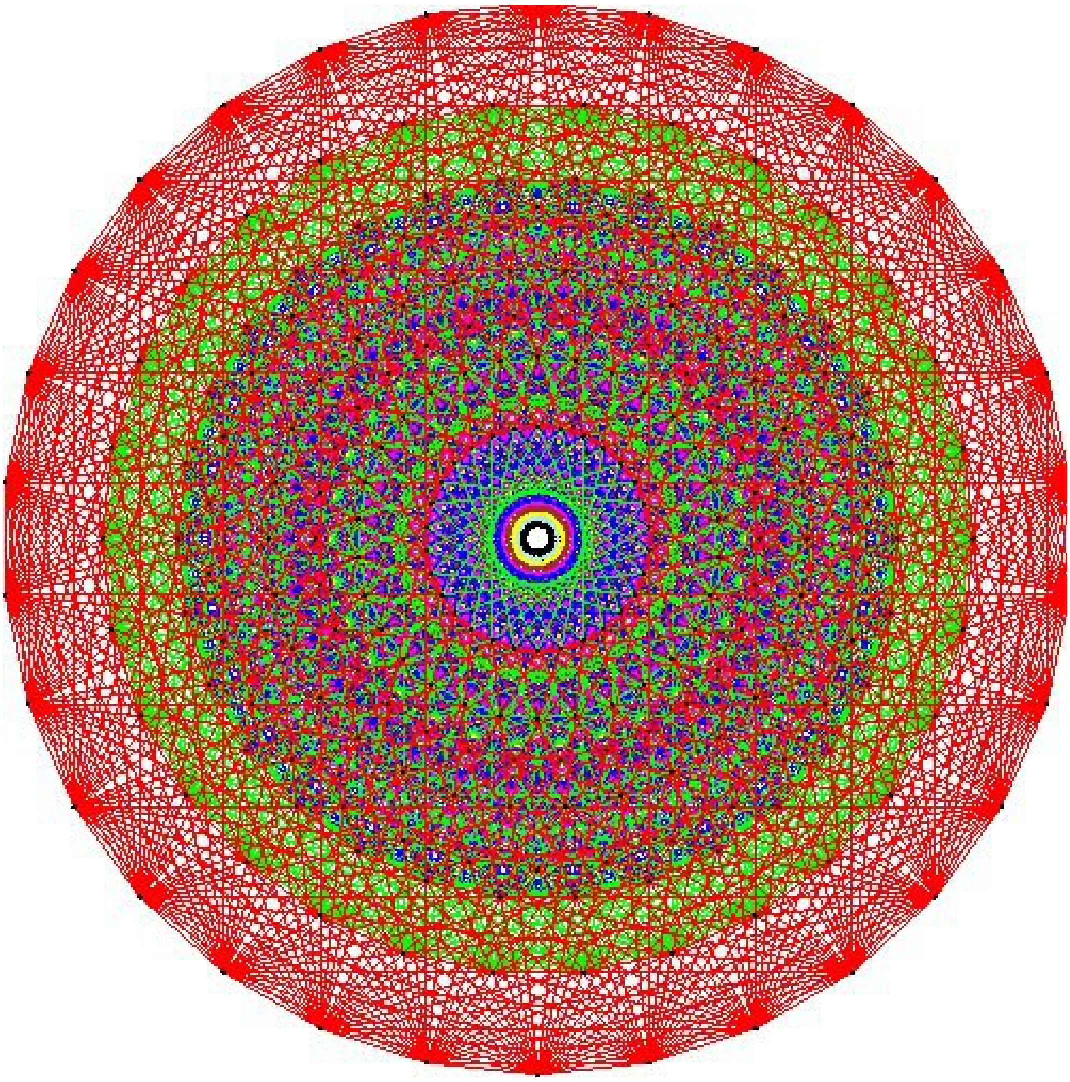
# Hopf Fibration and E8 in Vedic Physics

By John Frederick Sweeney



Abstract

A parallel construction exists in Vedic Nuclear Physics which appears to be the Exceptional Lie Algebra E8 and the Hopf Fibration. This paper describes the key sphere H7 in Vedic Physics and then attempts to draw isomorphic relationships between the structures. In this way, this paper attempts to explain the relationship between E8 and the Hopf Fibration, which is not well understood generally, in the hope that Vedic Nuclear Physics may provide a heuristic model. Along the way, this paper describes the life cycle of galaxies, from birth to final destruction into Dark Matter.

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# Introduction

In recent years the Exceptional Lie Algebra E8 has taken center stage as somehow central to a Theory of Everything (TOE), first under Garret Lisi, who brought E8 to international fame, and then others who have criticized or supported Lisi. No one really has a definitive answer, and much of what was written about E8 is pure conjecture, based on K Theory or M Theory or Super String or …none of which have much to support them in any event.

In contrast, Vedic Physics knows precisely what E8 is and what it does, and what comes after E8 in the sequence of the formation of matter, all included herein.

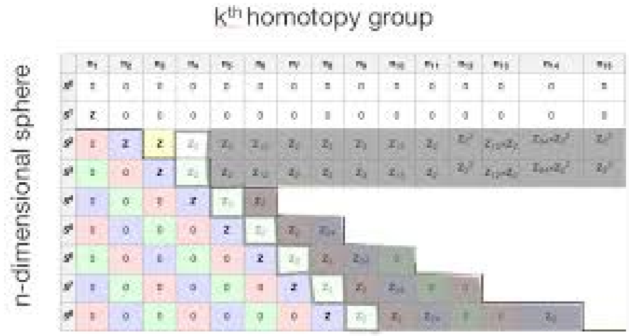
Western mathematical physics knows comparatively little, in fact, since it develops in linear fashion, moving from point A to point B to point C, etc. No one knows whether Point C was in fact the proper and correct next step, for it could lead to a blind alley or cul - de - sac, detaining scientific progress for a century or more.

For example, we have had Hamiltonians since around 1850, but failed to come up with Bott Periodicity until 1950, because late - Nineteenth Century mathematicians disregarded quarternions. Having devised Bott Periodicity, no mathematician considered to link that with Pisano (Fibonacci) Periodicity until the present author did so in a paper published on Vixra in 2013. In recent years Nobel Laureate Sir Roger Penrose has disdained the Octonions as “useless” in terms of physics, yet this criticism fortunately has failed to inhibit research into this area.

In fact, ancient science had already fully articulated quarternions, octonions and beyond, some 15,000 years ago in Egypt, Hindu culture and in China much later, but western mathematics has such short cultural memory that it never acknowledged the connections. Worse yet, western mathematics has taken claim for “discoveries” which originated in these ancient areas, or else ascribed their authorship to ancient Greece.

As western mathematics undertakes to re - invent the wheel, progress could be enhanced with reference to the more advanced science of extremely ancient Hindu culture. Hindu or Vedic Science is a fully - articulated, advanced science from a far more advanced society, which was intended to survive natural disasters such as Ice Ages, Pole Shift and sheer human ignorance. After all, it has taken this civilization, if it can be called that, some 13,000 years of evolution to reach the point where, by standing on tiptoe, we can begin to glimpse the marvels of ancient science - just prior to the next global debacle.

The author has noticed many parallel qualities between Exceptional Lie Algebras and what Vedic Nuclear Physics terms as Hyper Circles, which have been described in detail and given numerical quantities. Moreover, the Sanskrit text of the Rig Veda which describes Hyper Circles specifically contains the word for “fibre,” which heuristically suggests the Hopf Fibration. This paper attempts to draw isomorphic relationships between the Exceptional Lie Algebras and Hyper Circles, as well as the Hopf Fibration in order to shed light on the proper relationship between these concepts.




Jorge O’Farrell - Figueroa has done some interesting work in this area, but has since failed to follow the thread. John Baez re - worked this, but in a recent email admitted that he has never followed up on this thread since 2008. An email to O’Farrell - Figueroa awaits response, but it is likely that most mathematicians don’t like to be reminded about research promises they have failed to fulfill over the years. Nevertheless, this is but a small matter involving the Magic Square of Lie Algebras, and the author has done his best to ameliorate the lapses in the field.

# Magic Square of Exceptional Lie Algebras

| A \ B | R | C | H | O |
|---|---|---|---|---|
| **R** | $A_1$ | $A_2$ | $C_3$ | $F_4$ |
| **C** | $A_2$ | $A_2 \times A_2$ | $A_5$ | $E_6$ |
| **H** | $C_3$ | $A_5$ | $D_6$ | $E_7$ |
| **O** | $F_4$ | $E_6$ | $E_7$ | $E_8$ |

In mathematics, the **Freudenthal magic square** (or **Freudenthal–Tits magic square**) is a construction relating severalLie algebras (and their associated Lie groups). It is named after Hans Freudenthal and Jacques Tits, who developed the idea independently. It associates a Lie algebra to a pair of division algebras *A*, *B*. The resulting Lie algebras have Dynkin diagrams according to the table at right. The "magic" of the Freudenthal magic square is that the constructed Lie algebra is symmetric in *A* and *B*, despite the original construction not being symmetric, though Vinberg's symmetric method gives a symmetric construction; it is not a magic square as in recreational mathematics.
The Freudenthal magic square includes all of the exceptional Lie groups apart from $G_2$, and it provides one possible approach to justify the assertion that "the exceptional Lie groups all exist because of the octonions": $G_2$ itself is theautomorphism group of the octonions (also, it is in many ways like a classical Lie group because it is the stabilizer of a generic 3-form on a 7-dimensional vector space – see prehomogeneous vector space).

