_t+7u$
S_{1z}	$-2iu$	$2iu$	$2iu$	$-2iu$
-S_{2z}	$2iu$	$-2iu$	$-2iu$	$2iu$
-S_{3z}	$2iu$	$-2iu$	$-2iu$	$2iu$
S_{4z}	$-2iu$	$2iu$	$2iu$	$-2iu$
T_{1z}	$i(\omega+8u)$	$2iu$	$2iu$	$-2iu$
-T_{2z}	$2iu$	$i(\omega+8u)$	$-2iu$	$-2iu$
-T_{3z}	$2iu$	$-2iu$	$i(\omega+8u)$	$2iu$
T_{4z}	$-2iu$	$2iu$	$2iu$	$i(\omega+8u)$

Table 6: Table 1 continued

small compared to the exchange couplings considered so far. More precisely one has the natural hierarchy

$$e, f \sim O(MeV) \ll j, k \sim O(GeV) \ll \Lambda_F \alpha \ll \Lambda_F \tag{37}$$

so that the torsional couplings can indeed be expected to provide for the electron and the up- and down-quark mass with $m_{e,u,d}/m_{\mu,s,c} \sim 10^{-2}$. As will be shown in the next section neutrino masses are due to still smaller anisotropy effects among the torsional couplings.

The contributions (36) should be incorporated in the full 24×24 mass matrix given in tables 1-6. Then in principle the results can be used to accommodate all physical quark and lepton mass and mixing parameters. Due to lack of resources I have to leave the completion of these studies to future work. To get some understanding of the physics I will now concentrate on the lepton sector of the theory, which is somewhat easier to handle. During that course I will also describe in detail how the CKM and PMNS mixing matrices can be obtained from within the present framework.

4 (Not just) a Toy Model for Leptons

The mass problem for the leptons alone can be reduced from the tetrahedral configuration to the simple 1-dimensional structure depicted in fig. 4. As Eq. (28) indicates, the lepton masses do not depend on the couplings J_{SS} and J_{TT} but only on J_{ST}. As a matter of fact using some simple matrix algebra manipulations it may be shown that they can effectively be obtained from the configuration of fig. 4 with only 2 internal spin vectors $\vec{S}$ and $\vec{T}$ (instead of 8) which in the ground state point in the z-directions

$$\langle \vec{S} \rangle = \langle \vec{T} \rangle = (0, 0, 1) \tag{38}$$

To analyze the behavior of this system, I will start with the Heisenberg part of the interactions

$$V_H = -J \vec{S} \vec{T} \tag{39}$$

where J is identical to $4J_{ST}$ used in the last section. The factor of 4 is a geometrical factor arising from the reduction of the tetrahedral configuration.

Figure 4: The local ground state in the model for leptons, where the spin vectors $\vec{S}$ and $\vec{T}$ point in the z-direction. The parallel configuration is depicted here instead of the anti-parallel one in fig. 3. If only a Heisenberg interaction of the form (39) is included the combination $\vec{S}+\vec{T}$ remains static, thus giving rise to 3 zero modes corresponding to the 3 neutrinos. A fourth zero-mode appears, because the spin vectors will rotate only in the transversal (x, y)-plane, leaving torsional vibrations (i.e. in z-direction) as zero modes. If further interactions are added to the Heisenberg term, all 6 eigenmodes receive non-vanishing values.

The 6 d.o.f. of this system of 2 internal chiral spin vectors lead to 6 eigenmodes, and we are now going to show how the lepton masses and mixings, in particular the tiny neutrino masses may arise. The symmetry group of the ground state fig. 4 is $\{1, SR\}$, whose only non-trivial element is a reflection R at the x-y-plane followed by an internal time reversal S. This group has only singlet representations[27], according to which the 6 eigenmodes will be classified.

