Transport catastrophe analysis as an alternative to a fractal description: theory and application to financial crisis time series

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Abstract

The goal of this investigation was to overcome limitations of a persistency analysis, introduced by Benoit Mandelbrot for fractal Brownian processes: nondifferentiability, Brownian nature of process and a linear memory measure. We have extended a sense of a Hurst factor by consideration of a phase diffusion power law. It was shown that pre-catastrophic stabilization as an indicator of bifurcation leads to a new minimum of momentary phase diffusion, while bifurcation causes an increase of the momentary transport. Basic conclusions of a diffusive analysis have been compared to the Lyapunov stability model. An extended Reynolds parameter has been introduces as an indicator of phase transition. A combination of diffusive and Reynolds analysis has been applied for a description of a time series of Dow Jones Industrial weekly prices for a world financial crisis of 2007-2009. Diffusive and Reynolds parameters shown an extreme values in October 2008 when a mortgage crisis was fixed. A combined R/D description allowed distinguishing of short-memory and long memory shifts of a market evolution. It was stated that a systematic large scale failure of a financial system has begun in October 2008 and started fading in February 2009.

Key words: diffusive analysis, fractal Brownian process, bifurcation, catastrophe theory, world financial crisis, time series.

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1 Introduction

In 1955 the American researcher Hassler Whitney has created a mathematical foundation of a modern catastrophe theory – the theory of mapping singularities [1]. It includes investigations of peculiarity classes, that appear for mapping of one two dimensional surface to another one. A mapping was suggested to be smooth. An application of this instrument to dynamical system allowed introducing a mapping of characteristic surface into the surface of control parameters. Then a peculiarity appearance corresponds to a change of a polysemy multiplicity for mapping $(y_1, y_2) \to (x_1, x_2)$, where (x_1, x_2) defines control parameters and (y_1, y_2) is a characteristic surface. Whitney has found out two stable types of mappings – types that have not been destroyed after negligible deformations of surfaces or their projections. These types of mappings have been generalized for arbitrary manifolds with dimension up to 10 by Whitney's followers [2]. One of them led to the discrete change of a system's characteristic state – "cusp" catastrophe. It is represented for a one dimensional case at the Fig.1.

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An evolution of mapping leads to an occurrence of the Whitney's "cusp". The multiplicity or uncertainty is maximal in the unstable area of C – vicinity. According to [3] the disruption $D \to E$ appears as a fusion of stable and unstable regimes, marked by ovals. The characteristic distance between two stable attractors is proportional to the supercriticality level $x - x_{cr} = dx$: $H \propto \sqrt{|x - x_{cr}|}$.

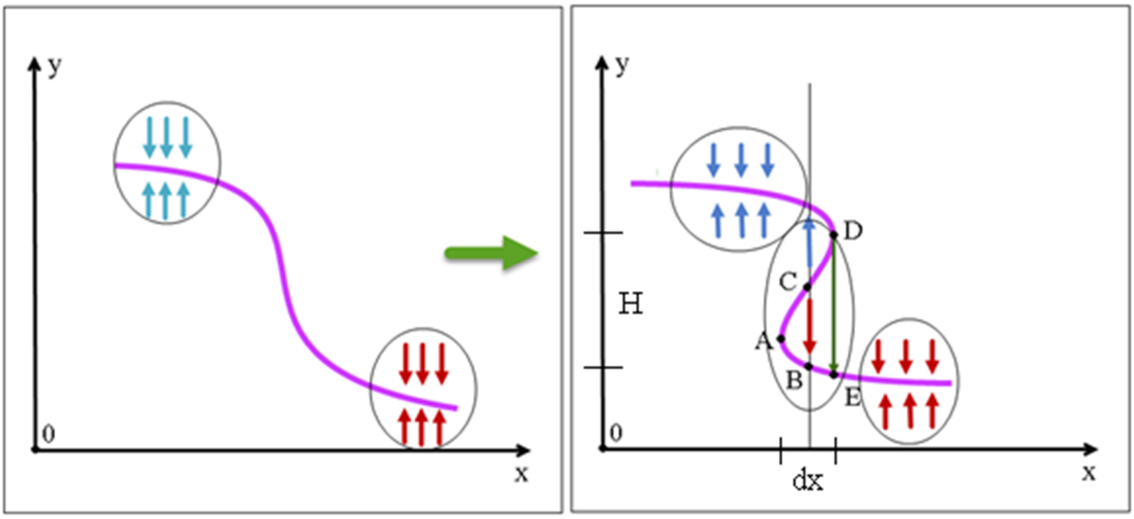

Fig.1. Appearance of Whitney's "cusp"

In terms of a bifurcation theory this one-dimensional evolution corresponds to the saddle-node fusion in a phase space. In the Fig.2 a fusion of one node and two saddles is represented. In fact this transformation is one dimensional as a node and a saddle has one dimensional peculiarity of trajectories direction change. In such a way, according to the Whitney's theorem "cusp" disruption is a consequence of attractor-repeller "annihilation" and a preliminary stabilization is as a necessary condition of the possible bifurcation. Another type of destabilization is a self-oscillating destabilization, considered by H. Poincare. In his revolutionary dissertation [4] he has shown that a birth of a new limited cycle in a phase space is realized by a transition through a stable equilibrium zone, i.e. the system should return to the stabilization before a new bifurcation occurs. A birth of a new cycle is preceded by the distortion and a death of a previous quasi stable regime. In such a way we have come to the same conclusion regarding a necessary condition of a catastrophe.

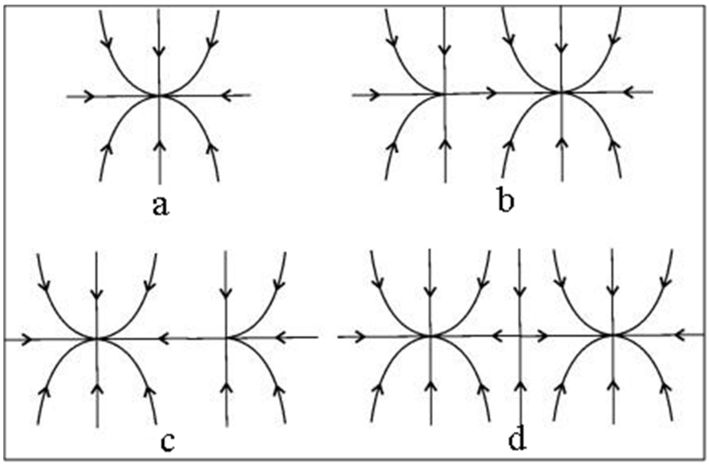

Fig.2. Fusion of one node and two saddles

A “calm before storm” or effect of small scale oscillations suppression before bifurcation has been noticed by M.M. Dubovikov and N.V. Starchenko in [5] as well. They have studied a behavior of financial time series by the use of Hurst fractal parameter of stability. Anatoly Neishtadt has shown that a delay of the dynamic bifurcation exists in case of all known analytical nonlinear systems [6] for adiabatic change of control parameter. It means that inertial properties resist a new synchronization – the system needs time for the restructuring as it happens in case of Ising model of magnetic domains. A delay depends on the clusters interaction and an intensity of the external “field”, i.e. macro scale influence. This model has been extended in the area of social complex systems by Callan E. and Shapiro O [7].