See history for context and motivation. These were originally constructed circa 1958 by Freudenthal and Tits, with more elegant formulations following in later years.[1]

**Tits' approach**[edit]

Tits' approach, discovered circa 1958 and published in (Tits 1966), is as follows.

Associated with any normed real division algebra $A$ (i.e., R, C, H or O) there is a Jordan algebra, $J_3(A)$, of 3 × 3 $A$-Hermitian matrices. For any pair ($A$, $B$) of such division algebras, one can define a Lie algebra

$$L = (\mathfrak{der}(A) \oplus \mathfrak{der}(J_3(B))) \oplus (A_0 \otimes J_3(B)_0)$$

where *der* denotes the Lie algebra of derivations of an algebra, and the subscript 0 denotes the trace-free part. The Lie algebra $L$ has $\mathfrak{der}(A) \oplus \mathfrak{der}(J_3(B))$ as a subalgebra, and this acts naturally on $A_0 \otimes J_3(B)_0$. The Lie bracket on $A_0 \otimes J_3(B)_0$ (which is not a subalgebra) is not obvious, but Tits showed how it could be defined, and that it produced the following table of compact Lie algebras.

| | B | R | C | H | O |
|---|---|---|---|---|---|
| **A** | ***der*(A/B)** | **0** | **0** | $\mathfrak{sp}_1$ | $\mathfrak{g}_2$ |
| **R** | **0** | $\mathfrak{so}_3$ | $\mathfrak{su}_3$ | $\mathfrak{sp}_3$ | $\mathfrak{f}_4$ |
| **C** | **0** | $\mathfrak{su}_3$ | $\mathfrak{su}_3 \oplus \mathfrak{su}_3$ | $\mathfrak{su}_6$ | $\mathfrak{e}_6$ |
| **H** | $\mathfrak{sp}_1$ | $\mathfrak{sp}_3$ | $\mathfrak{su}_6$ | $\mathfrak{so}_{12}$ | $\mathfrak{e}_7$ |
| **O** | $\mathfrak{g}_2$ | $\mathfrak{f}_4$ | $\mathfrak{e}_6$ | $\mathfrak{e}_7$ | $\mathfrak{e}_8$ |

Note that by construction, the row of the table with $A$=**R** gives **der**($J_3(B)$), and similarly vice versa.

## Vinberg's symmetric method[edit]

The "magic" of the Freudenthal magic square is that the constructed Lie algebra is symmetric in $A$ and $B$. This is not obvious from Tits' construction. Ernest Vinberg gave a construction which is manifestly symmetric, in (Vinberg 1966).

Instead of using a Jordan algebra, he uses an algebra of skew-hermitian trace-free matrices with entries in $A \otimes B$, denoted $\mathfrak{sa}_3(A \otimes B)$. Vinberg defines a Lie algebra structure on

$$\mathfrak{der}(A) \oplus \mathfrak{der}(B) \oplus \mathfrak{sa}_3(A \otimes B).$$

When $A$ and $B$ have no derivations (i.e., **R** or **C**), this is just the Lie (commutator) bracket on $\mathfrak{sa}_3(A \otimes B)$. In the presence of derivations, these form a subalgebra acting naturally on $\mathfrak{sa}_3(A \otimes B)$ as in Tits' construction, and the tracefree commutator bracket on $\mathfrak{sa}_3(A \otimes B)$ is modified by an expression with values in $\mathfrak{der}(A) \oplus \mathfrak{der}(B)$.

## Triality[edit]

A more recent construction, due to Pierre Ramond (Ramond 1976) and Bruce Allison (Allison 1978) and developed by Chris Barton and Anthony Sudbery, uses triality in the form developed by John Frank Adams; this was presented in (Barton & Sudbery 2000), and in streamlined form in (Barton & Sudbery 2003). Whereas Vinberg's construction is based on the automorphism groups of a division algebra $A$ (or rather their Lie algebras of derivations), Barton and Sudbery use the group of automorphisms of the corresponding triality. The triality is the trilinear map

$$A_1 \times A_2 \times A_3 \to \mathbf{R}$$

obtained by taking three copies of the division algebra *A*, and using the inner product on *A* to dualize the multiplication. The automorphism group is the subgroup of SO($A_1$) × SO($A_2$) × SO($A_3$) preserving this trilinear map. It is denoted Tri(*A*). The following table compares its Lie algebra to the Lie algebra of derivations.

| *A*: | R | C | H | O |
|---|---|---|---|---|
| $\mathfrak{der}(A)$ | 0 | 0 | $\mathfrak{sp}_1$ | $\mathfrak{g}_2$ |
| $\mathfrak{tri}(A)$ | 0 | $\mathfrak{u}_1 \oplus \mathfrak{u}_1$ | $\mathfrak{sp}_1 \oplus \mathfrak{sp}_1 \oplus \mathfrak{sp}_1$ | $\mathfrak{so}_8$ |

Barton and Sudbery then identify the magic square Lie algebra corresponding to (*A*,*B*) with a Lie algebra structure on the vector space

$$\mathfrak{tri}(A) \oplus \mathfrak{tri}(B) \oplus (A_1 \otimes B_1) \oplus (A_2 \otimes B_2) \oplus (A_3 \otimes B_3).$$

The Lie bracket is compatible with a $\mathbf{Z}_2 \times \mathbf{Z}_2$ grading, with **tri**(*A*) and **tri**(*B*) in degree (0,0), and the three copies of *A* ⊗ *B* in degrees (0,1), (1,0) and (1,1). The bracket preserves **tri**(*A*) and **tri**(*B*) and these act naturally on the three copies of *A* ⊗ *B*, as in the other constructions, but the brackets between these three copies are more constrained.

For instance when *A* and *B* are the octonions, the triality is that of Spin(8), the double cover of SO(8), and the Barton-Sudbery description yields

$$\mathfrak{e}_8 \cong \mathfrak{so}_8 \oplus \widehat{\mathfrak{so}}_8 \oplus (V \otimes \widehat{V}) \oplus (S_+ \otimes \widehat{S}_+) \oplus (S_- \otimes \widehat{S}_-)$$

where V, $S_+$ and $S_-$ are the three 8 dimensional representations of $\mathfrak{so}_8$ (the fundamental representation and the two spin representations), and the hatted objects are an isomorphic copy.

With respect to one of the $\mathbf{Z}_2$ gradings, the first three summands combine to give $\mathfrak{so}_{16}$ and the last two together form one of its spin representations $\Delta_+^{128}$ (the superscript denotes the dimension). This is a well known symmetric decomposition of E8.

The Barton-Sudbery construction extends this to the other Lie algebras in the magic square. In particular, for the exceptional Lie algebras in the last row (or column), the symmetric decompositions are:

$$\mathfrak{f}_4 \cong \mathfrak{so}_9 \oplus \Delta^{16}$$
$$\mathfrak{e}_6 \cong (\mathfrak{so}_{10} \oplus \mathfrak{u}_1) \oplus \Delta^{32}$$
$$\mathfrak{e}_7 \cong (\mathfrak{so}_{12} \oplus \mathfrak{sp}_1) \oplus \Delta_+^{64}$$
$$\mathfrak{e}_8 \cong \mathfrak{so}_{16} \oplus \Delta_+^{128}.$$

# Generalizations[edit]

## Split composition algebras[edit]

In addition to the normed division algebras, there are other composition algebras over **R**, namely the split-complex numbers, the split-quaternionsand the split-octonions. If one uses these instead of the complex numbers, quaternions, and octonions, one obtains the following variant of the magic square (where the split versions of the division algebras are denoted by a dash).

| A\B | R | C' | H' | O' |
|---|---|---|---|---|
| R | $\mathfrak{so}_3$ | $\mathfrak{sl}_3(\mathbf{R})$ | $\mathfrak{sp}_6(\mathbf{R})$ | $\mathfrak{f}_{4(4)}$ |
| C' | $\mathfrak{sl}_3(\mathbf{R})$ | $\mathfrak{sl}_3(\mathbf{R}) \oplus \mathfrak{sl}_3(\mathbf{R})$ | $\mathfrak{sl}_6(\mathbf{R})$ | $\mathfrak{e}_{6(6)}$ |
| H' | $\mathfrak{sp}_6(\mathbf{R})$ | $\mathfrak{sl}_6(\mathbf{R})$ | $\mathfrak{so}_{6,6}$ | $\mathfrak{e}_{7(7)}$ |
| O' | $\mathfrak{f}_{4(4)}$ | $\mathfrak{e}_{6(6)}$ | $\mathfrak{e}_{7(7)}$ | $\mathfrak{e}_{8(8)}$ |

Here all the Lie algebras are the split real form except for **so**$_3$, but a sign change in the definition of the Lie bracket can be used to produce the split form **so**$_{2,1}$. In particular, for the exceptional Lie algebras, the maximal compact subalgebras are as follows:

| Split form | $\mathfrak{f}_{4(4)}$ | $\mathfrak{e}_{6(6)}$ | $\mathfrak{e}_{7(7)}$ | $\mathfrak{e}_{8(8)}$ |
|---|---|---|---|---|
| Maximal compact | $\mathfrak{sp}_3 \oplus \mathfrak{sp}_1$ | $\mathfrak{sp}_4$ | $\mathfrak{su}_8$ | $\mathfrak{so}_{16}$ |