Adding a small universal torsional coupling $f \ll j$ the time evolution of the spin vectors is given by

$$\frac{d\vec{S}}{dt} = j\vec{S} \times \vec{T} + f(\vec{S} - \vec{T}) \qquad \frac{d\vec{T}}{dt} = j\vec{T} \times \vec{S} + f(\vec{T} - \vec{S}) \tag{40}$$

One then has to diagonalize the sum of the matrices given in tables 7 and 8 in order to obtain the eigenstates

$$\begin{aligned} \nu_e &= (0,0,1,0,0,1) \qquad e = (0,0,-1,0,0,1) \\ \nu_\mu &= (0,1,0,0,1,0) \qquad \mu = (-i,-1,0,i,1,0) \\ \nu_\tau &= (1,0,0,1,0,0) \qquad \tau = (i,-1,0,-i,1,0) \end{aligned} \tag{41}$$

	S_x	S_y	S_z	T_x	T_y	T_z
S_x	$i(\omega_f + v)$	$j + v$	0	$i(f - v)$	$-j - v$	0
S_y	$-j - v$	$i(\omega_f + v)$	0	$j + v$	$i(f - v)$	0
S_z	0	0	$i\omega_f$	0	0	if
T_x	$i(f - v)$	$-j - v$	0	$i(\omega_f + v)$	$j + v$	0
T_y	$j + v$	$i(f + v)$	0	$-j - v$	$i(\omega_f + v)$	0
T_z	0	0	if	0	0	$i\omega_f$

Table 7: The interaction matrix between the 2 chiral spin vectors $\vec{S}$ and $\vec{T}$ of figure 4 giving rise to electron-, muon- and tau-mass. In addition to the Heisenberg interaction eq. (39) inter-tetrahedral effects (v) and a universal torsional coupling (e) have been introduced. The neutrinos are still massless at this stage, corresponding to the 3 d.o.f. of $\vec{S} + \vec{T}$ which do not vibrate. The abbreviation $\omega_f = \omega - f$ is used.

and their masses/eigenfrequencies

$$\begin{aligned} \omega(\nu_e) &= 0 \qquad\qquad \omega(e) = 2f \\ \omega(\nu_\mu) &= 0 \qquad\qquad \omega(\mu) = 2(j + f) \\ \omega(\nu_\tau) &= 0 \qquad\qquad \omega(\tau) = 2(2v - j - f) \end{aligned} \tag{42}$$

In eq. (41) a 6-dimensional vector space of eigenvectors has been introduced in which the sum and difference $\vec{S} \pm \vec{T}$ are simply given by

$$\begin{aligned} S_x \pm T_x &= \frac{1}{\sqrt{2}}(1, 0, 0, \pm 1, 0, 0) \\ S_y \pm T_y &= \frac{1}{\sqrt{2}}(0, 1, 0, 0, \pm 1, 0) \\ S_z \pm T_z &= \frac{1}{\sqrt{2}}(0, 0, 1, 0, 0, \pm 1) \end{aligned} \tag{43}$$

An effective contribution $\sim v$ from the leading inter-tetrahedral interactions eq. (34) has been included in table 7 and eq. (42) which lifts the degeneracy between muon and τ-lepton. The natural hierarchy between these couplings can then be invoked to accommodate the lepton masses. The electron mass is naturally small as compared to m_μ and m_τ because the electron corresponds to a torsional vibration

	S_x	S_y	S_z	T_x	T_y	T_z
S_x	ie_{xy}	$-j_z$	j_{xy}	if_{xy}	$-l_z$	$-l_{xy}$
S_y	j_z	ie_{xy}	$-j_{xy}$	l_z	if_{xy}	l_{xy}
S_z	$-j_{xy}$	j_{xy}	ie_z	l_{xy}	$-l_{xy}$	if_z
T_x	if_{xy}	$-l_z$	$-l_{xy}$	ig_{xy}	$-m_z$	m_{xy}
T_y	l_z	if_{xy}	l_{xy}	m_z	ig_{xy}	$-m_{xy}$
T_z	l_{xy}	$-l_{xy}$	if_z	$-m_{xy}$	m_{xy}	ig_z

Table 8: The most general correction terms to the matrix in table 7.

(of $S_z - T_z$) in the z-direction and its mass gets only torsional contributions $\sim f \approx$ 0.25MeV. There is no contribution $\sim j$ to the electron mass because the Heisenberg interaction conserves each spin's fixed length and does not allow spin vibrations in the z-direction. Note the other mode in the z-direction, the one $\sim S_z + T_z$, is to describe the electron neutrino.