A comfortable choice of a macroscopic control parameter has been suggested in [8] on basis of Reynolds parameter for turbulent streams:

$$R(t) = f\left(\vec{\Pi}(t)\right) = \frac{q^{+}(t)}{q^{-}(t)} \tag{1}$$

Here $R(t)$ is basic phase parameter and $\vec{\Pi}(t)$ is set of microscopic control parameters - parametric vector. Quantities q^{+} and q^{-} correspond to power input and output per system unit. In given description bifurcation corresponds to the transition of an equilibrium state $R = 1$:

$$R(t_0) = 1 \Rightarrow \uparrow q^{+}(t) \Rightarrow \uparrow R(t) \Rightarrow R(t) \succ 1 \Rightarrow \uparrow q^{-}(t) \Rightarrow \uparrow R(t) \Rightarrow R(t_1) = 1 \tag{2}$$

$$R(t_0) = 1 \Rightarrow \downarrow q^{+}(t) \Rightarrow \downarrow R(t) \Rightarrow R(t) \prec 1 \Rightarrow \downarrow q^{-}(t) \Rightarrow \uparrow R(t) \Rightarrow R(t_1) = 1 \tag{3}$$

Here $\uparrow$ and $\downarrow$ show a finite increase and decrease of corresponding parameter for $t_1 \succ t \succ t_0$. The delay between a new cycle appearance and macro scale excitation is defined by inertial properties of system domains. We have to mark that a new bifurcation has to pass through an equilibrium quasi stable state of $R = 1$: in terms of statistical thermodynamics it corresponds to a minimum of Gibbs free energy $G(p,T)$. If a velocity of a control parameter change is higher then a minimal velocity of inertial processes, then a delay of rearrangement exists and is preceded by the change of relative excitation power $R(t)$.

2 Fractal analysis as an indicator of stability and its limitations

Inability to define strictly a set of control parameters or a global excitation balance obliged researchers to look for statistical measures of the system stability. One of possible approaches is a fractal analysis, suggested by Benoit Mandelbrot. He used the Hausdorff dimension as a basis for introduction a fractal object such that $D_H \succ D_T$. Here D_T is a topological dimension of a manifold, defined by a number of independent variables necessary for its description. D_H is Hausdorff dimension, determined by the relation (4):

$$D_H = \lim_{\varepsilon \to 0}\left(\frac{\ln N(\varepsilon)}{-\ln(\varepsilon)}\right) \tag{4}$$

Here $N(\varepsilon)$ is a number of elements, covering the given manifold, where ε is a characteristic size of an element. According to Mandelbrot [9] scale invariance is the necessary property of fractals. However we should note tha chaotic natural systems have scale characteristic limits. For example a turbulent flow has an internal micro scale, defined by inertial viscous forces and an external macro scale, defined by external hydrodynamic influence. Such type of system was denoted as quasi fractal by Mandelbrot, because they have a satisfactory fractal description only within given scale limits $\varepsilon_0 \prec \varepsilon \prec \mathrm{E}$.

An application of a fractal description to the time series $f(t_i)$ meant an investigation of statistical properties in case of several time resolutions $\varepsilon = {}_{\Delta}t$.

Covering elements can be defined as rectangles with heights, defined by function range in a given interval [5]: $h_i = \max(f(t_i),{}_\Delta t) - \min(f(t_i),{}_\Delta t)$. Then a fractional dimension can be expressed in the following way:

$$D_F = \lim_{{}_\Delta t\to 0}\left(\frac{\ln R({}_\Delta t)}{-\ln({}_\Delta t)}\right)+1 \qquad R({}_\Delta t)=\sum_i h_i({}_\Delta t) \tag{5}$$

According to this relation it is defined by the true range of its integral element $h_i({}_\Delta t)$. It is important to underline that nondifferentiability as a necessary property of scale invariance is obligatory for a fractal time series but is impossible for natural time series. In fact it would mention an infinite energy of such range and a violation of energy conservation law.

Another fractal characteristic, introduced by Mandelbrot for the description of stochastic time series is a fractal Hurst factor. It was induced through the relation of Fractional Brownian Motion (FBM). An idea of FBM introduction was inability to explain deviations from normal distributions in some natural systems, for example financial markets. Pareto – Levy distributions have been obtained as particular cases of such abnormal behavior. Mandelbrot decided to make generalization.

Let us consider a Standard Brownian Motion (SBM) time series $B(t)$, which satisfies a normal distribution. Then an FBM increment can be expressed in the following way [10]:

$$B_H(t_2)-B_H(t_1)=\frac{1}{\Gamma(H+1/2)}\int_{-\infty}^{t_2}(s-t_2)^{1/2-H}dB(s)-\frac{1}{\Gamma(H+1/2)}\int_{-\infty}^{t_1}(s-t_1)^{1/2-H}dB(s) \tag{6}$$

A given increment is expressed through the fractal derivatives of SBM with a factor $k=(0.5-H), 0\prec H\prec 1$. This factor defines a deviation from a standard markovian Brownian motion ($H=0.5$). FBH allows obtaining anomalous distributions with "thick tails", and a flexible explaining of flights. According to [10] an expectation of FBM deviation is self-affine:

$$E\left([B_H(t+T)-B_H(t)]^2\right)=V_H\cdot T^{2H} \tag{7a}$$

$$V_H=\left(\frac{1}{\Gamma(H+1/2)}\right)^2\cdot\left[\int_{-\infty}^{0}\left[(1-s)^{H-1/2}-(-s)^{H-1/2}\right]^2 ds+\frac{1}{2H}\right] \tag{7b}$$

This means that a probable amplitude of the deviation depends on a time scale and a Hurst factor H of system's memory. If $H=0.5$ then relation (7a) corresponds to the Einstein's law of Brownian walks:

$$E\left([B(t+T)-B(t)]^2\right)=V_{0.5}\cdot T \tag{8}$$

If $H\neq 0.5$ we achieve an anomalous transport, that includes Levy flights and "thick tails" of distribution for $H\succ 0.5$. Despite a charm of this approach it has several limitations, enumerated below.

a) <u>FBM is achieved as weighted averaged Brownian motion</u>.