A non-symmetric version of the magic square can be obtained by combining the split algebras with the usual division algebras. According to Barton and Sudbery, the resulting table of Lie algebras is as follows.

| A\B | R | C | H | O |
|---|---|---|---|---|
| R | $\mathfrak{so}_3$ | $\mathfrak{su}_3$ | $\mathfrak{sp}_3$ | $\mathfrak{f}_4$ |
| C' | $\mathfrak{sl}_3(\mathbf{R})$ | $\mathfrak{sl}_3(\mathbf{C})$ | $\mathfrak{sl}_3(\mathbf{H})$ | $\mathfrak{e}_{6(-26)}$ |
| H' | $\mathfrak{sp}_6(\mathbf{R})$ | $\mathfrak{su}_{3,3}$ | $\mathfrak{so}^*_6(\mathbf{H})$ | $\mathfrak{e}_{7(-25)}$ |
| O' | $\mathfrak{f}_{4(4)}$ | $\mathfrak{e}_{6(2)}$ | $\mathfrak{e}_{7(-5)}$ | $\mathfrak{e}_{8(-24)}$ |

The real exceptional Lie algebras appearing here can again be described by their maximal compact subalgebras.

| | | | | | |
|---|---|---|---|---|---|
| **Lie algebra** | $\mathfrak{e}_{6(2)}$ | $\mathfrak{e}_{6(-26)}$ | $\mathfrak{e}_{7(-5)}$ | $\mathfrak{e}_{7(-25)}$ | $\mathfrak{e}_{8(-24)}$ |
| **Maximal compact** | $\mathfrak{su}_6 \oplus \mathfrak{sp}_1$ | $\mathfrak{f}_4$ | $\mathfrak{su}_{12} \oplus \mathfrak{sp}_1$ | $\mathfrak{e}_6 \oplus \mathfrak{u}_1$ | $\mathfrak{e}_7 \oplus \mathfrak{sp}_1$ |

# Arbitrary fields[edit]

The split forms of the composition algebras and Lie algebras can be defined over any field **K**. This yields the following magic square.

| | | | |
|---|---|---|---|
| $\mathfrak{so}_3(\mathbf{K})$ | $\mathfrak{sl}_3(\mathbf{K})$ | $\mathfrak{sp}_6(\mathbf{K})$ | $\mathfrak{f}_4(\mathbf{K})$ |
| $\mathfrak{sl}_3(\mathbf{K})$ | $\mathfrak{sl}_3(\mathbf{K}) \oplus \mathfrak{sl}_3(\mathbf{K})$ | $\mathfrak{sl}_6(\mathbf{K})$ | $\mathfrak{e}_6(\mathbf{K})$ |
| $\mathfrak{sp}_6(\mathbf{K})$ | $\mathfrak{sl}_6(\mathbf{K})$ | $\mathfrak{so}_{12}(\mathbf{K})$ | $\mathfrak{e}_7(\mathbf{K})$ |
| $\mathfrak{f}_4(\mathbf{K})$ | $\mathfrak{e}_6(\mathbf{K})$ | $\mathfrak{e}_7(\mathbf{K})$ | $\mathfrak{e}_8(\mathbf{K})$ |

There is some ambiguity here if **K** is not algebraically closed. In the case **K** = **C**, this is the complexification of the Freudenthal magic squares for **R** discussed so far.

# More general Jordan algebras[edit]

The squares discussed so far are related to the Jordan algebras $J_3(A)$, where $A$ is a division algebra. There are also Jordan algebras $J_n(A)$, for any positive integer $n$, as long as $A$ is associative. These yield split forms (over any field **K**) and compact forms (over **R**) of generalized magic squares.

| | | |
|---|---|---|
| $\mathfrak{so}_n(\mathbf{K})$ | $\mathfrak{sl}_n(\mathbf{K})$ or $\mathfrak{su}_n$ | $\mathfrak{sp}_{2n}(\mathbf{K})$ or $\mathfrak{sp}_n$ |
| $\mathfrak{sl}_n(\mathbf{K})$ or $\mathfrak{su}_n$ | $\mathfrak{sl}_n(\mathbf{K}) \oplus \mathfrak{sl}_n(\mathbf{K})$ or $\mathfrak{su}_n \oplus \mathfrak{su}_n$ | $\mathfrak{sl}_{2n}(\mathbf{K})$ or $\mathfrak{su}_{2n}$ |
| $\mathfrak{sp}_{2n}(\mathbf{K})$ or $\mathfrak{sp}_n$ | $\mathfrak{sl}_{2n}(\mathbf{K})$ or $\mathfrak{su}_{2n}$ | $\mathfrak{so}_{4n}(\mathbf{K})$ |

For $n$=2, $J_2(O)$ is also a Jordan algebra. In the compact case (over **R**) this yields a magic square of orthogonal Lie algebras.

| A\B | R | C | H | O |
| --- | --- | --- | --- | --- |
| R | $\mathfrak{so}_2$ | $\mathfrak{so}_3$ | $\mathfrak{so}_5$ | $\mathfrak{so}_9$ |
| C | $\mathfrak{so}_3$ | $\mathfrak{so}_4$ | $\mathfrak{so}_6$ | $\mathfrak{so}_{10}$ |
| H | $\mathfrak{so}_5$ | $\mathfrak{so}_6$ | $\mathfrak{so}_8$ | $\mathfrak{so}_{12}$ |
| O | $\mathfrak{so}_9$ | $\mathfrak{so}_{10}$ | $\mathfrak{so}_{12}$ | $\mathfrak{so}_{16}$ |

The last row and column here are the orthogonal algebra part of the isotropy algebra in the symmetric decomposition of the exceptional Lie algebras mentioned previously.These constructions are closely related to hermitian symmetric spaces – cf. prehomogeneous vector spaces.

## Symmetric spaces[edit]

Riemannian symmetric spaces, both compact and non-compact, can be classified uniformly using a magic square construction, in (Huang & Leung 2011). The irreducible compact symmetric spaces are, up to finite covers, either a compact simple Lie group, a Grassmannian, a Lagrangian Grassmannian, or a double Lagrangian Grassmannian of subspaces of $(\mathbf{A}\otimes\mathbf{B})^n,$ for normed division algebras **A** and **B**. A similar construction produces the irreducible non-compact symmetric spaces.

# History[edit]

## Rosenfeld projective planes[edit]

Following Ruth Moufang's discovery in 1933 of the Cayley projective plane or "octonionic projective plane" $\mathbf{P}^2(\mathbf{O})$, whose symmetry group is the exceptional Lie group $F_4$, and with the knowledge that $G_2$ is the automorphism group of the octonions, it was proposed by Rozenfeld (1956) that the remaining exceptional Lie groups $E_6$, $E_7$, and $E_8$ are isomorphism groups of projective planes over certain algebras over the octonions:[1]

- the **bioctonions**, $\mathbf{C}\otimes\mathbf{O}$,
- the **quateroctonions**, $\mathbf{H}\otimes\mathbf{O}$,
- the **octooctonions**, $\mathbf{O}\otimes\mathbf{O}$.

This proposal is appealing, as there are certain exceptional compact Riemannian symmetric spaces with the desired symmetry groups and whose dimension agree with that of the putative projective planes ($\dim(\mathbf{P}^2(\mathbf{K}\otimes\mathbf{K}')) = 2\dim(\mathbf{K})\dim(\mathbf{K}')$), and this would give a uniform construction of the exceptional Lie groups as symmetries of naturally occurring objects (i.e., without an a priori knowledge of the exceptional Lie groups). The Riemannian symmetric spaces were classified by Cartan in 1926 (Cartan's labels are used in sequel); see classification for details, and the relevant spaces are:

- the octonionic projective plane – FII, dimension 16 = 2 × 8, $F_4$ symmetry, Cayley projective plane $\mathbf{P}^2(\mathbf{O})$,
- the bioctonionic projective plane – EIII, dimension 32 = 2 × 2 × 8, $E_6$ symmetry, complexified Cayley projective plane, $\mathbf{P}^2(\mathbf{C} \otimes \mathbf{O})$,
- the "**quateroctonionic projective plane**"[2] – EVI, dimension 64 = 2 × 4 × 8, $E_7$ symmetry, $\mathbf{P}^2(\mathbf{H} \otimes \mathbf{O})$,
- the "**octooctonionic projective plane**"[3] – EVIII, dimension 128 = 2 × 8 × 8, $E_8$ symmetry, $\mathbf{P}^2(\mathbf{O} \otimes \mathbf{O})$.

The difficulty with this proposal is that while the octonions are a division algebra, and thus a projective plane is defined over them, the bioctonions, quarteroctonions and octooctonions are not division algebras, and thus the usual definition of a projective plane does not work. This can be resolved for the bioctonions, with the resulting projective plane being the complexified Cayley plane, but the constructions do not work for the quarteroctonions and octooctonions, and the spaces in question do not obey the usual axioms of projective planes,[1] hence the quotes on "(putative) projective plane".

