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Vector Operations Transform Into Matrix Operations

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Abstract. An efficient technique is developed to simplify the computations in the field of vector analysis. The evaluation of vector algebraic and differential operations becomes more simple and straightforward by simply transforming the vector operations into matrix operations. The matrix operations are especially useful when there are mixed coordinate basis involved in the vector operations.

Keywords: Vectors, Matrices, Vector differential operators, Vector algebraic operators, Matrix multiplication, Coordinate transformation.

1 Introduction

Many problems in science and engineering require substantial amount of vector and matrix analysis. In many situations, it is generally more efficient to work with matrix operations than to deal with vector operations. In this paper a useful technique, "Vector operation Transforms into Matrix operation" (VTM), is developed to simplify the manipulation of vector algebraic and differential operations. Generally, in vector operations, components and basis unit vectors are inseparable and must stick together; while in VTM technique, components and basis unit vectors can be dealt with separately and usually, only the vector components are involved by skipping the basis unit vectors [1]. The matrix manipulation is especially more favorable when there are different coordinate basis involved in the vector operations [2].

As an illustration, for a vector differential operation

$$\mathbf{H} = \nabla \times \mathbf{A}$$

the explicit expression of both sides in spherical coordinates may be written in vector expression as

$$\hat{r}H_r + \hat{\theta}H_\theta + \hat{\phi}H_\phi = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ A_r & rA_\theta & r\sin\theta\,A_\phi \end{vmatrix}$$

$$= \hat{r}\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial}{\partial\phi}A_\theta\right) + \hat{\theta}\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}A_r - \frac{\partial}{\partial r}(rA_\phi)\right) + \hat{\phi}\frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial}{\partial\theta}A_r\right)$$

or converted into matrix form as

$$\begin{bmatrix} H_\theta \\ H_\phi \\ H_r \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{r}\frac{\partial}{\partial r}r & \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \\ \frac{1}{r}\frac{\partial}{\partial r}r & 0 & -\frac{1}{r}\frac{\partial}{\partial\theta} \\ -\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} & \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\sin\theta & 0 \end{bmatrix}\begin{bmatrix} A_\theta \\ A_\phi \\ A_r \end{bmatrix}$$

Obviously, the matrix form is more simplified and favorable than the vector expression, since in the matrix form, the unit vectors $(\hat{r},\hat{\theta},\hat{\phi})$ are not required to be shown and can be omitted. Also, the operator $\nabla\times$ and the operand A are clearly separated, which makes the operation easier to manipulate. Therefore, any vector operations can be performed more efficiently by simply converting them into matrix operations.

To derive explicit formulas of double vector differential operations in spherical coordinates, such as $\nabla\nabla\cdot\mathbf{A}$, $\nabla\times\nabla\times\mathbf{A}$, $\nabla^2 f \equiv \nabla\cdot\nabla f$, $\nabla^2\mathbf{A} \equiv \nabla\nabla\cdot\mathbf{A} - \nabla\times\nabla\times\mathbf{A}$, it may generally take several hours to get the explicit results through the conventional approach, while it can take only few minutes to accomplish the same results by using the VTM technique.

Some interesting examples related to VTM will be demonstrated in the later sections. In addition, an application in electromagnetic field solutions is provided to show the merit of VTM technique presented.

2 Basic Operations

A matrix representing a vector or vector operator are to be enclosed by brackets, possibly with superscript indicating its coordinate basis. The superscripts *c*, *d*, and *s* are respectively used to denote the Cartesian, cylindrical, and spherical coordinate basis.

These superscripts may be omitted, inside or outside of the brackets as needed. For example, $[\,\mathbf{A}^c],[\,\mathbf{A}^c\cdot],$ and $[\,\mathbf{A}^c\times]$ represent the vector, dot-product, and cross-product operators in the Cartesian basis; while $[\,\nabla\,]^s, [\,\nabla\cdot\,]^s, [\,\nabla\times]^s, [\nabla\cdot\nabla]^s$ and $[\,\nabla\nabla\cdot\,-\nabla\times\nabla\times]^s$ represent the gradient, divergent, curl, scalar and vector Laplacian operators in the spherical basis. Also, because of the particular scheme relating to coordinate transformations [2], the order of the spherical coordinate variables and components is purposely arranged as (θ, ϕ, r) instead of the conventional (r, θ, ϕ).