According to the original work of B. Mandelbrot [10] "FBM of the exponent H is a moving average of $dB(t)$ in which past increments of $B(t)$ are weighted by the kernel $(t-s)^{H-1/2}$". The weights are defined on the basis of time distance between current moment and previous states $\Delta=(s-t_j)$. The intensity of a history influence is determined by a memory factor $0\prec H\prec 1$. However FBM operator assumes SBM kernel for weighted average. It means that FBM is considered as dynamical moving weighted integration of standard Brownian process. If we calibrate FBM such that $B_H(t_0)=0$ then an absolute value can be expressed in the following way:

$${}_\Delta B_H(t)=\frac{1}{\Gamma(H+1/2)}\int_{-\infty}^{t}(s-t)^{1/2-H}\cdot dB(s) \tag{7}$$

If we consider a motion only in negligible time range $(t-ds,t)$ then this relation can be simplified:

$$dB_H(t)=\frac{1}{\Gamma(H+1/2)}ds^{1/2-H}\cdot B'(s)\cdot ds=\frac{ds^{1/2-H}}{\Gamma(H+1/2)}\cdot dB \tag{8}$$

Up to a constant factor this relation corresponds to the SBM increment. In such a way FBM assumes limitations of Markovian process that should be satisfied for small time deviations. It should be mentioned that small time deviations allow to state connection between fractal dimension of time series and memory factor: $D_F=2-H$. However in many works, for example [5] and [11] this relation is applied for macroscopic time scales. This approach creates a logical conflict for FBM was initially introduced to explain a non Gaussian process.

b) <u>An increment of the FBM has an infinite exact energy</u>.

As it was stated by Mandelbrot [10] a first fractal derivative of FBM and consequently its energy diverges for the range $0\prec H\prec 1$. To overcome this obstacle he has introduced a smoothed derivative where a range of smoothing δ is defined artificially:

$$B'_H(t,\delta)=-\int_{-\infty}^{\infty}B_H(s)d\varphi(t-s) \tag{9}$$

$$\varphi(t)=\frac{1}{\delta},t\in[0,\delta]\quad \varphi(t)=0,t\prec 0\quad \varphi(t)=0,t\succ\delta$$

However that is not the only procedure to introduce "physical" derivative (we may use a weighted derivative as well) and that's why the universality of a dynamic description is lost;

c) <u>Hurst factor expresses a linear measure of memory and is not applicable for nonlinear cases</u>.

This remark needs a certain clarification. According to [10] a linear autocorrelation function of a first derivative can be expressed in the following way:

$$C_H(\tau,\delta)=E\left(\frac{\partial B_H}{\partial t}(t,\delta)\middle|\frac{\partial B_H}{\partial t}(t+\tau,\delta)\right)=V_H\cdot H\cdot(2H-1)|\tau|^{2(H-1/2)} \tag{10}$$

It is a quadratic function of Hurst factor and depends on the artificial smoothing parameter δ. Cases of $H\succ 0.5$ and $H\prec 0.5$ correspond to the persistent and antipersistent trends correspondingly [5,10]. In case of markovian SBM $H=0.5$ and $C_H(\tau,\delta)=0$. However that does not mean an absence of system memory. For markovian process a probability connection is stated by Chapman - Kolmogorov equation which is not linear in general:

$$W(x_3,t_3|x_1,t_1)=\int dx_2 W(x_3,t_3|x_2,t_2)\cdot W(x_2,t_2|x_1,t_1) \tag{11}$$

In fact a Hurst factor as a memory measure can be applied for the characterization of linear trends in regard to the function ${}_\Delta B_H(t)$, but according to a standard FBM model it can't be used generally for the indication of pre-catastrophic stabilization, considered in the section 1.

3 Extended interpretation of Hurst parameter

In this section we shall consider a diffusive approach to the characterization of pre-catastrophic stabilization effect (PS-effect), noted in the Section 1. This description assumes an introduction of a second transport factor, used in a standard Fokker-Planck equation (12). This equation has been derived on a basis of Chapman - Kolmogorov equation (11) and thus is applied for markovian processes:

$$\frac{\partial P(x,t)}{\partial t}=\frac{1}{2}\cdot\frac{\partial}{\partial x}\left(D(x)\cdot\frac{\partial P(x,t)}{\partial x}\right)\quad D(x)=\lim_{{}_\Delta t\to 0}\left(\frac{\langle\langle {}_\Delta x^2\rangle\rangle}{{}_\Delta t}\right)=\frac{\langle\langle {}_\Delta x^2\rangle\rangle}{{}_\Delta t_{\min}} \tag{12}$$

Here double brackets designate an averaging of an initial coordinate:

$$b(x,{}_\Delta t)=\int(x-x_0)^2\cdot W(x,x_0,{}_\Delta t)dx_0=\langle\langle {}_\Delta x^2\rangle\rangle \tag{13}$$

It is significant to note that a factor $b(x,{}_{\Delta}t)$ is equal to the left part of an equation (7a) for the expectation of FBM shift. It has been shown in [12] that systems of phase mixing have second factor with explicit time dependence, expressed through the specific energy of characteristic vector (Fig.14, Fig.15).

$$D(x,t)=\frac{\left\langle\left\langle {}_{\Delta}x^{2}\right\rangle\right\rangle}{{}_{\Delta}t_{\min}}={}_{\Delta}t_{\min}\cdot\int\varepsilon(x,t)\cdot W(x,x_{0},{}_{\Delta}t)dx_{0}={}_{\Delta}t_{\min}\cdot\left\langle\left\langle\varepsilon(x,t)\right\rangle\right\rangle \tag{14}$$

$$\varepsilon(x,t)=\lim_{{}_{\Delta}t\to 0}\left(\frac{{}_{\Delta}x(x,t)}{{}_{\Delta}t}\right)^{2}=\left(\frac{{}_{\Delta}x(x,t)}{{}_{\Delta}t_{\min}}\right)^{2} \tag{15}$$