However, the tangent space at each point of these spaces can be identified with the plane $(\mathbf{H} \otimes \mathbf{O})^2$, or $(\mathbf{O} \otimes \mathbf{O})^2$further justifying the intuition that these are a form of generalized projective plane.[2][3] Accordingly, the resulting spaces are sometimes called **Rosenfeld projective planes** and notated as if they were projective planes. More broadly, these compact forms are the **Rosenfeld elliptic projective planes**, while the dual non-compact forms are the **Rosenfeld hyperbolic projective planes**. A more modern presentation of Rosenfeld's ideas is in (Rosenfeld 1997), while a brief note on these "planes" is in (Besse 1987, pp. 313–316).[4]

The spaces can be constructed using Tit's theory of buildings, which allows one to construct a geometry with any given algebraic group as symmetries, but this requires starting with the Lie groups and constructing a geometry from them, rather than constructing a geometry independently of a knowledge of the Lie groups.[1]

## Magic square[edit]

While at the level of manifolds and Lie groups, the construction of the projective plane $\mathbf{P}^2(\mathbf{K} \otimes \mathbf{K}')$ of two normed division algebras does not work, the corresponding construction at the level of Lie algebras *does* work. That is, if one decomposes the Lie algebra of infinitesimal isometries of the projective plane $\mathbf{P}^2(\mathbf{K})$ and applies the same analysis to $\mathbf{P}^2(\mathbf{K} \otimes \mathbf{K}')$, one can use this decomposition, which holds when $\mathbf{P}^2(\mathbf{K} \otimes \mathbf{K}')$ can actually be defined as a projective plane, as a *definition* of a "magic square Lie algebra" $M(\mathbf{K},\mathbf{K}')$ This definition is purely algebraic, and holds even without assuming the existence of the corresponding geometric space. This was done independently circa 1958 in (Tits 1966) and by Freudenthal in a series of 11 papers, starting with (Freudenthal 1954) and ending with (Freudenthal 1963), though the simplified construction outlined here is due to (Vinberg 1966).[1]

# Hopf Fibration

## Wikipedia describes the Hopf Fibration in this way:

In the mathematical field of topology, the **Hopf fibration** (also known as the **Hopf bundle** or **Hopf map**) describes a 3-sphere (a hypersphere in four-dimensional space) in terms of circles and an ordinary sphere. Discovered by Heinz Hopf in 1931, it is an influential early example of a fiber bundle. Technically, Hopf found a many-to-one continuous function (or "map") from the 3-sphere onto the 2-sphere such that each distinct *point* of the 2-sphere comes from a distinct *circle* of the 3-sphere (Hopf 1931). Thus the 3-sphere is composed of fibers, where each fiber is a circle — one for each point of the 2-sphere.

This fiber bundle structure is denoted

$$S^1 \hookrightarrow S^3 \xrightarrow{p} S^2,$$

meaning that the fiber space $S^1$ (a circle) is embedded in the total space $S^3$ (the 3-sphere), and $p : S^3 \to S^2$ (Hopf's map) projects $S^3$ onto the base space $S^2$ (the ordinary 2-sphere). The Hopf fibration, like any fiber bundle, has the important property that it is locally a product space. However it is not a *trivial*fiber bundle, i.e., $S^3$ is not *globally* a product of $S^2$ and $S^1$ although locally it is indistinguishable from it.

This has many implications: for example the existence of this bundle shows that the higher homotopy groups of spheres are not trivial in general. It also provides a basic example of a principal bundle, by identifying the fiber with the circle group.

Stereographic projection of the Hopf fibration induces a remarkable structure on $\mathbf{R}^3$, in which space is filled with nested tori made of linking Villarceau circles. Here each fiber projects to a circle in space (one of which is a line, thought of as a "circle through infinity"). Each torus is the stereographic projection of the inverse image of a circle of latitude of the 2-sphere. (Topologically, a torus is the product of two circles.) These tori are illustrated in the images at right. When $\mathbf{R}^3$ is compressed to a ball, some geometric structure is lost although the topological structure is retained (see Topology and geometry). The loops are homeomorphic to circles, although they are not geometric circles.
There are numerous generalizations of the Hopf fibration. The unit sphere in complex coordinate space $\mathbf{C}^{n+1}$fibers naturally over the complex projective space $\mathbf{CP}^n$ with circles as fibers, and there are also real,quaternionic, and octonionic versions of these fibrations. In particular, the Hopf fibration belongs to a family of four fiber bundles in which the total space, base space, and fiber space are all spheres:

$$\begin{aligned} S^0 &\hookrightarrow S^1 \to S^1, \\ S^1 &\hookrightarrow S^3 \to S^2, \\ S^3 &\hookrightarrow S^7 \to S^4, \\ S^7 &\hookrightarrow S^{15} \to S^8. \end{aligned}$$

By Adams' theorem such fibrations can occur only in these dimensions.

# Definition and construction[edit]

For any natural number $n$, an $n$-dimensional sphere, or n-sphere, can be defined as the set of points in an ($n$+1)-dimensional space which are a fixed distance from a central point. For concreteness, the central point can be taken to be the origin, and the distance of the points on the sphere from this origin can be assumed to be a unit length. With this convention, the $n$-sphere, $S^n$, consists of the points $(x_1, x_2, \ldots, x_{n+1})$ in $\mathbf{R}^{n+1}$ with $x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1$. For example, the 3-sphere consists of the points $(x_1, x_2, x_3, x_4)$ in $\mathbf{R}^4$ with $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$.

The Hopf fibration $p$: $S^3 \to S^2$ of the 3-sphere over the 2-sphere can be defined in several ways.

## Direct construction[edit]

Identify $\mathbf{R}^4$ with $\mathbf{C}^2$ and $\mathbf{R}^3$ with $\mathbf{C}\times\mathbf{R}$ (where $\mathbf{C}$ denotes the complex numbers) by writing:

$(x_1, x_2, x_3, x_4)$ as $(z_0 = x_1 + ix_2, z_1 = x_3 + ix_4)$; and
$(x_1, x_2, x_3)$ as $(z = x_1 + ix_2, x = x_3)$.

Thus $S^3$ is identified with the subset of all $(z_0, z_1)$ in $\mathbf{C}^2$ such that $|z_0|^2 + |z_1|^2 = 1$, and $S^2$ is identified with the subset of all $(z, x)$ in $\mathbf{C}\times\mathbf{R}$ such that $|z|^2 + x^2 = 1$. (Here, for a complex number $z = x + iy$, $|z|^2 = z z^* = x^2 + y^2$, where the star denotes the complex conjugate.) Then the Hopf fibration $p$ is defined by

$$p(z_0, z_1) = (2z_0z_1^*, |z_0|^2 - |z_1|^2).$$

The first component is a complex number, whereas the second component is real. Any point on the 3-sphere must have the property that $|z_0|^2 + |z_1|^2 = 1$. If that is so, then $p(z_0, z_1)$ lies on the unit 2-sphere in $\mathbf{C}\times\mathbf{R}$, as may be shown by squaring the complex and real components of $p$

Furthermore, if two points on the 3-sphere map to the same point on the 2-sphere, i.e., if $p(z_0, z_1) = p(w_0, w_1)$, then $(w_0, w_1)$ must equal $(\lambda z_0, \lambda z_1)$ for some complex number $\lambda$ with $|\lambda|^2 = 1$. The converse is also true; any two points on the 3-sphere that differ by a common complex factor $\lambda$ map to the same point on the 2-sphere. These conclusions follow, because the complex factor $\lambda$ cancels with its complex conjugate $\lambda^*$ in both parts of $p$: in the complex $2z_0z_1^*$ component and in the real component $|z_0|^2 - |z_1|^2$.

Since the set of complex numbers $\lambda$ with $|\lambda|^2 = 1$ form the unit circle in the complex plane, it follows that for each point $m$ in $S^2$, the inverse image $p^{-1}(m)$ is a circle, i.e., $p^{-1}m \cong S^1$. Thus the 3-sphere is realized as a disjoint union of these circular fibers.

#### Geometric interpretation using the complex projective line[edit]

A geometric interpretation of the fibration may be obtained using the complex projective line, $\mathbf{CP}^1$, which is defined to be the set of all complex one-dimensional subspaces of $\mathbf{C}^2$. Equivalently, $\mathbf{CP}^1$ is the quotient of $\mathbf{C}^2\backslash\{0\}$ by the equivalence relation which identifies $(z_0, z_1)$ with $(\lambda z_0, \lambda z_1)$ for any nonzero complex number $\lambda$. On any complex line in $\mathbf{C}^2$ there is a circle of unit norm, and so the restriction of the quotient map to the points of unit norm is a fibration of $S^3$ over $\mathbf{CP}^1$.

$\mathbf{CP}^1$ is diffeomorphic to a 2-sphere: indeed it can be identified with the Riemann sphere $\mathbf{C}_\infty = \mathbf{C} \cup \{\infty\}$, which is the one point compactification of $\mathbf{C}$ (obtained by adding a point at infinity). The formula given for $p$ above defines an explicit diffeomorphism between the complex projective line and the ordinary

2-sphere in 3-dimensional space. Alternatively, the point ($z_0$, $z_1$) can be mapped to the ratio $z_1/z_0$ in the Riemann sphere$\mathbf{C}_\infty$.