In the following basic expressions for the curvilinear orthogonal coordinate system, a vector **A** is in the $\mathbf{u}=(u_1,u_2,u_3)$ coordinate basis with $\hat{\mathbf{u}}=(\hat{u}_1,\hat{u}_2,\hat{u}_3)$ as basis unit vectors and $h=(h_1,h_2,h_3)$ as metric coefficients. A scalar quantity is denoted as s which may be either a 1 x 1 single element matrix as in $[\mathbf{A}]\,s$ or a 3 x 3 diagonal matrix as in $s\,[\mathbf{A}]$, following the rule of matrix multiplication.

$$s = [\;s\;] \quad \text{or} \quad \begin{bmatrix} s & & \\ & s & \\ & & s \end{bmatrix}$$

$$[\;\mathbf{A}\;] = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}$$

$$[\;\mathbf{A}\cdot\;] = \begin{bmatrix} A_1 & A_2 & A_3 \end{bmatrix}$$

$$[\;\mathbf{A}\times\;] = \begin{bmatrix} 0 & -A_3 & A_2 \\ A_3 & 0 & -A_1 \\ -A_2 & A_1 & 0 \end{bmatrix}$$

$$[\;\nabla\;] = \begin{bmatrix} \frac{1}{h_1}\frac{\partial}{\partial u_1} \\ \frac{1}{h_2}\frac{\partial}{\partial u_2} \\ \frac{1}{h_3}\frac{\partial}{\partial u_3} \end{bmatrix}$$

$$[\;\nabla\cdot\;] = \begin{bmatrix} \frac{1}{h_1h_2h_3}\frac{\partial}{\partial u_1}h_2h_3 & \frac{1}{h_1h_2h_3}\frac{\partial}{\partial u_2}h_3h_1 & \frac{1}{h_1h_2h_3}\frac{\partial}{\partial u_3}h_1h_2 \end{bmatrix}$$

$$[\;\nabla\times\;] = \begin{bmatrix} 0 & -\frac{1}{h_3h_2}\frac{\partial}{\partial u_3}h_2 & \frac{1}{h_2h_3}\frac{\partial}{\partial u_2}h_3 \\ \frac{1}{h_3h_1}\frac{\partial}{\partial u_3}h_1 & 0 & -\frac{1}{h_1h_3}\frac{\partial}{\partial u_1}h_3 \\ -\frac{1}{h_2h_1}\frac{\partial}{\partial u_2}h_1 & \frac{1}{h_1h_2}\frac{\partial}{\partial u_1}h_2 & 0 \end{bmatrix}$$

The single and double vector differential operators may be expressed in compact form as

$$
\begin{aligned}
\nabla &= h^{-1}\vec{\partial} \\
\nabla\cdot &= \sigma^{-1}\vec{\partial}\cdot\,\sigma h^{-1} \\
\nabla\times &= \sigma^{-1}h\,\vec{\partial}\times h
\end{aligned}
$$

$$
\begin{aligned}
\nabla\times\nabla &= \sigma^{-1}\,h\;\vec{\partial}\times\vec{\partial} = 0 \\
\nabla\cdot\nabla\times &= \sigma^{-1}\,\vec{\partial}\cdot\vec{\partial}\times h = 0 \\
\nabla\cdot\nabla &= \sigma^{-1}\,\vec{\partial}\cdot\sigma\,h^{-2}\,\vec{\partial} \\
\nabla\;\nabla\cdot &= h^{-1}\,\vec{\partial}\,\sigma^{-1}\,\vec{\partial}\cdot\sigma\,h^{-1} \\
\nabla\times\nabla\times &= \sigma^{-1}h\,\vec{\partial}\times h^{2}\sigma^{-1}\,\vec{\partial}\times h
\end{aligned}
$$

where

$$
\vec{\partial} = \begin{bmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{bmatrix}, \qquad \vec{\partial}\cdot = \lfloor\, \partial_1 \;\; \partial_2 \;\; \partial_3 \,\rfloor, \qquad \vec{\partial}\times = \begin{bmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{bmatrix},
$$

$$
h = \begin{bmatrix} h_1 & & \\ & h_2 & \\ & & h_3 \end{bmatrix}, \qquad \sigma = h_1\, h_2\, h_3
$$

$$
\partial_i \equiv \frac{\partial}{\partial u_i}\,, \qquad h_i = \sqrt{\left(\frac{\partial x}{\partial u_i}\right)^2 + \left(\frac{\partial y}{\partial u_i}\right)^2 + \left(\frac{\partial z}{\partial u_i}\right)^2}\,, \qquad i = 1, 2, 3
$$