Let's introduce a variable time lag $T=t-t_{0}$ and a power regression of the transport factor that will be denoted as a dynamic diffusion below:

$$D(t_{0},T)=D_{0}(t_{0})\cdot T^{\kappa} \tag{16}$$

Then an expectation of the stochastic shift can be represented in the following way:

$$E\left([x(t_{0}+T)-x(t_{0})]^{2}\right)=D_{0}(t_{0})\cdot T^{\kappa+1} \tag{17}$$

A comparison of this relation with an equation (7a) allows expressing the Hurst factor through a stability coefficient $\kappa: H=\frac{\kappa+1}{2}$. Unlike FBM procedure we made no assumptions regarding a micro scale probability distribution function. That's why a generalized Hurst parameter has no obligatory preliminary limitations in frame of this model $-\infty\prec H\prec\infty$. It doesn't create an artificial energy divergence and may be applied to natural systems directly without smoothing. However an extended Hurst analysis still has a limitation of markovian processes (11). A critical value of $H_{cr}=0.5$ corresponds to the constant diffusion case and is a boundary of the diffusive expansion ($H\succ 0.5$) and the diffusive contraction ($H\prec 0.5$) of a characteristic area. Thus a generalized Hurst factor is a measure of the attractor stability. We may introduce a potential of attraction on the basis of a diffusive scale:

$$\Lambda_{D}=\sqrt{\left\langle\left\langle {}_{\Delta}x^{2}\right\rangle\right\rangle}=\sqrt{D_{0}(t_{0})}\cdot T^{H} \tag{18}$$

Then a diffusive acceleration can be represented in the following way:

$$\frac{\partial^{2}\Lambda}{\partial T^{2}}=\sqrt{D_{0}(t_{0})}H(H-1)\cdot T^{H-2}=-\frac{\partial V_{eff}}{\partial x} \tag{19}$$

Here V_{eff} is an efficient volume potential of attraction of a given phase space. In regard to the time series analysis it characterizes a volatility of a considered time series:

$$\int\sqrt{D_{0}(t_{0})}H(1-H)\cdot T^{H-2}dx=V_{eff} \tag{20}$$

It is an integral characteristic of a manifold internal interaction. According to the relation (20) the value of an efficient potential of attraction tends to zero for $t\to\infty$ - a "deliquescence of the phase drop" in terms of G.M.Zaslavsky [13]. It means that a system may loose memory even for "persistent" case of $H\succ 0.5$. This may happen if a characteristic time period $T\succ\frac{1}{h_{d}}$, where h_{d} is dynamic entropy of Kolmogorov [13]:

$$h_{d}=\frac{\partial S}{\partial t}=\frac{\partial}{\partial t}\ln\left({}_{\Delta}\Gamma\right) \tag{21}$$

Here $S=\ln\left({}_{\Delta}\Gamma\right)$ expresses Gibbs entropy through a phase space area ${}_{\Delta}\Gamma$. Dynamic entropy shows instability of phase trajectories and their exponential expansion intensity. That's why, as it was mentioned in the Section 2 FBM Hurst factor can't characterize a nonlinear system memory in general case.

A sense of a generalized Hurst factor can be clarified with a use of spectral description. If we introduce a characteristic frequency $_{\Delta}t = 1/f$, then according to the relation (16) a following formula may be represented for a dynamic diffusion spectrum:

$$D(t_0, f) = D_0(t_0) \cdot f^{-\kappa} \tag{22}$$

We may note that the case of $H \succ H_{cr}$ corresponds to the large scale transport, while if $H \prec H_{cr}$ a micro scale energy absorption is more intense. A shift of a basic absorption band from micro scale range to macro scale range corresponds to catastrophic behavior, when a coherent motion and resonances appear. A most stable case of attractor corresponds to a Brownian transport law: $H = H_{cr} = 0.5$, when domain shifts are assumed to be independent and normally distributed. An artificial Gaussian mixing, for example random permutations of financial time series have shown that a Hurst factor tends to a critical value [11]. However we should mention that relations (16) and (22) assume a reverse power law. This type of transport is typical for a uniform Kolmogorov turbulence [14] and flicker electrical noise [15], when large scale fluctuations are more intense and more weakly dissipated. An experimental research of $D(t_0, T)$ allows finding out an evolution of a natural scale-law, corresponding to the investigated complex system.

4 Diffusive analysis and R – bifurcations: equivalence of descriptions

In Section 1 we have considered a pre-catastrophic stabilization effect (*PS* - effect) as a first necessary condition of the bifurcation. In regard to the diffusive analysis that means a preliminary intensification of small scale fluctuations with a following large scale catastrophe. If we consider $f^{-\kappa}$ systems like turbulent flow or a flicker noise, then according to (22) *PS* – effect corresponds to a formation of a κ local minimum. Correspondingly an extended Hurst coefficient $H = \dfrac{\kappa + 1}{2}$ forms a new minimal value. For an arbitrary system of undefined scale law an empirical dependence $D(t_0, f) = g(f)\big|_{t_0}$ should be primarily stated.

Generally it should be marked that a PS-effect leads to the small scale spectrum band intensification with a following transition to a large scale transport. One of possible ways, that can be used to compare macro/micro transport properties, is represented by the relation (23).

$$I(t_0) = \frac{\int_{T_{\min}}^{T_{\max}/2} D(t_0, T)\,dT}{\int_{T_{\min}}^{T_{\max}/2} D(t_0, T)\,dT} \tag{23}$$

Here an integral stabilization factor *I* is expressed through the relation of small frequency and high frequency integrals. A total integration range $\left(T_{\min}, T_{\max}\right)$ is defined with an account of measurement resolution.