#### Fiber bundle structure[edit]

The Hopf fibration defines a fiber bundle, with bundle projection *p*. This means that it has a "local product structure", in the sense that every point of the 2-sphere has some neighborhood *U* whose inverse image in the 3-sphere can be identified with the product of *U* and a circle: $p^{-1}(U) \cong U \times S^1$.

Such a fibration is said to be locally trivial.

For the Hopf fibration, it is enough to remove a single point *m* from $S^2$ and the corresponding circle $p^{-1}(m)$ from $S^3$; thus one can take $U = S^2 \setminus \{m\}$, and any point in $S^2$ has a neighborhood of this form.

## Geometric interpretation using rotations[edit]

Another geometric interpretation of the Hopf fibration can be obtained by considering rotations of the 2-sphere in ordinary 3-dimensional space. The rotation group SO(3) has a double cover, the spin group Spin(3), diffeomorphic to the 3-sphere. The spin group acts transitively on $S^2$ by rotations. The stabilizer of a point is isomorphic to the circle group. It follows easily that the 3-sphere is a principal circle bundle over the 2-sphere, and this is the Hopf fibration.

To make this more explicit, there are two approaches: the group Spin(3) can either be identified with the group Sp(1) of unit quaternions, or with the special unitary group SU(2).

In the first approach, a vector ($x_1$, $x_2$, $x_3$, $x_4$) in $\mathbf{R}^4$ is interpreted as a quaternion $q \in \mathbf{H}$ by writing

$$q = x_1 + \mathbf{i}x_2 + \mathbf{j}x_3 + \mathbf{k}x_4.$$

The 3-sphere is then identified with the quaternions of unit norm, i.e., those $q \in \mathbf{H}$ for which $|q|^2 = 1$, where $|q|^2 = q\,q^*$, which is equal to $x_1^2 + x_2^2 + x_3^2 + x_4^2$ for *q* as above.

On the other hand, a vector ($y_1$, $y_2$, $y_3$) in $\mathbf{R}^3$ can be interpreted as an imaginary quaternion

$$p = \mathbf{i}y_1 + \mathbf{j}y_2 + \mathbf{k}y_3.$$

Then, as is well-known since Cayley (1845), the mapping

$$p \mapsto qpq^*$$

is a rotation in $\mathbf{R}^3$: indeed it is clearly an isometry, since $|q\,p\,q^*|^2 = q\,p\,q^*\,q\,p^*\,q^* = q\,p\,p^*\,q^* = |p|^2$, and it is not hard to check that it preserves orientation.

In fact, this identifies the group of unit quaternions with the group of rotations of $\mathbf{R}^3$, modulo the fact that the unit quaternions *q* and −*q*determine the same rotation. As noted above, the rotations act transitively on $S^2$, and the set of unit quaternions *q* which fix a given unit imaginary quaternion *p* have the form $q = u + v\,p$, where *u* and *v* are real numbers with $u^2 + v^2 = 1$. This is a circle subgroup. For concreteness, one can take $p = \mathbf{k}$, and then the Hopf fibration can be defined as the map sending a unit quaternion ω to $\omega\,\mathbf{k}\,\omega^*$. All the quaternions $\omega q$, where *q* is one of the circle of unit quaternions that fix *k*, get mapped to the same thing (which happens to be one of the two 180° rotations rotating $k$ to the same place as ω does).

Another way to look at this fibration is that every unit quaternion ω moves the plane spanned by {1, $k$} to a new plane spanned by {ω, ω$k$}. Any quaternion ω$q$, where $q$ is one of the circle of unit quaternions that fix $k$, will have the same effect. We put all these into one fibre, and the fibres can be mapped one-to-one to the 2-sphere of 180° rotations which is the range of $\omega k \omega^*$.

This approach is related to the direct construction by identifying a quaternion $q = x_1 + \mathbf{i}\, x_2 + \mathbf{j}\, x_3 + \mathbf{k}\, x_4$ with the 2×2 matrix:

$$\begin{bmatrix} x_1 + \mathbf{i}x_2 & x_3 + \mathbf{i}x_4 \\ -x_3 + \mathbf{i}x_4 & x_1 - \mathbf{i}x_2 \end{bmatrix}.$$

This identifies the group of unit quaternions with SU(2), and the imaginary quaternions with the skew-hermitian 2×2 matrices (isomorphic to**C**×**R**).

**Explicit formulae**[edit]

The rotation induced by a unit quaternion $q = w + \mathbf{i}\, x + \mathbf{j}\, y + \mathbf{k}\, z$ is given explicitly by the orthogonal matrix

$$\begin{bmatrix} 1-2(y^2+z^2) & 2(xy-wz) & 2(xz+wy) \\ 2(xy+wz) & 1-2(x^2+z^2) & 2(yz-wx) \\ 2(xz-wy) & 2(yz+wx) & 1-2(x^2+y^2) \end{bmatrix}.$$

Here we find an explicit real formula for the bundle projection. For, the fixed unit vector along the $z$ axis, (0,0,1), rotates to another unit vector,

$$\Big(2(xz+wy), 2(yz-wx), 1-2(x^2+y^2)\Big),$$

which is a continuous function of ($w$,$x$,$y$,$z$). That is, the image of $q$ is where it aims the $z$ axis. The fiber for a given point on $S^2$ consists of all those unit quaternions that aim there.

To write an explicit formula for the fiber over a point ($a$,$b$,$c$) in $S^2$, we may proceed as follows. Multiplication of unit quaternions produces composition of rotations, and

$$q_\theta = \cos\theta + \mathbf{k}\sin\theta$$

is a rotation by 2θ around the $z$ axis. As θ varies, this sweeps out a great circle of $S^3$, our prototypical fiber. So long as the base point, ($a$,$b$,$c$), is not the antipode, (0,0,−1), the quaternion

$$q_{(a,b,c)} = \frac{1}{\sqrt{2(1+c)}}(1+c-\mathbf{i}b+\mathbf{j}a)$$

will aim there. Thus the fiber of ($a$,$b$,$c$) is given by quaternions of the form $q_{(a,b,c)}q_\theta$, which are the $S^3$ points

$$\frac{1}{\sqrt{2(1+c)}}\Big((1+c)\cos(\theta), a\sin(\theta)-b\cos(\theta), a\cos(\theta)+b\sin(\theta), (1+c)\sin(\theta)\Big).$$

Since multiplication by $q_{(a,b,c)}$ acts as a rotation of quaternion space, the fiber is not merely a topological circle, it is a geometric circle. The final fiber, for (0,0,−1), can be given by using $q_{(0,0,-1)} = \mathbf{i}$, producing

$$\Big(0, \cos(\theta), -\sin(\theta), 0\Big),$$

which completes the bundle.

Thus, a simple way of visualizing the Hopf fibration is as follows. Any point on the 3-sphere is equivalent to a quaternion, which in turn is equivalent to a particular rotation of a Cartesian coordinate frame in three dimensions. The set of all possible quaternions produces the set of all possible rotations, which moves the tip of one unit vector of such a coordinate frame (say, the **z** vector) to all possible points on a unit 2-sphere. However, fixing the tip of the **z** vector does not specify the rotation fully; a further rotation is possible about the **z**-axis. Thus, the 3-sphere is mapped onto the 2-sphere, plus a single rotation.

## Fluid Mechanics[edit]

If the Hopf fibration is treated as a vector field in 3 dimensional space then there is a solution to the (compressible, non-viscous) Navier-Stokes equations of fluid dynamics in which the fluid flows along the circles of the projection of the Hopf fibration in 3 dimensional space. The size of the velocities, the density and the pressure can be chosen at each point to satisfy the equations. All these quantities fall to zero going away from the centre. If a is the distance to the inner ring, the velocities, pressure and density fields are given by:

$$\mathbf{v}(x,y,z) = A\left(a^2+x^2+y^2+z^2\right)^{-2}\left(2(-ay+xz), 2(ax+yz), a^2-x^2-y\right.$$

$$p(x,y,z) = -A^2B\left(a^2+x^2+y^2+z^2\right)^{-3},$$

$$\rho(x,y,z) = 3B\left(a^2+x^2+y^2+z^2\right)^{-1}$$

for arbitrary constants A and B. Similar patterns of fields are found as soliton solutions of magnetohydrodynamics:[1]

# Generalizations[edit]

The Hopf construction, viewed as a fiber bundle $p$: $S^3 \to \mathbf{CP}^1$, admits several generalizations, which are also often known as Hopf fibrations. First, one can replace the projective line by an $n$-dimensional projective space. Second, one can replace the complex numbers by any (real)division algebra, including (for $n$ = 1) the octonions.

## Real Hopf fibrations[edit]

A real version of the Hopf fibration is obtained by regarding the circle $S^1$ as a subset of $\mathbf{R}^2$ in the usual way and by idenitifying antipodal points. This gives a fiber bundle $S^1 \to \mathbf{RP}^1$ over the real projective line with fiber $S^0$ = {1, -1}. Just as $\mathbf{CP}^1$ is diffeomorphic to a sphere, $\mathbf{RP}^1$is diffeomorphic to a circle.

More generally, the $n$-sphere $S^n$ fibers over real projective space $\mathbf{RP}^n$ with fiber $S^0$.