It should be noted that the expressions of the Vector operations Transform into Matrix operations (VTM) presented here are somewhat different from those of the author's previous expressions, Matrix Formulation of Vector Operations (MFVO) [1] and from those of the Generalized Differential Matrix Operators (GDMO) [3,4]. The vector differential operators, such as gradient, divergent and curl, are represented, respectively, by (∇), {∇} and [∇] in [1], and by |∇⟩, ⟨∇·| and [∇×] in [3]. In the present article, they are denoted by [∇], [∇·], and [∇×], respectively. The brackets for these expressions may be removed provided no confusion exists.

It is worthwhile to mention that the vector differential operations may be written as

$$
\begin{aligned}
\text{grad } f = \nabla f &= \sum_i \frac{\hat{\mathbf{u}}_i}{h_i}\frac{\partial f}{\partial u_i} \\
\text{div } \mathbf{F} = \nabla\cdot\mathbf{F} &= \sum_i \frac{\hat{\mathbf{u}}_i}{h_i}\cdot\frac{\partial \mathbf{F}}{\partial u_i} \\
\text{curl } \mathbf{F} = \nabla\times\mathbf{F} &= \sum_i \frac{\hat{\mathbf{u}}_i}{h_i}\times\frac{\partial \mathbf{F}}{\partial u_i}
\end{aligned}
$$

These are straightforward expressions, however, they deal with basis unit vectors as those in the usual vector expressions.

The vector operations transform into matrix operations in Cartesian, cylindrical and spherical coordinate systems are specially tabulated in the next section.

3. VTM in Three Major Coordinate Basis

Cartesianl Coordinates : $\mathrm{c} = (x,\ y,\ z),\quad (h_x, h_y, h_z) = (1,1,1)$

$$[\ \mathbf{A}^c\ ] = \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

$$[\ \mathbf{A}^c\cdot\ ] = \begin{bmatrix} A_x & A_y & A_z \end{bmatrix}$$

$$[\ \mathbf{A}^c\times\ ] = \begin{bmatrix} 0 & -A_z & A_y \\ A_z & 0 & -A_x \\ -A_y & A_x & 0 \end{bmatrix}$$

$$[\ \nabla^c\ ] = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$$

$$[\ \nabla^c\cdot\ ] = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix}$$

$$[\ \nabla^c\times\ ] = \begin{bmatrix} 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix}$$

$$[\ \nabla^c\cdot\ \nabla^c\times] = [\ 0 \quad 0 \quad 0\ ]$$

$$[\ \nabla^c\times\ \nabla^c] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[\ \nabla^c\cdot\ \nabla^c\ ] = [\ \Delta^c\ ]$$

$$\Delta^c = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$[\ \nabla^c\ \ \nabla^c\cdot\ ] = \begin{bmatrix} \frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial x\,\partial y} & \frac{\partial^2}{\partial x \partial z} \\ \frac{\partial^2}{\partial x\,\partial y} & \frac{\partial^2}{\partial y^2} & \frac{\partial^2}{\partial y \partial z} \\ \frac{\partial^2}{\partial x \partial z} & \frac{\partial^2}{\partial y \partial z} & \frac{\partial^2}{\partial z^2} \end{bmatrix}$$

$$[\ \nabla^c \times \nabla^c\times\ ] = \begin{bmatrix} -\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} & \frac{\partial^2}{\partial x\,\partial y} & \frac{\partial^2}{\partial x \partial z} \\ \frac{\partial^2}{\partial x\,\partial y} & -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial z^2} & \frac{\partial^2}{\partial y \partial z} \\ \frac{\partial^2}{\partial x \partial z} & \frac{\partial^2}{\partial y \partial z} & -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \end{bmatrix}$$

$$[\ \nabla^c\ \ \nabla^c\cdot\ -\ \nabla^c \times \nabla^c\times\ ] = \begin{bmatrix} \Delta^c & 0 & 0 \\ 0 & \Delta^c & 0 \\ 0 & 0 & \Delta^c \end{bmatrix}$$