Another alternative is a momentary transport analysis of the uniform markovian time series $x_i = x(t_i)$. It can be outlined on a basis of the expression (24). This relation allows defining an average transport for the period $T = t_i - t_{i-N}$. This factor reaches a new minimum during a pre-catastrophic stabilization phase: $_{\Delta}D(x_i, T) = 0$. A disruption leads to an increase of the momentary transport due to a large scale motion.

$$D(x_i, T) = \frac{\left\langle\left\langle {}_{\Delta}x_i^{\,2} \right\rangle\right\rangle}{t_i - t_{i-N}} = \frac{\sum_{j=i-N}^{i} \left(x_i - x_j\right)^2}{T} \qquad {}_{\Delta}D(x_i, T) = 0 \Rightarrow {}_{\Delta}D(x_i, T) \succ 0 \tag{24}$$

An alternative choice of a control parameter has been suggested in a Section 1 on the basis of an extended Reynolds factor. It may be denoted as a basic phase parameter $R(t)$. According to mechanisms (2) and (3) a bifurcation corresponds to the state $R(t) \neq 1$, and a following formation of a new quasi cycle such that $R(t) \to 1$. As it was shown in [8] two principle types of disruption are possible. Let's use auxiliary dynamic entropy of Kolmogorov h_d [13]: $h_d = \langle h(\vec{x}(t)) \rangle$. Here averaging in phase space is designated as $\langle\ \rangle$ and averaged quantity can be expressed as sum of positive Lyapunov factors h_i^+ for each dimension of generalized phase space:

$$h = \sum_{i=N}^{K} h_i^+ = \ln\left(\prod_{i=N}^{K} \sigma_i^+\right) \qquad \sigma_i^+(t) = \frac{|\delta x_i(t)|}{|\delta x_i^0|} \tag{25}$$

We may denote $\vec{x}(t)$ is characteristic phase vector of system state. Factor σ_i^+ shows distance growth $\delta x_i(t)$ in i direction for two infinitely closely located points in phase space. Condition of stationary state then is equal to $h = 0$ or $\sigma_i^+ = 1$ $(i = \overline{N,K})$.

Let's introduce a relation for specific system power:

$$q(t) = \frac{\delta}{\delta t}\left(\sum_{i=1}^{K} \frac{v_i^2}{2}\right) = \sum_{i=1}^{K} v_i \cdot \dot{v}_i = q^+(t) - q^-(t) \tag{26}$$

System stability condition leads to $h_i \le 0$ and $\sigma_i \le 1$ if we consider all Lyapunov factors. Given inequalities lead to expression (27) for velocity components v_i $\left(i = \overline{1,K}\right)$.

$$\frac{|\delta x_i(t)|}{|\delta x_i^0|} = \frac{|\delta x_i(t)/\delta t|}{|\delta x_i^0/\delta t|} = \frac{|v_i(t)|}{|v_i^0|} = \alpha(t) \qquad 0 \prec \alpha(t) \le 1 \tag{27}$$

Relation (27) in fact allows receiving components of acceleration $\dot{v}_i(t)$.

$$\lim_{_\Delta t \to 0}\left(\frac{|v_i(t)| - |v_i^0|}{_\Delta t}\right) \le 0 \qquad \dot{v}_i(t) = \lim_{_\Delta t \to 0}\left(\frac{v_i(t) - v_i^0}{_\Delta t}\right) \tag{28}$$

Indeed, consideration of specific power $q(t)$ can be reduced to two cases: a) $\delta x_i(t) \succ 0$ and $|\delta x_i(t)| = \delta x_i(t)$; b) $\delta x_i(t) \le 0$ and $|\delta x_i(t)| = -\delta x_i(t)$. Signs of $\delta x_i(t)$ and δx_i^0 match - this condition is obligatory for definition of Lyapunov factors. Then $\alpha(t)$ doesn't depend on initial sign of coordinate shift δx_i^0.

For cases a) and b) we then receive: a) $\dot{v}_i \prec 0$ and $v_i \succ 0$; b) $\dot{v}_i \ge 0$ and $v_i \le 0$. In both cases with use of relation (26) we receive that $q(t) \le 0$. According to definition (1) of basic phase parameter this means that $R(t) \le 1$ for $h_d \le 0$. Condition of R_f, i.e. $R(t) = 1$ corresponds to a new stable regime. Thus use of $R(t)$ as control parameter must be delimited for two types of system [8]: a) accelerator - $\dot{v}_i(t) \succ 0$; b) decelerator - $\dot{v}_i(t) \le 0$. For first type of system motion stability loss and bifurcation are realized for $R(t) \prec 1$, while decelerator comes to transition only for $R(t) \succ 1$. In both cases a transition corresponds to the following requirement:

$$R(t) = 1 \Rightarrow R(t) \neq 1 \qquad {}_\Delta\varepsilon(t) = 0 \Rightarrow {}_\Delta\varepsilon(t) \neq 0 \tag{29}$$

Let's define a connection between a diffusive condition (23) and the relation (28) for a particular choice of a control parameter $R(t)$. A condition ${}_\Delta D(x_i, T) = 0$ can be easily simplified for a time series in the following way:

$$\sum_{j=i-N}^{i}(x_i - x_j)^2 = \sum_{k=i-N-1}^{i-1}(x_{i-1} - x_k)^2 \qquad \varepsilon_i^j = \frac{1}{2}\left(\frac{x_i - x_j}{t_i - t_j}\right)^2 \tag{30}$$

A reduced energy tensor of a characteristic vector is represented through the second part of a given expression. Then a diffusive equilibrium requirement can be represented in the following way:

$$\sum_{j=i-N}^{i}\varepsilon_i^j \cdot (t_i - t_j)^2 = \sum_{k=i-N-1}^{i-1}\varepsilon_{i-1}^k \cdot (t_{i-1} - t_k)^2 \tag{31}$$

This requirement is equal to the relation (31):

$$\varepsilon_i^j = \varepsilon_{i-1}^k \quad j \in [i-N, i] \quad k = j-1 \tag{32}$$

For $T \to 0$ we come to the conditions (29), which means that for small time increments both descriptions of a PS – effect are equivalent.

5 R/D analysis: application to financial markets

In this section we shall consider several illustrations of diffusive and R-analysis applications to a financial market description. A deep trading history of basic stock indexes and an availability of data make financial time series comfortable examples for a nonlinear analysis.

Let's consider a time period of 16.07.2006-21.02.2010 which includes two significant phases – a beginning of the world financial crisis and a gradual recovering. According to the analysis of George Soros [16] a preliminary origin of the crisis corresponds to the falling of bank liquidity in August 2007. In September 2008 it caused a mortgage crisis and a failure of greatest American mortgage agencies: Lehman Brothers, Fannie Mae, Freddie Mac. All world stock indexes reacted sharply – speculators observed greatest indexes collapse since "Black Monday" of 1987. Dow Jones Industrial (DJI) Average was not an exception. This index is formed as an average price of 30 industrial US stocks. During the period of August 2007 – January 2009 it has lost 51% of its initial price. We should mark that in January 2009 US Federal Reserve has started the fourth supporting program of financial stabilization (QE4) that led to a preliminary stabilization.