## Complex Hopf fibrations[edit]

The Hopf construction gives circle bundles $p$ : $S^{2n+1} \to \mathbf{CP}^n$ over complex projective space. This is actually the restriction of the tautological line bundle over $\mathbf{CP}^n$ to the unit sphere in $\mathbf{C}^{n+1}$.

## Quaternionic Hopf fibrations[edit]

Similarly, one can regard $S^{4n+3}$ as lying in $\mathbf{H}^{n+1}$ (quaternionic $n$-space) and factor out by unit quaternion (= $S^3$) multiplication to get $\mathbf{HP}^n$. In particular, since $S^4 = \mathbf{HP}^1$, there is a bundle $S^7 \to S^4$ with fiber $S^3$.

## Octonionic Hopf fibrations[edit]

A similar construction with the octonions yields a bundle $S^{15} \to S^8$ with fiber $S^7$. But the sphere $S^{31}$ does not fiber over $S^{16}$ with fiber $S^{15}$. One can regard $S^8$ as the octonionic projective line $\mathbf{OP}^1$. Although one can also define an octonionic projective plane $\mathbf{OP}^2$, the sphere $S^{23}$ does not fiber over $\mathbf{OP}^2$ with fiber $S^7$.[2][3]

## Fibrations between spheres

Sometimes the term "Hopf fibration" is restricted to the fibrations between spheres obtained above, which are

$S^1 \to S^1$ with fiber $S^0$
$S^3 \to S^2$ with fiber $S^1$
$S^7 \to S^4$ with fiber $S^3$
$S^{15} \to S^8$ with fiber $S^7$

As a consequence of Adams' theorem, fiber bundles with spheres as total space, base space, and fiber can occur only in these dimensions. Fiber bundles with similar properties, but different from the Hopf fibrations, were used by John Milnor to construct exotic spheres.

The Hopf fibration has many implications, some purely attractive, others deeper. For example, stereographic projection $S^3 \to \mathbf{R}^3$ induces a remarkable structure in $\mathbf{R}^3$, which in turn illuminates the topology of the bundle (Lyons 2003). Stereographic projection preserves circles and maps the Hopf fibers to geometrically perfect circles in $\mathbf{R}^3$ which fill space. Here there is one exception: the Hopf circle containing the projection point maps to a straight line in $\mathbf{R}^3$ — a "circle through infinity".

The fibers over a circle of latitude on $S^2$ form a torus in $S^3$ (topologically, a torus is the product of two circles) and these project to nested toruses in $\mathbf{R}^3$ which also fill space. The individual fibers map to linkingVillarceau circles on these tori, with the exception of the circle through the projection point and the one through its opposite point: the former maps to a straight line, the latter to a unit circle perpendicular to, and centered on, this line, which may be viewed as a degenerate torus whose radius has shrunken to zero. Every other fiber image encircles the line as well, and so, by symmetry, each circle is linked through *every* circle, both in $\mathbf{R}^3$ and in $S^3$. Two such linking circles form a Hopf link in $\mathbf{R}^3$

Hopf proved that the Hopf map has Hopf invariant 1, and therefore is not null-homotopic. In fact it generates the homotopy group $\pi_3(S^2)$ and has infinite order.

In quantum mechanics, the Riemann sphere is known as the Bloch sphere, and the Hopf fibration describes the topological structure of a quantum mechanical two-level system or qubit. Similarly, the topology of a pair of entangled two-level systems is given by the Hopf fibration

$$S^3 \hookrightarrow S^7 \to S^4.$$

(Mosseri & Dandoloff 2001).

# Exceptional Lie Algebra E8

Wikipedia describes the Exceptional Lie Algebra E8 in this way:

In mathematics, $\mathbf{E_8}$ is any of several closely related exceptional simple Lie groups, linear algebraic groupsor Lie algebras of dimension 248; the same notation is used for the corresponding root lattice, which hasrank 8. The designation $E_8$ comes from the Cartan–Killing classification of the complex simple Lie algebras, which fall into four infinite series labeled $A_n$, $B_n$, $C_n$, $D_n$, and five exceptional cases labeled $E_6$, $E_7$, $E_8$,$F_4$, and $G_2$. The $E_8$ algebra is the largest and most complicated of these exceptional cases.

Wilhelm Killing (1888a, 1888b, 1889, 1890) discovered the complex Lie algebra $E_8$ during his classification of simple compact Lie algebras, though he did not prove its existence, which was first shown by Élie Cartan. Cartan determined that a complex simple Lie algebra of type $E_8$ admits three real forms. Each of them gives rise to a simple Lie group of dimension 248, exactly one of which is compact. Chevalley (1955) introducedalgebraic groups and Lie algebras of type $E_8$ over other fields: for example, in the case of finite fieldsthey lead to an infinite family of finite simple groups of Lie type.

The Lie group $E_8$ has dimension 248. Its rank, which is the dimension of its maximal torus, is 8. Therefore the vectors of the root system are in eight-dimensional Euclidean space: they are described explicitly later in this article. The Weyl group of $E_8$, which is the group of symmetries of the maximal torus which are induced byconjugations in the whole group, has order $2^{14}\ 3^{5}\ 5^{2}\ 7 =$ 696729600.

The compact group $E_8$ is unique among simple compact Lie groups in that its non-trivial representation of smallest dimension is the adjoint representation (of dimension 248) acting on the Lie algebra $E_8$ itself; it is also the unique one which has the following four properties: trivial center, compact, simply connected, and simply laced (all roots have the same length).

There is a Lie algebra $E_n$ for every integer $n \geq 3$, which is infinite dimensional if $n$ is greater than 8.

## Real and complex forms[edit]

There is a unique complex Lie algebra of type $E_8$, corresponding to a complex group of complex dimension 248. The complex Lie group $E_8$ of complex dimension 248 can be considered as a simple real Lie group of real dimension 496. This is simply connected, has maximal compact subgroup the compact form (see below) of $E_8$, and has an outer automorphism group of order 2 generated by complex conjugation.

As well as the complex Lie group of type $E_8$, there are three real forms of the Lie algebra, three real forms of the group with trivial center (two of which have non-algebraic double covers, giving two further real forms), all of real dimension 248, as follows:

- The compact form (which is usually the one meant if no other information is given), which is simply connected and has trivial outer automorphism group.
- The split form, EVIII (or $E_{8(8)}$), which has maximal compact subgroup Spin(16)/(**Z**/2**Z**), fundamental group of order 2 (implying that it has adouble cover, which is a simply connected Lie real group but is not algebraic, see below) and has trivial outer automorphism group.
- EIX (or $E_{8(-24)}$), which has maximal compact subgroup $E_7 \times SU(2)/(-1,-1)$, fundamental group of order 2 (again implying a double cover, which is not algebraic) and has trivial outer automorphism group.

For a complete list of real forms of simple Lie algebras, see the list of simple Lie groups.

## $E_8$ as an algebraic group[edit]

By means of a Chevalley basis for the Lie algebra, one can define $E_8$ as a linear algebraic group over the integers and, consequently, over any commutative ring and in particular over any field: this defines the so-called split (sometimes also known as “untwisted”) form of $E_8$. Over an algebraically closed field, this is the only form; however, over other fields, there are often many other forms, or “twists” of $E_8$, which are classified in the general framework of Galois cohomology (over a perfect field $k$) by the set $H^1(k,\mathrm{Aut}(E_8))$ which, because the Dynkin diagram of $E_8$(see below) has no automorphisms, coincides with $H^1(k,E_8)$.[1]

Over **R**, the real connected component of the identity of these algebraically twisted forms of $E_8$ coincide with the three real Lie groups mentionedabove, but with a subtlety concerning the fundamental group: all forms of $E_8$ are simply connected in the sense of algebraic geometry, meaning that they admit no non-trivial algebraic coverings; the non-compact and simply connected real Lie group forms of $E_8$ are therefore not algebraic and admit no faithful finite-dimensional representations.

Over finite fields, the Lang–Steinberg theorem implies that $H^1(k,E_8)=0$, meaning that $E_8$ has no twisted forms: see below.

## Representation theory[edit]

The characters of finite dimensional representations of the real and complex Lie algebras and Lie groups are all given by the Weyl character formula. The dimensions of the smallest irreducible representations are (sequence A121732 in OEIS):

1, 248, 3875, 27000, 30380, 147250, 779247, 1763125, 2450240, 4096000, 4881384, 6696000, 26411008, 70680000, 76271625, 79143000, 146325270, 203205000, 281545875, 301694976, 344452500, 820260000, 1094951000, 2172667860, 2275896000, 2642777280, 2903770000, 3929713760, 4076399250, 4825673125, 6899079264, 8634368000 (twice), 12692520960…

The 248-dimensional representation is the adjoint representation. There are two non-isomorphic irreducible representations of dimension 8634368000 (it is not unique; however, the next integer with this property is 175898504162692612600853299200000 (sequence A181746 in OEIS)). The fundamental representations are those with dimensions 3875, 6696000, 6899079264, 146325270, 2450240, 30380, 248 and 147250 (corresponding to the eight nodes in the Dynkin diagram in the order chosen for the Cartan matrix below, i.e., the nodes are read in the seven-node chain first, with the last node being connected to the third).