Cylindrical Coordinates : $\mathrm{d} = (\rho,\ \phi,\ z),\quad (h_\rho, h_\phi, h_z) = (1, \rho, 1)$

$$[\ \mathbf{A}^d\ ] = \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

$$[\ \mathbf{A}^d\cdot\ ] = \begin{bmatrix} A_\rho & A_\phi & A_z \end{bmatrix}$$

$$[\ \mathbf{A}^d\times\ ] = \begin{bmatrix} 0 & -A_r & A_\phi \\ A_r & 0 & -A_\rho \\ -A_\phi & A_\rho & 0 \end{bmatrix}$$

$$[\ \nabla^d\ ] = \begin{bmatrix} \dfrac{\partial}{\partial\rho} \\ \dfrac{1}{\rho}\dfrac{\partial}{\partial\phi} \\ \dfrac{\partial}{\partial z} \end{bmatrix}$$

$$[\ \nabla^d\cdot\ ] = \begin{bmatrix} \dfrac{\partial}{\partial\rho}+\dfrac{1}{\rho} & \dfrac{1}{\rho}\dfrac{\partial}{\partial\phi} & \dfrac{\partial}{\partial z} \end{bmatrix}$$

$$[\ \nabla^d\times\ ] = \begin{bmatrix} 0 & -\dfrac{\partial}{\partial z} & \dfrac{1}{\rho}\dfrac{\partial}{\partial\phi} \\ \dfrac{\partial}{\partial z} & 0 & -\dfrac{\partial}{\partial\rho} \\ -\dfrac{1}{\rho}\dfrac{\partial}{\partial\phi} & \dfrac{\partial}{\partial\rho}+\dfrac{1}{\rho} & 0 \end{bmatrix}$$

$$[\ \nabla^d\cdot\ \nabla^d\times] = [\ 0 \quad 0 \quad 0\ ]$$

$$[\ \nabla^d\times\ \nabla^d] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[\ \nabla^d\cdot\ \nabla^d\ ] = [\ \Delta^d\ ]$$

$$\Delta^d = \frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}+\frac{\partial^2}{\partial z^2}$$

$$[\ \nabla^d\ \ \nabla^d\cdot\ ] = \begin{bmatrix} \dfrac{\partial^2}{\partial\rho^2}+\dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}-\dfrac{1}{\rho^2} & \dfrac{1}{\rho}\ \dfrac{\partial^2}{\partial\rho\,\partial\phi}-\dfrac{1}{\rho^2}\dfrac{\partial}{\partial\phi} & \dfrac{\partial^2}{\partial\rho\partial z} \\ \dfrac{1}{\rho}\ \dfrac{\partial^2}{\partial\rho\,\partial\phi}+\dfrac{1}{\rho^2}\dfrac{\partial}{\partial\phi} & \dfrac{1}{\rho^2}\dfrac{\partial^2}{\partial\phi^2} & \dfrac{1}{\rho}\dfrac{\partial^2}{\partial z\partial\phi} \\ \dfrac{\partial^2}{\partial\rho\partial z}+\dfrac{1}{\rho}\dfrac{\partial}{\partial z} & \dfrac{1}{\rho}\dfrac{\partial^2}{\partial z\partial\phi} & \dfrac{\partial^2}{\partial z^2} \end{bmatrix}$$

$$[\ \nabla^d\times\nabla^d\times\ ] = \begin{bmatrix} -\dfrac{1}{\rho^2}\dfrac{\partial^2}{\partial\phi^2}-\dfrac{\partial^2}{\partial z^2} & \dfrac{1}{\rho}\ \dfrac{\partial^2}{\partial\rho\,\partial\phi}+\dfrac{1}{\rho^2}\dfrac{\partial}{\partial\phi} & \dfrac{\partial^2}{\partial\rho\partial z} \\ \dfrac{1}{\rho}\ \dfrac{\partial^2}{\partial\rho\,\partial\phi}-\dfrac{1}{\rho^2}\dfrac{\partial}{\partial\phi} & -\dfrac{\partial^2}{\partial\rho^2}-\dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}+\dfrac{1}{\rho^2}-\ \dfrac{\partial^2}{\partial z^2} & \dfrac{1}{\rho}\dfrac{\partial^2}{\partial z\partial\phi} \\ \dfrac{\partial^2}{\partial\rho\partial z}+\dfrac{1}{\rho}\dfrac{\partial}{\partial z} & \dfrac{1}{\rho}\dfrac{\partial^2}{\partial z\partial\phi} & -\dfrac{\partial^2}{\partial\rho^2}-\dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}-\dfrac{1}{\rho^2}\dfrac{\partial^2}{\partial\phi^2} \end{bmatrix}$$

$$[\ \nabla^d\ \ \nabla^d\cdot\ -\ \nabla^d\times\nabla^d\times\ ] = \begin{bmatrix} \Delta^d-\dfrac{1}{\rho^2} & -\dfrac{2}{\rho^2}\dfrac{\partial}{\partial\phi} & 0 \\ \dfrac{2}{\rho^2}\dfrac{\partial}{\partial\phi} & \Delta^d-\dfrac{1}{\rho^2} & 0 \\ 0 & 0 & \Delta^d \end{bmatrix}$$