Let's analyze a time series of DJI weekly prices – an index price was defined at the trading end of each Friday (end of the trading week). In Fig.3 an upward trend, marked as rising corridor, has been broken in August 2007.

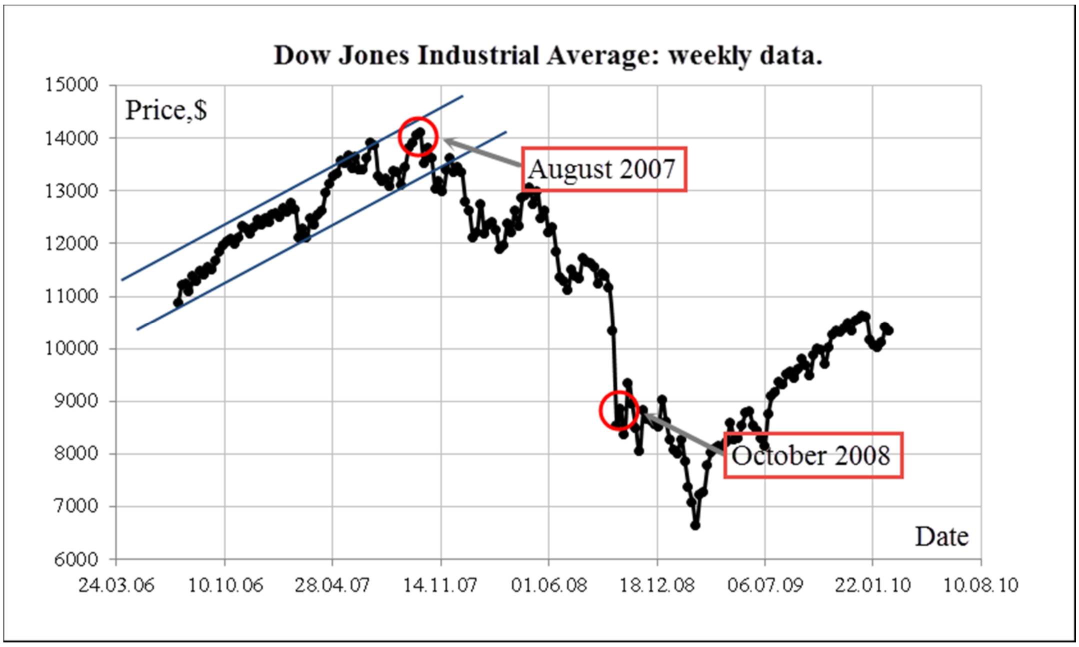

Fig. 3. DJI weekly time series. Two stages of crisis – a failure of bank liquidity in August 2007 and the collapse of key mortgage agencies in September 2008.

However a total collapse of this index corresponds to the failure of mortgage agencies in October 2008. Let's demonstrate an application of two approaches that were considered above – analysis of a basic phase parameter $R(t)$, R - analysis and a diffusive analysis, D - analysis. Both will be applied for the discovering of new system disruptions, corresponding to macro scale shifts of price.

$$\varepsilon_i = \frac{1}{2}\left(\frac{x_i - x_{i-1}}{{}_\Delta t_i}\right)^2 \quad {}_\Delta\varepsilon_i = \varepsilon_i - \varepsilon_{i-1} \quad {}_\Delta t_i = {}_\Delta t = const \quad i = \overline{1, N} \tag{33}$$

The set of relations above (33) expresses a difference approximation of the parameter ${}_\Delta\varepsilon_i(t_i)$. According to the conditions, considered above, an attraction sign should be defined simultaneously: $\dot{v}_i(t) \succ 0$ or $\dot{v}_i(t) \prec 0$ scenario must be chosen for a correct indication of the bifurcation: a) $\dot{v}_i(t) \succ 0$ and ${}_\Delta\varepsilon_i(t_i) \prec 0$ or b) $\dot{v}_i(t) \prec 0$ and ${}_\Delta\varepsilon_i(t_i) \succ 0$. Let's compose a time dependence of a product $B_i = {}_\Delta\varepsilon_i \cdot \dot{v}_i$, where acceleration is expressed by formulas (34).

$$v_i = \frac{x_i - x_{i-1}}{{}_\Delta t_i} \qquad \dot{v}_i = \frac{v_i - v_{i-1}}{{}_\Delta t_i} \qquad i = \overline{1, N} \tag{34}$$

An appearance of a disruption then corresponds to the transition $B_i = 0 \Rightarrow B_i \prec 0$. A graph of a normalized bifurcation indicator $B(t)$ is represented in Fig.4. Bifurcations are indicated at the points of OX intersection. A minimal value corresponds to the largest index fall of September-October 2008 - the failure of greatest American mortgage agencies and a mortgage crisis in US.

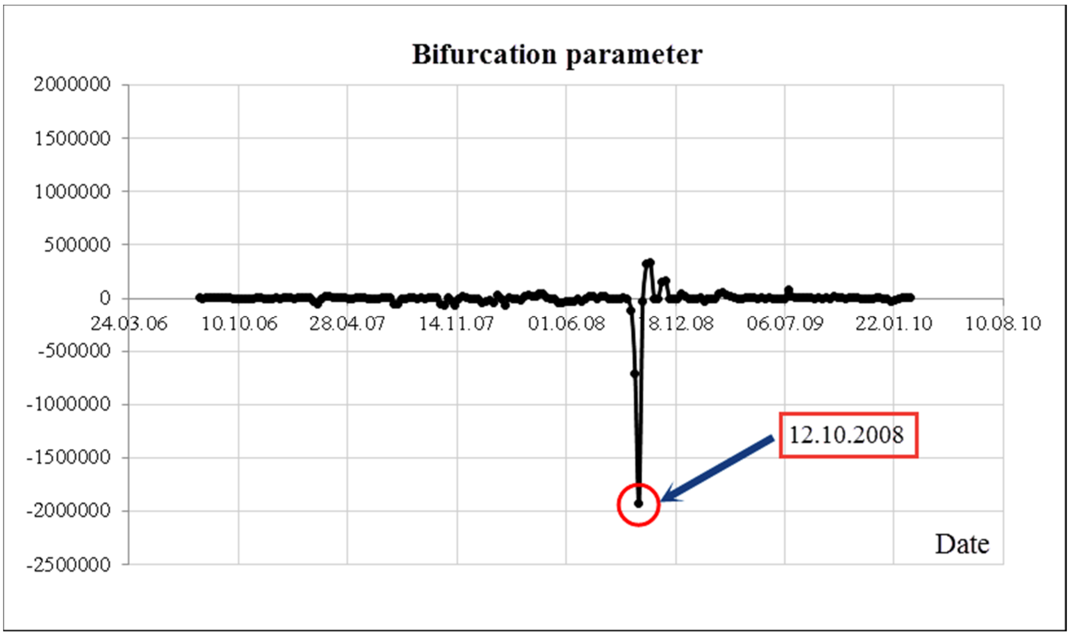

Fig. 4. Bifurcation parameter time series: 16.07.2006-21.02.2010.