The Hamiltonian on which eq. (40) is based furthermore conserves the total spin $\vec{S}+\vec{T}$ of the system so that the 3 d.o.f. corresponding to this quantity do not vibrate. They give the neutrino states ν_e, ν_μ and ν_τ and can be interpreted as the Goldstone modes arising from the breaking of the internal Heisenberg spin SU(2)-symmetry.

Goldstone bosons? This sounds strange in view of the fact that neutrinos are fermions. The point is that one has to distinguish the dynamics in internal from that in physical space. In physical space the neutrinos are fermions, but in internal space they are described by (bosonic) excitations of the internal angular momentum $\vec{S} + \vec{T}$ which is the conserved quantity associated with the internal rotational symmetry. Applying Goldstone's theorem to the internal dynamics then yields 3 internal massless excitations - just as magnons are obtained as Goldstone bosons of the broken rotational symmetry in ordinary magnetic systems.

In order to obtain non-zero neutrino masses I have written down the most general interaction matrix including anisotropic and torsional forces in table 8. Compared to the leading terms in table 7 the new contributions must be tiny, as are the neutrino masses. More precisely, they should be of order at most $O(m_\nu/m_e) \approx 10^{-7}$.

It seems clear that corrections of such minuteness are difficult to handle quantitatively. Nevertheless, the present approach allows to analyze the question from which of the various sources appearing in table 8 the observed neutrino masses and mixings[1] actually arise. For example, an inverted hierarchy of neutrino masses which seems to be slightly favored by the present data can be accommodated quite easily, and measured values of the mixing angles will further determine the coupling parameters in table 8.

A complete numerical analysis of the whole parameter space is not undertaken in the present paper. Rather, one realistic example will be discussed now to see whether the neutrino measured parameters can be reproduced. Namely, a simple solution to the case of the inverted hierarchy is obtained by putting $j_{xy} = l_{xy} = m_{xy} \equiv \delta_j$ and $f_{xy} \equiv \delta_e$ and all other parameters in table 8 to zero. The result for the masses then is

$$\begin{aligned} \omega(\nu_e) &= -\delta_e & \omega(e) &= 2f \\ \omega(\nu_\mu) &= -\frac{1}{2}\{\delta_e + \sqrt{\delta_e^2 + 32\delta_j^2}\} & \omega(\mu) &= 2(j+f) + \delta_e \\ \omega(\nu_\tau) &= -\frac{1}{2}\{\delta_e - \sqrt{\delta_e^2 + 32\delta_j^2}\} & \omega(\tau) &= 2(2v + j - f) + \delta_e \end{aligned} \tag{44}$$

Assuming $\delta_e \gg \delta_j$ one can indeed reproduce the 'inverted hierarchy' of neutrino masses. To be definite we choose the values $\delta_j = 0.002$eV, $\delta_e = 0.05$eV, $f = 0.25$MeV $j = 50$MeV $v = 0.4$GeV in order to reproduce the current data, i.e.

$$m_2^2 - m_1^2 = 0.000063 \; eV^2 \tag{45}$$

$$m_3^2 - \frac{1}{2}(m_1^2 + m_2^2) = -0.0025 \; eV^2 \tag{46}$$

Next one has to check whether the mixing angles for the neutrino come out right. This is then the appropriate place to discuss the general strategy how mixing matrices can be obtained in the present framework. To determine the CKM quark mixing elements the eigenvectors of a complicated 24×24 problem have to be fixed. For the case of leptons with the 6×6 matrix in tables 7 and 8 the calculation of eigenvectors and PMNS mixing matrix elements is a relatively simple exercise.