Spherical Coordinates: $s = (\theta,\ \phi,\ r), \quad (h_\theta, h_\phi, h_r) = (r, r\sin\theta, 1)$

$$\left[\ \mathbf{A}^s\ \right] = \begin{bmatrix} A_\theta \\ A_\phi \\ A_r \end{bmatrix}$$

$$\left[\ \mathbf{A}^s\cdot\ \right] = \begin{bmatrix} A_\theta & A_\phi & A_r \end{bmatrix}$$

$$\left[\ \mathbf{A}^s\times\ \right] = \begin{bmatrix} 0 & -A_r & A_\phi \\ A_r & 0 & -A_\theta \\ -A_\phi & A_\theta & 0 \end{bmatrix}$$

$$\left[\ \nabla^s\ \right] = \begin{bmatrix} \frac{1}{r}\frac{\partial}{\partial\theta} \\ \frac{1}{r}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} \\ \frac{\partial}{\partial r} \end{bmatrix}$$

$$\left[\ \nabla^s\cdot\ \right] = \begin{bmatrix} \frac{1}{r}\frac{\partial}{\partial\theta}+\frac{1}{r}\frac{\cos\theta}{\sin\theta} & \frac{1}{r}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} & \frac{\partial}{\partial r}+\frac{2}{r} \end{bmatrix}$$

$$\left[\ \nabla^s\times\ \right] = \begin{bmatrix} 0 & -\frac{\partial}{\partial r}-\frac{1}{r} & \frac{1}{r}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} \\ \frac{\partial}{\partial r}+\frac{1}{r} & 0 & -\frac{1}{r}\frac{\partial}{\partial\theta} \\ -\frac{1}{r}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} & \frac{1}{r}\frac{\partial}{\partial\theta}+\frac{1}{r}\frac{\cos\theta}{\sin\theta} & 0 \end{bmatrix}$$

$$\left[\ \nabla^s\cdot\ \nabla^s\times\right] = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

$$\left[\ \nabla^s\times\ \nabla^s\right] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\ \nabla^s\cdot\ \nabla^s\ \right] = \left[\ \Delta^s\ \right]$$

$$\Delta^s = \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{1}{r^2}\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}+\frac{1}{r^2}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}+\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}$$

$$\left[\ \nabla^s\ \ \nabla^s\cdot\ \right] = \begin{bmatrix} \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{1}{r^2}\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}-\frac{1}{r^2}\frac{1}{\sin^2\theta} & \frac{1}{r^2}\ \frac{1}{\sin\theta}\frac{\partial^2}{\partial\theta\,\partial\phi}-\frac{1}{r^2}\frac{\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\phi} & \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}+\frac{2}{r^2}\frac{\partial}{\partial\theta} \\ \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial^2}{\partial\theta\,\partial\phi}+\frac{1}{r^2}\frac{\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\phi} & \frac{1}{r^2}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} & \frac{1}{r}\frac{1}{\sin\theta}\frac{\partial^2}{\partial r\partial\phi}+\frac{2}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} \\ \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}+\frac{1}{r}\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial r}-\frac{1}{r^2}\frac{\partial}{\partial\theta}-\frac{1}{r^2}\frac{\cos\theta}{\sin\theta} & \frac{1}{r}\frac{1}{\sin\theta}\frac{\partial^2}{\partial r\partial\phi}-\frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} & \frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}-\frac{2}{r^2} \end{bmatrix}$$