Let's mark the date points of a critical bifurcation parameter $B_i \prec 0$. They are denoted by red points of date axis in Fig.5. We have marked four clusters of critical points, corresponding to bifurcations: 14.10.07-24.02.2008, 04.05.2008-22.06.2008, 17.08.2008 – 05.10.2008 and 11.01.2000 – 15.02.2009. First cluster defines a delay between a bank crisis and a market reaction – we may observe inertial properties of a market system. However R – analysis shows preliminary signals before a trend channel breakthrough. A third cluster corresponds to the mortgage crisis and a minimal B_i of Fig.4.

A second approach that should be considered is a diffusive markovian analysis, represented above. The transport factor approximation has been calculated on the basis of relations (24). Its normalized values are displayed in Fig.6. Highest amplitude of D_i fluctuation again corresponds to October 2008, the mortgage crisis. An each intersection of a date axis indicates a new markovian bifurcation. Points of intersections have been defined through the following condition:

$$_{\Delta}D(x_{i-1},T)\cdot_{\Delta}D(x_i,T) \prec 0 \tag{35}$$

This relation allows determining of derivative sign change: $_{\Delta}D(x_{i-1},T) \succ 0 \Rightarrow _{\Delta}D(x_i,T) \prec 0$ or $_{\Delta}D(x_{i-1},T) \prec 0 \Rightarrow _{\Delta}D(x_i,T) \succ 0$. In Fig, 7 principal clusters are marked in the OX axis as it has been done in Fig.5. It should be marked that in both cases of R – analysis and D - analysis we have used a natural criterion of clustering indication: 5 consequent date points of switched bifurcation indicator.

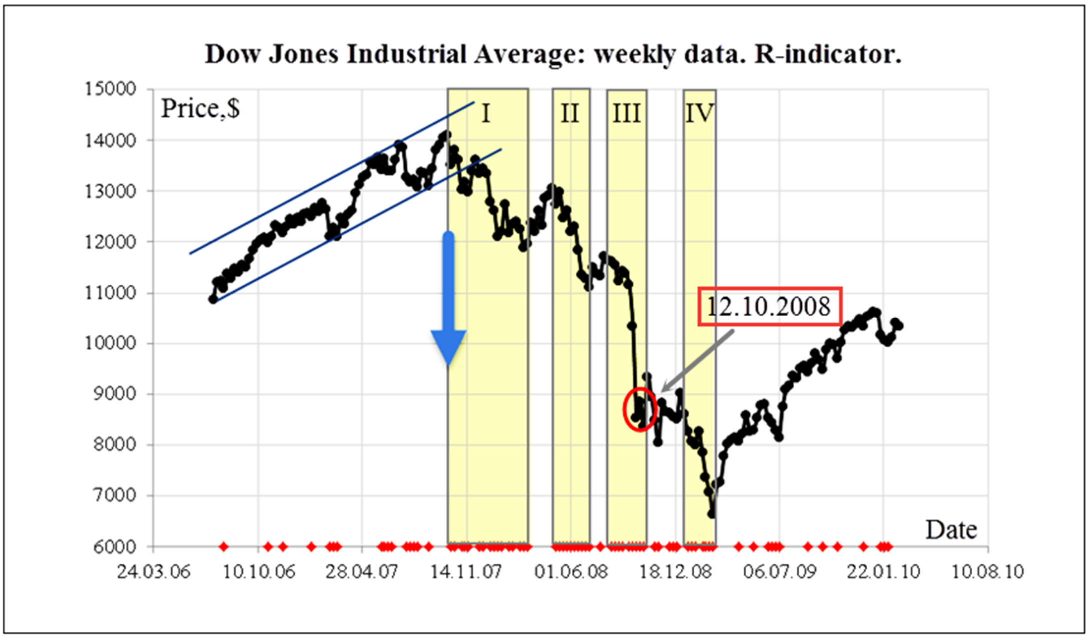

Fig. 5. Clusters of a critical bifurcation: *R* – analysis.

A period of 5 weeks has been chosen for this statistical date is necessary for a correct definition of the $B_i = _{\Delta}\varepsilon_i \cdot \dot{v}_i$ series. However if one uses these types of analysis separately then a time period is introduces subjectively or in accordance with practical needs. It is significant to emphasize that an *R* – analysis has a wider area of application since it is not limited by markovian processes and may be applied to processes with long memory, like inertial trends. In Fig.7 areas of disruption, defined by both *R* – analysis and *D* – analysis, are marked by a blue color. Thus a diffusive analysis has demonstrated 50% efficiency in relation the *R* – description. Let's designate a combination of *R* – analysis and *D* – analysis as *R/D* - analysis.

This combined type of a description allows distinguishing of short-memory, markovian stages of a market evolution and long memory processes. Considered historical period of 2006-2010 allowed finding out of two short memory periods – 14.10.07-24.02.2008, 04.05.2008-22.06.2008 and two long memory stages – 17.08.2008 – 05.10.2008, 11.01.2009 – 15.02.2009. If we define a financial crisis as a systematic, inertial large scale failure of basic economic indexes, then, according to *R/D* – analysis, its beginning corresponds to October 2008. This is a date of propagation beginning through adjacent developed markets like markets of European Union.