One just has to remember that the mixing matrix is defined as the unitary transition matrix $U = (U_{\alpha a})$ between the mass eigenstates $\nu_\alpha, \alpha = e, \mu, \tau$, and the weak interaction eigenstates $\nu_a, a = 1, 2, 3$:

$$\nu_\alpha = \sum_a U_{\alpha a} \nu_a \longleftrightarrow \nu_a = \sum_\alpha U^*_{\alpha a} \nu_\alpha \tag{47}$$

Here the mass eigenstates can be directly identified with the eigenvectors of the given eigenvalue problem. In the above example they are given by

$$\begin{aligned}
\nu_e &= (-0.07 - 0.16i, -0.07 + 0.16i, -0.67, -0.07 - 0.16i, -0.07 + 0.16i, -0.67) \\
e &= (0, 0, 0.71, 0, 0, -0.71) \\
\nu_\mu &= (-0.08 - 0.47i, -0.48, 0.17 + 0.14i, -0.08 - 0.47i, -0.48, 0.17 + 0.14i) \\
\mu &= (-0.50i, -0.50, 0, 0.50i, 0.50, 0) \\
\nu_\tau &= (0.03 + 0.49i, 0.50, -0.07 + 0.08i, 0.03 + 0.49i, -0.50, -0.07 + 0.08i) \\
\tau &= (-0.50i, 0.50, 0, 0.50i, -0.50, 0) \qquad (48)
\end{aligned}$$

where the 6-dimensional notation of eqs. (43) has been used.

According to eq. (10) the difference $\vec{S} - \vec{T}$ gives the axial part of the spin vectors which after the SSB defines the longitudinal modes of the $\vec{W}$-bosons. Therefore the transformation from the mass eigenstates to the interaction eigenstates is given by the projection matrix of the eigenvectors (48) to the axial spin vector components:

$$U_{\alpha a} = (\nu_\alpha).\frac{S_a - T_a}{\sqrt{2}} = \begin{pmatrix} 0.94 & 0.093 - 0.223i & 0.093 + 0.223i \\ -0.23 - 0.20i & 0.67 & 0.111 + 0.66i \\ 0.101 - 0.108i & 0.70 & -0.048 - 0.70i \end{pmatrix} \tag{49}$$

where the dot denotes the euclidean scalar product in the 6-dimensional space of eigenvectors. There is a CP violating effect in this matrix, because the Jarlskog invariant[33] J_{CP} is non-zero and given by $J_{CP} = -0.023$.

Unfortunately, the result does not really reproduce the observed mixing angles[1, 30]. A complete scan of the full parameter space seems unavoidable. This effort will be undertaken in future work.

5 Conclusions

In the present paper a microscopic model for the SM Higgs mechanism has been applied to determine the quark and lepton masses and mixing angles. A discrete tetrahedral structure within a dynamical internal space has managed to fill the gap between the phenomenological hierarchy of mass scales. The underlying physical picture is that the universe resembles a huge crystal of internal molecules, each 'molecule' of tetrahedral form and arranged in such a way that certain symmetries are (spontaneously) broken. For such a model to be consistent, a (6+1)-dimensional space time has been introduced in ref.[2], i.e. the 'molecules' extend to 3 internal dimensions orthogonal to physical space, and they interact via a (6+1)-dimensional QED featuring the necessary 'iso-magnetic' forces. The strong correlations within this system provide for the Higgs particle and the weak vector bosons as bound states. Furthermore, internal spin excitations turn out to generate the correct quark and lepton spectrum. Then, it happens that an excitation in one internal tetrahedron is able to excite an excitation in the neighboring internal space and thus can travel as a quasi-particle through Minkowski space with a certain wave vector $\vec{k}$ which is to be interpreted as the physical momentum of the quark or lepton.

In this model, the SM symmetry breaking can be understood to proceed in 2 steps:

- the formation of a tetrahedron due to a new internal interaction force, which is 'antiferromagnetic' at small distances and leads to a frustrated configuration of isospin vectors fig. 1. The frustrated tetrahedron breaks the internal spin-SU(2) as well as internal parity and time reversal to the Shubnikov group $A_4 + S(S_4 - A_4)$. This symmetry breaking is not spontaneous but arises from the internal antiferromagnetic exchange interaction which avoids parallel spin states. The resulting local ground state fig. 1 is a chiral configuration, i.e. it violates internal and (as shown in ref.[2]) external parity, and the whole system is left $SU(2)_L$-symmetric in the following sense:

- each local tetrahedral ground state can rotate independently of the others, i.e. it can freely rotate as a rigid body over its base point in Minkowski space, and this rotational symmetry of the rigid chiral spin vector system corresponds to a $SO(3)$ symmetry group, whose covering group is taken to define $SU(2)_L$. As a matter of fact it is a local symmetry, because the rotation can be different for tetrahedrons over different base points. Furthermore, due to the $V - A$ structure of the interactions induced by the chiral tetrahedral structure, it is a symmetry involving only left handed particles[2]. At large distances of the order of the Fermi scale the new interactions are (internally) ferromagnetic in nature and give rise to the global ferromagnetic order shown in fig. 2. Finally, the non-vanishing vev for the Higgs field $\sim \bar{\psi}\psi$ is due to a pairing mechanism as described in ref.[2].

In order to analyze the mass problem of the fermions within this model, the most general $SU(2)_L \times SU(2)_R$ invariant NJL Lagrangian has been used[18] as an effective approximation. Afterwards, a Heisenberg Hamiltonian for the internal spin vector interactions has been derived from that Lagrangian. This is justified because at the stage when the internal tetrahedron is formed chiral symmetry is still valid, so that one can describe the internal spin vibrations in terms of the chiral spin vectors $\vec{S}$ and $\vec{T}$.

Concerning the 'ferromagnetic' attraction between different tetrahedrons at large distances responsible for the SSB, it was noticed in section 3 that the gauge structure enforces an additional term which resembles the so called Dzyaloshinskii-Moriya interactions[10] of solid state physics. This term was interpreted quite naturally as a curvature effect of the $SU(2)_L$ gauge fields induced in the fiber bundle formed by the system of all tetrahedrons. The DM term simply takes care of this curvature to effectively maintain the 'ferromagnetic' order fig. 2.

Based on the prescribed model expressions for the quark and lepton mass spectrum were derived. It turned out that the extreme hierarchy in this spectrum can be attributed to the fact that the masses of different fermions get contributions from physically different sources, namely

- the top mass is dominated by a contribution of order Λ_F which stems from the SSB *inter*-tetrahedral DM interactions. Physically it arises because the top quark corresponds to the 3 eigenmodes which 'disturb' the global ground state in the strongest possible way. This disturbance is also responsible for the hierarchy observed in the CKM matrix elements.

- strange-, charm- and muon-mass are dominated by antiferromagnetic exchange couplings within one tetrahedron, and thus can be obtained from the *inner*-tetrahedral Heisenberg exchange couplings alone.

- down-quark, up-quark and electron get their relatively small masses from energetically favored torsion contributions, which only concern 'radial' excitations of the internal spin vectors.

- neutrino masses are protected by symmetry, because they correspond to vibrations in the valleys of the potential where all Heisenberg and even most of the torsional energy contributions vanish. This was shown to be related to a Goldstone effect of the internal rotational symmetry obeyed by the leading Heisenberg spin interactions.

Furthermore, the question of quark and lepton mixing was considered, albeit not in a very elaborate way. The quark mixing is a complicated 24×24 eigenvector problem with many parameters and a detailed analysis therefore postponed to future work. An attempt to determine the lepton mixing parameters was made. It turned out that the phenomenological values for the PMNS mixing angles cannot be obtained in a straightforward manner. The upshot of the discussion presented in this paper is that an accommodation to the measured neutrino properties is a non-trivial calculational exercise, because a complete scan over the parameter space of the anisotropic torsional couplings is needed.

To summarize, the quark and lepton masses have been successfully reduced to couplings among the internal spin vectors. Using these results the poor man's strategy (applied in this paper) is to choose the couplings so that the fermion masses and mixings come out right. The reader may rightfully object that everything done here is to replace one set of free parameters by another set. However, as shown in ref.[2] the internal couplings can be calculated as exchange integrals over internal space just as in ordinary magnetism the exchange couplings of the Heisenberg model is in principle calculable from exchange integrals of electronic wave functions over physical space. A more ambitious program therefore is to determine all internal couplings from a fundamental theory like the higher dimensional QED which has been considered in [2].

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