$$\left[\ \nabla^s\times\nabla^s\times\ \right] = \begin{bmatrix} -\frac{\partial^2}{\partial r^2}-\frac{2}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} & \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial^2}{\partial\theta\,\partial\phi}+\frac{1}{r^2}\frac{\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\phi} & \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta} \\ \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial^2}{\partial\theta\,\partial\phi}-\frac{1}{r^2}\frac{\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\phi} & -\frac{\partial^2}{\partial r^2}-\frac{2}{r}\frac{\partial}{\partial r}-\ \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}-\frac{1}{r^2}\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}+\frac{1}{r^2}\frac{1}{\sin^2\theta} & \frac{1}{r}\frac{1}{\sin\theta}\frac{\partial^2}{\partial r\partial\phi} \\ \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}+\frac{1}{r}\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial}{\partial\theta}+\frac{1}{r^2}\frac{\cos\theta}{\sin\theta} & \frac{1}{r}\frac{1}{\sin\theta}\frac{\partial^2}{\partial r\partial\phi}+\frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} & -\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}-\frac{1}{r^2}\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}-\frac{1}{r^2}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \end{bmatrix}$$

$$\left[\ \nabla^s\ \ \nabla^s\cdot\ -\ \nabla^s\times\nabla^s\times\ \right] = \begin{bmatrix} \Delta^s-\frac{1}{r^2}\frac{1}{\sin^2\theta} & -\frac{2}{r^2}\frac{\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\phi} & \frac{2}{r^2}\frac{\partial}{\partial\theta} \\ \frac{2}{r^2}\frac{\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\phi} & \Delta^s-\frac{1}{r^2}\frac{1}{\sin^2\theta} & \frac{2}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} \\ -\frac{2}{r^2}\frac{\partial}{\partial\theta}-\frac{2}{r^2}\frac{\cos\theta}{\sin\theta} & -\frac{2}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} & \Delta^s-\frac{2}{r^2} \end{bmatrix}$$

4 Basic Vector Operation Identities

The printing area is In matrix formulation of vector operations [1], it is often necessary to manipulate entry operations, such as removal of parenthesis, exchange of entry order, etc. It should be emphasized that $A\times B\times C = A\times(B\times C)$, but $\neq (A\times B)\times C$, and $A\times B\cdot C = A\times(B\cdot C)$, but $\neq (A\times B)\cdot C$. This is sometimes contradictory to many of the vector analysis textbooks. The identity operations must always follow the rules of matrix multiplication. The following operation identities for two entries are fairly simple to derive and are very expendable to manipulate the vector algebraic and differential operations. Applying the basic vector operation identities can derive all the vector differential operation formulas extensively listed in the next section. For completeness of presentation, the operation identities for three entries are also included. Some of the operation identities listed are quite similar to those of "Symbolic vector expressions" presented in [5].

1. $r\, s = s\, r$
2. $A\, s = s\, A$
3. $A\cdot s = s\; A\cdot$
4. $A\times s = s\; A\times$
5. $A\cdot B = B\cdot A$
$$= AB\cdot - B\times A\times = AB\cdot - A\times B\times + (A\times B)\times$$
$$= BA\cdot - A\times B\times = BA\cdot - B\times A\times + (B\times A)\times$$
6. $A\times B = -B\times A$
7. $A\cdot B\times = -B\cdot A\times$
8. $A\;\; B\cdot = B\times A\times + \;\; B\cdot A$
9. $A\times B\times = B\; A\cdot \;\; - B\cdot A$
10. $(A\times B)\;\; = A\times B = -B\times A$
11. $(A\times B)\cdot\;\; = A\cdot B\;\;\times = -B\cdot A\times$
12. $(A\times B)\times = A\cdot B - A\,B\cdot + A\times B\times = -B\cdot A + B\,A\cdot - B\times A\times$
$$= -A\,B\cdot + B\,A\cdot = A\times B\times - B\times A\times$$

13. $A\cdot B\times C=B\cdot C\times A=C\cdot A\times B$
$=-B\cdot A\times C=-C\cdot B\times A=-A\cdot C\times B$
$=B\times CA\cdot+C\times AB\cdot+A\times BC\cdot$
$=BC\cdot A\times+CA\cdot B\times+AB\cdot C\times$

14. $AB\cdot C=B\cdot CA=C\cdot AB+C\times A\times B$
$=AC\cdot B=C\cdot BA=B\cdot AC+B\times A\times C$

15. $A\cdot BC\cdot=C\cdot A\cdot B=B\cdot C\cdot A+B\cdot C\times A\times$
$=B\cdot AC\cdot=C\cdot B\cdot A=A\cdot C\cdot B+A\cdot C\times B\times$

16. $A\cdot BC=CA\cdot B=B\cdot CA+B\times C\times A$
$=B\cdot AC=CB\cdot A=A\cdot CB+A\times C\times B$