The coefficients of the character formulas for infinite dimensional irreducible representations of $E_8$ depend on some large square matrices consisting of polynomials, the Lusztig–Vogan polynomials, an analogue of Kazhdan–Lusztig polynomials introduced for reductive groups in general by George Lusztig and David Kazhdan (1983). The values at 1 of the Lusztig–Vogan polynomials give the coefficients of the matrices relating the standard representations (whose characters are easy to describe) with the irreducible representations.

These matrices were computed after four years of collaboration by a group of 18 mathematicians and computer scientists, led by Jeffrey Adams, with much of the programming done by Fokko du Cloux. The most difficult case (for exceptional groups) is the split real form of $E_8$ (see above), where the largest matrix is of size 453060×453060. The Lusztig–Vogan polynomials for all other exceptional simple groups have been known for some time; the calculation for the split form of $E_8$ is far longer than any other case. The announcement of the result in March 2007 received extraordinary attention from the media (see the external links), to the surprise of the mathematicians working on it.

The representations of the $E_8$ groups over finite fields are given by Deligne–Lusztig theory.

## Constructions[edit]

One can construct the (compact form of the) $E_8$ group as the automorphism group of the corresponding $\mathbf{e}_8$ Lie algebra. This algebra has a 120-dimensional subalgebra **so**(16) generated by $J_{ij}$ as well as 128 new generators $Q_a$ that transform as a Weyl–Majorana spinor of **spin**(16). These statements determine the commutators

$$[J_{ij}, J_{k\ell}] = \delta_{jk}J_{i\ell} - \delta_{j\ell}J_{ik} - \delta_{ik}J_{j\ell} + \delta_{i\ell}J_{jk}$$

as well as

$$[J_{ij}, Q_a] = \frac{1}{4}(\gamma_i\gamma_j - \gamma_j\gamma_i)_{ab}Q_b,$$

while the remaining commutator (not anticommutator!) is defined as

$$[Q_a, Q_b] = \gamma^{[i}_{ac}\gamma^{j]}_{cb}J_{ij}.$$

It is then possible to check that the Jacobi identity is satisfied.

## Geometry[edit]

The compact real form of $E_8$ is the isometry group of the 128-dimensional exceptional compact Riemannian symmetric space EVIII (in Cartan'sclassification). It is known informally as the "octooctonionic projective plane" because it can be built using an algebra that is the tensor product of the octonions with themselves, and is also known as a Rosenfeld projective plane, though it does not obey the usual axioms of a projective plane. This can be seen systematically using a construction known as the *magic square*, due to Hans Freudenthal and Jacques Tits(Landsberg & Manivel 2001).

# H7 Hyper Circle in Vedic Physics

Vedic Nuclear Physics contains the concept of Hyper Circles, which describe the atomic nucleus, one layer or ring, upon another. The hyper circles exist or live within a series of spaces known as Lokas, or the dimensional spaces of regular bodies. The fourteen spaces are known as Bhuvanas - the places of the regular bodies which are developed in their respective multi - dimensional spaces of the Lokas. In

The full Sanskrit term is “Sapta Nava - Asva   (p. 117 top)

The regular body of the Sapta Nava - Asva is made by Hyper Circle 7 of the RTA in the 7 - dimensional space of Dyou. The structure of this regular body begins from the one - dimensional space of Bhuhu - Loka when the content of the central seat of the nuclear affined space (God Vishnu) starts inhaling the Svadha, or the Brahma content from the surrounding area, which creates its own gravitational force.

A different writer on Vedic Physics states that gravity forms when one larger body passes near another. It may be the case here that the content of the central seat of the nuclear affined space is proportionately larger than the Svadha, or becomes so as the seat inhales the Svadha. In this case the statement of one writer on Vedic Physics would support that of another, reflecting no contradiction between the views of the two distinct writers, both independently arriving at similar conclusions after analyzing different sources.

This then becomes H1 in Bhuhu Loka, then H2 in Bhuhava Loka, etc. until it reaches the seven - dimensional space of Satya Loka where it forms H7. This is the place where it gets the name of Sapta Nava - Asva. During the exhalation process, this stuff reverses and gets pushed back into the lesser Lokas, where the Prayatithi is exhaled from the Paraha state of the god Vishnu, to the Avaraha state of Brahma, which becomes the Retaha of god Vishnu.

This process may best be envisioned by examining photos of rotating galaxies in space. The central condensed part inhales the ground state Dark Matter content through three white channels of flow from a finite three - dimensional space. The inhaled surplus content of the functioning Dark Matter is ejected through three holes in three - dimensional space. The ejected content contains the will of desire to create the universe, which is why the ejected part contains a spotted structure in the flow of its channels.

The flow channels of Dark Matter are called Isani of god Rudra. The word Rudra is formed from the root Rudhra, meaning resistance and a hole. This meaning is expressed through the erection of the Adi Deva structure, told in the Na Particle. The content of the functioning state of Brahma meets resistance in the Paraha state, in the central part of the affined space of Na.

When this resisted part exhales the surplus inhaled content of the Brahma through breathing, then the central condensed part ejects very fine units of Brahma into the Avaraha state by making a very fine hole in the confined space of the Paraha state for the ejection process. Rudra thus contains the concept of making a hole. The clear figure can be observed in the structure of a rotating galaxy in space. The condensed state of the resisted content of the functioning state of Dark Matter can be clearly seen as white in the central part of a rotating galaxy.

The mature state of galactic structure can be clearly seen since the solid structures of the bodies made by the outgoing flow of the channels of functional Brahma. The immature state of galactic structure proves invisible since it is incapable of erecting the solid white structures of the bodies of the outgoing flow of the functioning state of Brahma through its flow channels. The immature state of galactic structure is what western contemporary science terms “Black Holes,” the Dyou space.

All new stars are formed by immature galaxies. When some mature galaxy ages beyond a certain point, then it receives a comparatively smaller amount of energy supply delivered to the solid structures. In this way, solid bodies disintegrate and dissolve into the functioning state of Dark Matter. These paragraphs thus describe the life cycle of galaxies in the universe, and so Rudra is known as the god of evolution and dissolution.

The matured state of galactic structure is called Akasha Ganga in Vedic Science because the flow of the functioning state of Dark Matter is named by the word Ganga. The flow of divine Ganga goes first into the head of the god Rudra through pure white channels of Brahma, then comes back to the regular spotted bodies of the universe to nourish them. All of this a product of the will of god Vishnu sitting at the center.

## Vishnu At Center

At the initial stage, there was complete darkness inside the Ground State of Dark Matter, in the state of stasis with no function, before the process of evolution began in the universe. When the whole of the space filled with the ground state of Dark Matter began to flow within the confined space of an enlarged A (Aha), Dark Matter transformed into a fluid state to cover all of the dimensional spaces of the Lokas.

A small quantity of functional Brahma covered the seat of the enlarged A form (Abhu), which erected the structure of the seven - dimensional space Satya Loka.

| Dimension | Name | Alternate Name | Hyper | |
|---|---|---|---|---|
| 1 | | | | |
| 2 | | | | |
| 3 | | | | |
| 4 | Abhu | | | |
| 5 | | | | |
| 6 | Tapaha Loka | Savita | H6 | |
| 7 | Satya Loka | | H7 | |
| 8 | Abhu | | | |
| 9 | Hiranya Garbha | | | Center = 8 |
| | Abhu | Spirit | | 4 dimensions |
| | Brahma Tattva | Clay lamp | | |
| | Atma Tattva | Na Particles ejected | | Ajam (born) |

Five Elements

| | |
|---|---|
| Akasha | Aether |
| Vayu | Wind |
| Agni | Fire |
| Apaha | Water |
| Prithivi | Earth |

Rig Veda (1 - 189 - 6) pp. 36, 88 - 89,

# Leyman Series

In physics and chemistry, the **Lyman series** is the series of transitions and resulting ultraviolet emission lines of the hydrogen atom as an electron goes from *n* ≥ 2 to *n* = 1 (where *n* is the principal quantum number) the lowest energy level of the electron. The transitions are named sequentially by Greek letters: from *n* = 2 to *n* = 1 is called Lyman-alpha, 3 to 1 is Lyman-beta, 4 to 1 is Lyman-gamma, etc. The series is named after its discoverer, Theodore Lyman. The greater the difference in the principal quantum numbers, the higher the energy of the electromagnetic emission.

## The Lyman series[edit]

The version of the Rydberg formula that generated the Lyman series was:[2]

$$\frac{1}{\lambda} = R_{\mathrm{H}} \left(1 - \frac{1}{n^2}\right) \qquad \left(R_{\mathrm{H}} = 1.0968{\times}10^7\,\mathrm{m}^{-1} = \frac{13.6\,\mathrm{eV}}{hc}\right)$$

Where *n* is a natural number greater than or equal to 2 (i.e. *n* = 2,3,4,...).

Therefore, the lines seen in the image above are the wavelengths corresponding to *n*=2 on the right, to $n = \infty$ on the left (there are infinitely many spectral lines, but they become very dense as they approach to $n = \infty$ (Lyman limit), so only some of the first lines and the last one appear).

The wavelengths (nm) in the Lyman series are all ultraviolet:

| $n$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | $\infty$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **Wavelength (nm)** | 121.6 | 102.6 | 97.3 | 95.0 | 93.8 | 93.1 | 92.6 | 92.3 | 92.1 | 91.9 | 91.18 (Lyman limit) |

# Spheres

A fibre and a sphere connect to create a Hopf Bundle or Sphere in decreasing volume and heat in terms of joules per second.