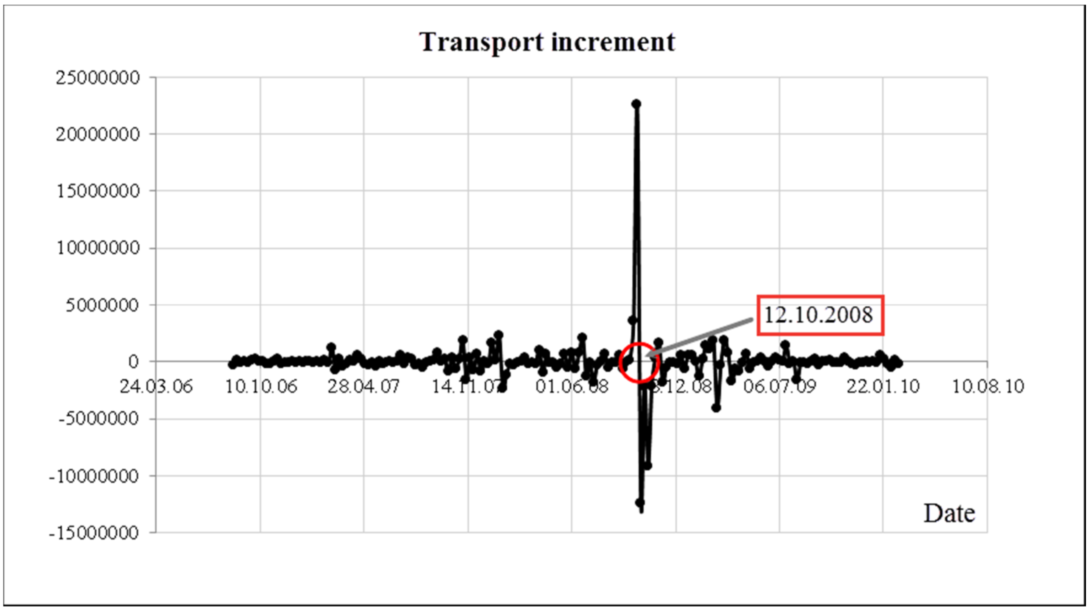

Fig. 6. A transport increment time series: 16.07.2006-21.02.2010.

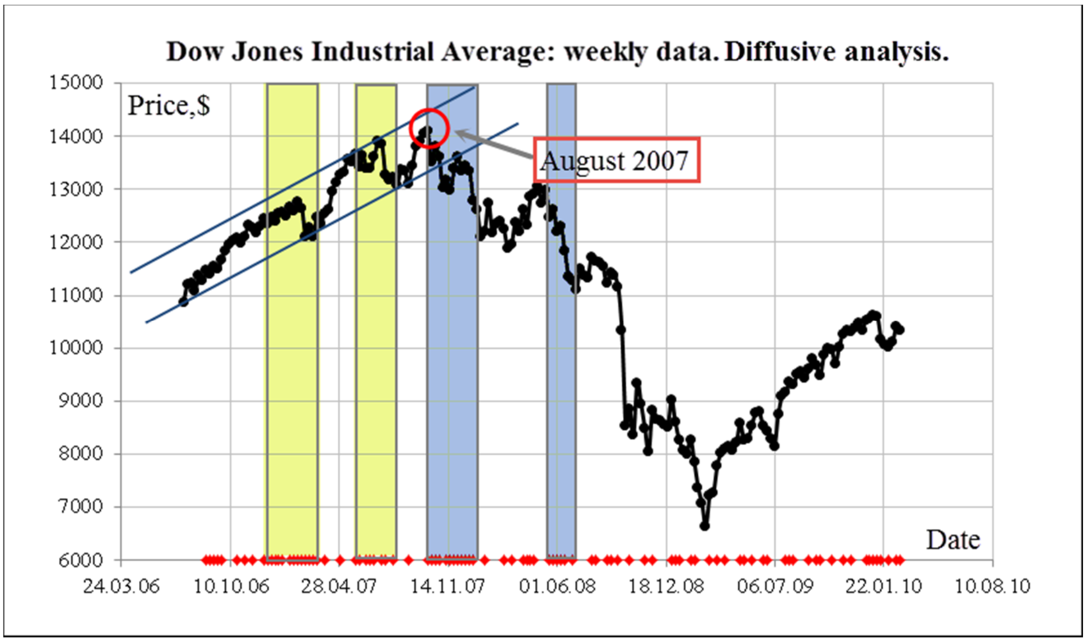

Fig. 7. Clusters of a critical bifurcation: diffusive analysis.

6 Conclusions

In this paper a combined transport analysis has been considered for the description of a pre-catastrophic stabilization. This effect has been stated as a fusion of stable and unstable parametric areas. A delay between a new cycle appearance and a macro scale excitation is defined by inertial properties of system domains. That's why an indication of transport anomalies may help to forecast a catastrophe. We have shown that a standard fractal analysis introduce artificial properties into physical process: nondifferentiability, Brownian nature and linear memory measure. In this frame a Hurst factor can not be used for the indication of nonlinear pre-catastrophic stabilization. Additionally a self affinity of a fractal time series makes impossible a direct calculation of the process energy.

A sense of a Hurst factor has been extended by the consideration of a phase diffusion power law. This relation characterizes a shift of a basic absorption band from a micro scale to a macro scale range when a coherent motion and resonances appear. However it was stated that such type of description assumes a power frequency spectrum. To overcome this limitation momentary phase

diffusion has been introduced. An experimental research of diffusion allows finding out the evolution of a natural scale-law, corresponding to the investigated complex system. This factor reaches a new minimum during a pre-catastrophic stabilization phase: $_{\Delta}D(x_i,T)=0$. A disruption leads to an increase of the momentary transport due to a large scale motion. In regard to the diffusive analysis that means a preliminary intensification of small scale fluctuations with a following large scale shift, i.e. catastrophe. We have applied a Lyapunov-Reynolds analysis for the verification of a diffusive description. It was found out that for an accelerated phase motion a stability loss and the bifurcation are realized for $R(t)\prec 1$, while a decelerator comes to the transition only for $R(t)\succ 1$. Basic conclusions of a diffusive analysis have been compared to the Lyapunov stability model and verification has been found for the small time increments.
A combined diffusive and Reynolds analysis has been applied for a description of a time series of Dow Jones Industrial weekly prices during a world financial crisis of 2007-2009. Diffusive and Reynolds parameters shown extreme values in October 2008 - the failure of greatest American mortgage agencies. This combined type of a description has allowed distinguishing of short-memory markovian stages and long memory processes of a market evolution. The considered historical period of 2006-2010 allowed finding out of two short memory periods – 14.10.07-24.02.2008, 04.05.2008-22.06.2008 and two long memory stages – 17.08.2008-05.10.2008, 11.01.2009-15.02.2009. It was stated that a systematic large scale failure of a financial market began in October 2008 and started fading in February 2009.

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