## Fibrations between spheres

Sometimes the term "Hopf fibration" is restricted to the fibrations between spheres obtained above, which are

- $S^1 \to S^1$ with fiber $S^0$
- $S^3 \to S^2$ with fiber $S^1$
- $S^7 \to S^4$ with fiber $S^3$
- $S^{15} \to S^8$ with fiber $S^7$

As a consequence of Adams' theorem, fiber bundles with spheres as total space, base space, and fiber can occur only in these dimensions. Fiber bundles with similar properties, but different from the Hopf fibrations, were used by John Milnor to construct exotic spheres.

The author believes that Hopf Fibrations should be strictly limited to those described above - mathematicians and physicists play too loosely with language, especially when discussion the Hopf Fibration. It is important here to distinguish clearly between the Hopf Fibrations and spheres, whether exotic or not. The Hopf Fibrations should refer only to those which match the RCHO projective planes.

The measurement of hyper - circles continues to grow up to E8 and beyond, reaching its maximum between H7 and H8 but never attaining the complete size of H8. Measurement decreases after maxima to approach zero in the sense of calculus. Two sequences comprise the magnitude of the Asva body, one increasing, one decreasing. The Asva body organs, to shorten the Asva body, decrease the body magnitude in the decreasing function.

The point from where the structure of the regular body of any hyper - circle starts to be erected in its dimensional space and the structure of the hyper - circle of the previous dimensional space ends: that point is the proper point of the joint between two distinct hyper - circles, where the cutting knife should enter to cut the organs of the Asva to increasingly shorten the magnitude. The number of hyper circles in Hn is greater than seven, is always a natural number such as 8,9,10…when its sequence of function continues to decrease.

Each distinct Hyper - circle should be given its own name.

The Asva Sukta mantra of the Rig Veda (RG - 1 - 162) describes the 8 - hyper circle, where the process of decreasing function starts changing its sequence from increasing to decreasing function after attaining the maxima. At the maxima, the cutting knife begins to reduce the magnitude of the hyper - circles.

(RG - 1 - 189 - 6)

Two vertices keep the interval of the magnitude of the RTA energy quantum of the Asva H8. The H8 vector called the Asva of the Tvasta Rsi remains under the command of the centre, which keeps the eight radii stretched in the Dyou space. The Tvasta Rishi is the force of the RTA energy quantum which erects the regular body of H8. The driving force continues to move in the Dyou space which is filled with RTA.

The one point of maxima lies between two vertices of the regular body H8, and changes the function from increase to decrease. This point then becomes the cutter that cuts the Asva organs in a particular way, the H8 Asva of the Tvasta Rishi. These are the well - defined functions. The Rishi makes the special effort of the Yajna to decrease the energy of the Asva body by sacrificing those units of organs of bonding units of RTA energy of the total quantum of the full body of the Asva into the Agni.

Agni is born from RTA (RG - 1 - 189 - 6), which is why it is called Rtvija Agni. When the magnitude of the RTA body energy quantum of the body of Asva decreases by cutting energy units of the bonding organs, then the RTA energy units are sacrificed into the Agni made of RTA so that it may absorb that released functioning energy into its more condensed state of matter.

Energy is absorbed by those mass - containing bodies which are more condense than the energy - releasing unit. This is the general rule defined by the Tettry Upanishad, which states that the mass of the body of matter is created by RTA energy making itself gradually condensed. Then Agni absorbs the RTA energy because Agni was created by RTA first.

In 1937, Neils Bohr calculated the liquid Drop Model of the nucleus to find the inter - nucleon distance at 6 x 10 ^-15 meter. The difference between H7 and H8 is0.603665 units, which appears to be a multiple of the Bohr number.

The measurement of H15 is 5.7216492 units and this approaches the Stan Boltzmann Constant of 5.668 x 10^ - 8 watt/m2K4. This number seems to be a multiple of 5.7216492 with some other factor of minute variance such as the temperature of K4.

When H7 tries to convert its formation into the regular body of H8, it inhales some RTA to reach the maxima, and then increases its size to reach the maxima. After reaching maxima it exhales and releases the RTA content into space. Volume is decreased by releasing RTA until it reaches the size of H7 (E8).

The regular body of H7 functions with the breathing process by inhaling and exhaling RTA content while maintaining its structure in a non - decaying state. Then it palpitates and vibrates at very high frequency while it sucks up RTA from one side and ejects it from the other, like a jet engine, flying in the Dyou space with wave motion. It is said to make the sound of the wings of a flying hawk and the feet of deer. H7 acquires the force necessary to fly in the Dyou space by wave transmission, through the inhalation and exhalation of RTA.

# Exceptional Lie Algebras, Fibres, Spheres, Bundles, Projective Spaces and Hyper Circles

If H2 + H2 = G2

S7 + S8 = S15, and these = B4, F4 and E8, then

| **Fibre** | **Sphere** | **Bundle** | | **Hyper Circle** | **Value** | **Difference** | **Projective Space** |
|---|---|---|---|---|---|---|---|
| | | | | H8 | 32.469697 | | |
| | | | | | 33.1323046 | 0.6623076 | **Planck Const** |
| $S^7$ | + $S^8$ | = $S^{15}$ | | | 33.073362 | 0.589426 | $OP^1$ |
| B4 | F4 | | E8 | H7 | | | |
| $S^3$ | $S^4$ | $S^7$ | | | 31.006277 | 2.067085 | $HP^1$ |
| B3 | D4 | | E7 | H6 | | | |
| $S^1$ | $S^2$ | $S^3$ | | | 26.318945 | 4.687332 | $CP^n$ |
| D4 | G2 | | E6 | H5 | 19.739209 | 6.579736 | |
| $S^0$ | $S^1$ | $S^1$ | | | 12.566371 | 7.172838 | $RP^1$ |
| H2 | G2 | | D4 | H4 | | | |
| | | | | | 6.2831853 | 6.2831857 | Circle |
| H2 | H2 | | G2 | H3 | | | |
| | | | | | 3.1415927 | 3.141593 | Radius |
| H1 | H1 | | H2 | H2 | | | |

# Spheres

These spheres may simply refer to spheres or to the Exotic Milner Spheres.

| Fibre | Sphere | Bundle | Hyper Circle | Value | Difference | Projective Space |
|---|---|---|---|---|---|---|
| | | | H8 | 32.469697 | | |
| | | | Max | 33.1323046 | | |
| | | | H9 | 29.68658 | | |
| | | | H10 | 25.50164 | | |
| | | | H11 | 20.725143 | | |
| | | | H12 | 16.023153 | | |
| | | | H13 | 11.838174 | | |
| | | | H14 | 8.3897034 | | |
| | | | H15 | 5.7216492 | | |
| | | | H16 | 3.765290 | | |
| | | | H17 | 2.3966788 | | |
| | | | H18 | 1.478626 | | |
| | | | H19 | 0.44290823 | | |
| | | | H20 | 0.258 | | |

The author has consulted a number of professional mathematicians, none of whom recognizes this series of numbers except as n - spheres. The author suspects a deeper connection to Taylor Series and Bernoulli numbers but lacks the computer skills to run these through programs. This bit of research shall need to wait for future papers.

# Exceptional Lie Algebras and Platonic Solids

S.M. Phillips gives the following chart to indicate the close relationships between Exceptional Lie Algebras and Platonic Solids.

$G_2$: 12 roots ↔ 120 internal points in first four Platonic solids;

$F_4$: **48** roots ↔ 480 internal points & lines in first four Platonic solids;

$E_6$: **72** roots ↔ 720 internal points & lines in five Platonic solids;

$E_7$: 126 roots ↔ 1260 internal points, lines & triangles in five Platonic solids;

$E_8$: **248** roots ↔ 2480 points, lines & triangles in five Platonic solids.

# Conclusion

This is what happens after E8. There is no more, just this. The author sincerely hopes you weren't disappointed. If this amounts to a Theory of Everything, then the author surely feels delighted. Send in the dancing girls, please.

But this is not the end, only a choice highlight of the middle. There is much more to Vedic Particle Physics, and the author will try to reveal these aspects. The original upon which this paper is based was poorly written, where the original author tends to repeat every statement at least four times, uses the passive tense and uses a style of English that belongs to the early 20th Century. Most readers would perhaps throw their hands in the air and give up, since the author is guilty of "burying the lead" in describing Vedic Physics. Astounding statements appear from nowhere in the text.

The author, a professional writer, has tried to modernize this language for the benefit of the reader and science at large, which should not pass up the secrets of Vedic Science over poorly written books.

There it is, an end to all that speculation, provided that H7 proves isomorphic to E8. Remaining work includes identifying the logarithms and numerical series which tie this group of fibres and spheres into an entire series. It would be good if a professional mathematician could complete the work that O'Farrell - Figueroa began six years ago. Yet six years have passed, nothing has been done, so the gap is now filled. What appears here constitutes the best - guess of the author, a non - specialist word man, and corrections are welcome.

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'Some men see things as they are and say, why? I dream things that never were and say, why not?'

So let us dedicate ourselves to what the Greeks wrote so long ago: to tame the savageness of man and make gentle the life of this world.

**Robert Francis Kennedy**