17. $A\cdot B\cdot C=B\cdot CA\cdot=C\cdot A\cdot B+C\cdot A\times B\times$
$=A\cdot C\cdot B=C\cdot BA\cdot=B\cdot A\cdot C+B\cdot A\times C\times$

18. $A\cdot B\times C\times=B\cdot C\cdot A-B\cdot CA\cdot=C\cdot AB\cdot-C\cdot A\cdot B-C\cdot A\times B\times$
$=-B\cdot A\times C\times=-A\cdot C\cdot B+A\cdot CB\cdot=-C\cdot BA\cdot+C\cdot B\cdot A+C\cdot B\times A\times$

19. $A\times B\times C=C\cdot AB-CA\cdot B=BC\cdot A-B\cdot CA-B\times C\times A$
$=-A\times C\times B=-B\cdot AC+BA\cdot C=-CB\cdot A+C\cdot BA+C\times B\times A$

20. $A\cdot BC\times=C\times A\cdot B=B\cdot C\times A+B\cdot CA\times-BC\cdot A\times+B\times C\times A\times+B\times C\cdot A-B\times CA\cdot$
$=B\cdot AC\times=C\times B\cdot A=A\cdot C\times B+A\cdot CB\times-AC\cdot B\times+A\times C\times B\times+A\times C\cdot B-A\times CB\cdot$

21. $A\times B\cdot C=B\cdot CA\times=C\cdot A\times B+C\times A\cdot B-C\times AB\cdot+C\times A\times B\times+C\cdot AB\times-CA\cdot B\times$
$=A\times C\cdot B=C\cdot BA\times=B\cdot A\times C+B\times A\cdot C-B\times AC\cdot+B\times A\times C\times+B\cdot AC\times-BA\cdot C\times$

22. $AB\cdot C\times=-C\cdot AB\times-C\times A\times B\times=B\cdot C\times A+B\cdot CA\times-BC\cdot A\times+B\times C\times A\times$
$=-AC\cdot B\times=B\cdot AC\times+B\times A\times C\times=-C\cdot B\times A-C\cdot BA\times+CB\cdot A\times-C\times B\times A\times$

23. $A\times BC\cdot=-B\times C\cdot A-B\times C\times A\times=C\cdot A\times B+C\times A\cdot B-C\times AB\cdot+C\times A\times B\times$
$=-B\times AC\cdot=A\times C\cdot B+A\times C\times B\times=-C\cdot B\times A-C\times B\cdot A+C\times BA\cdot-C\times B\times A\times$

24. $A\times B\times C\times=-C\times B\cdot A-C\cdot BA\times-C\times B\times A\times$
$=BA\cdot C\times-B\cdot AC\times=-C\cdot A\times B-C\cdot AB\times+CA\cdot B\times-C\times A\times B\times-C\times A\cdot B$
$=A\times CB\cdot-A\times C\cdot B=-B\cdot C\times A-B\cdot CA\times-B\times C\times A\times-B\times C\cdot A+B\times CA\cdot$

Employing some basic operation identities may solve equations involving vectors.

Example: Solve the following equation for Z:

$$sZ + A\times Z + B\cdot ZC + D\times Z\times E + Z\times F\times G = W$$

By some necessary rearrangement, we have

$$MZ = W \qquad \Rightarrow \qquad Z = M^{-1}\,W$$

where

$$M = (s + A\times + CB\cdot - D\times E\times + F\,G\cdot - G\,F\cdot)$$

Some special cases that the solutions may be directly expressed in explicit closed form without going through matrix inversion:

$sZ + A\times Z = W$	$\Rightarrow$	$Z = \dfrac{(s^2 - sA\times + AA\cdot)W}{s(s^2 + A\cdot A)}$
$sZ + AB\cdot Z = W$	$\Rightarrow$	$Z = \dfrac{(s - B\times A\times)W}{s(s + A\cdot B)}$
$sZ + A\times B\times Z = W$	$\Rightarrow$	$Z = \dfrac{(s - BA\cdot)\,W}{s(s - A\cdot B)}$
$sA\times Z + BC\cdot Z = W$	$\Rightarrow$	$Z = \dfrac{(s\,AA\cdot + C\times A\times B\times)W}{s\,A\cdot B\,A\cdot C}$
$\begin{cases} A\cdot Z = s \\ B\times Z = V, \end{cases} \quad (B\cdot V = 0)$	$\Rightarrow$	$Z = \dfrac{sB - A\times V}{A\cdot B}$
$\begin{cases} A\cdot Z = p \\ B\cdot Z = q \\ C\cdot Z = r \end{cases}$	$\Rightarrow$	$Z = \dfrac{pB\times C + qC\times A + rA\times B}{A\cdot B\times C}$
$Ax + By + Cz = W$	$\Rightarrow$	$\begin{cases} x = \dfrac{B\cdot C\times W}{B\cdot C\times A} \\ y = \dfrac{C\cdot A\times W}{C\cdot A\times B} \\ z = \dfrac{A\cdot B\times W}{A\cdot B\times C} \end{cases}$
$B\times Cx + C\times Ay + A\times Bz = W$	$\Rightarrow$	$\begin{cases} x = \dfrac{A\cdot W}{A\cdot B\times C} \\ y = \dfrac{B\cdot W}{B\cdot C\times A} \\ z = \dfrac{C\cdot W}{C\cdot A\times B} \end{cases}$

5 Vector Differential Operation Formulas

1. ∇u
2. $\nabla \cdot A$
3. $\nabla \times A$
4. $C \cdot \nabla A$
5. $B \cdot C \times \nabla A$
6. $\nabla uv = v\nabla u + u\nabla v$
7. $\nabla A \cdot B = B \times \nabla \times A + B \cdot \nabla A + A \times \nabla \times B + A \cdot \nabla B$
8. $\nabla \cdot Au = u\nabla \cdot A + A \cdot \nabla u$
9. $\nabla \times Au = u\nabla \times A - A \times \nabla u$
10. $\nabla \cdot A \times B = B \cdot \nabla \times A - A \cdot \nabla \times B$
11. $\nabla \times A \times B = A\nabla \cdot B - A \cdot \nabla B - B\nabla \cdot A + B \cdot \nabla A$
12. $C \cdot \nabla Au = AC \cdot \nabla u + uC \cdot \nabla A$
13. $C \cdot \nabla A \times B = A \times C \cdot \nabla B - B \times C \cdot \nabla A$
14. $C \cdot \nabla A \cdot B = A \cdot C \cdot \nabla B + B \cdot C \cdot \nabla A$
15. $(\nabla \times A) \times B = -B \times \nabla \times A - A \times \nabla \times B + A\nabla \cdot B - A \cdot \nabla B$
16. $(\nabla \times A) \cdot B = B \cdot \nabla \times A - A \cdot \nabla \times B$
17. $(\nabla \times A)u = u\nabla \times A - A \times \nabla u$
18. $\nabla \cdot AB = B\nabla \cdot A + A \cdot \nabla B$
19. $\nabla \cdot ABu = uB\nabla \cdot A + uA \cdot \nabla B + BA \cdot \nabla u$
20. $\nabla \cdot Au = u\nabla \cdot A + A \cdot \nabla u$
21. $(C \times \nabla) \times A = C \times \nabla \times A - C\nabla \cdot A + C \cdot \nabla A$
22. $(C \times \nabla) \cdot A = C \cdot \nabla \times A$
23. $(C \times \nabla)u = C \times \nabla u$
24. $(C \cdot \nabla)A = C \cdot \nabla A$
25. $(C \cdot \nabla)u = C \cdot \nabla u$
26. $\nabla uvw = vw\nabla u + wu\nabla v + uv\nabla w$
27. $\nabla \cdot Auv = uv\nabla \cdot A + vA \cdot \nabla u + uA \cdot \nabla v$
28. $\nabla \times Auv = uv\nabla \times A - vA \times \nabla u - uA \times \nabla v$
29. $\nabla A \cdot Bu = A \cdot B\nabla u + uB \times \nabla \times A + uB \cdot \nabla A + uA \times \nabla \times B + uA \cdot \nabla B$
30. $\nabla \cdot A \times Bu = uB \cdot \nabla \times A - uA \cdot \nabla \times B + A \cdot B \times \nabla u$
31. $\nabla \times A \times Bu = uA\nabla \cdot B - uA \cdot \nabla B - uB\nabla \cdot A + uB \cdot \nabla A - A \cdot B\nabla u + uAB \cdot \nabla u - A \times B \times \nabla u$
32. $\nabla \times AB \cdot C = B \cdot C\nabla \times A - A \times C \times \nabla \times B - A \times B \times \nabla \times C - AC \cdot \nabla B - AB \cdot \nabla C$
33. $\nabla \cdot A \cdot BC = C \cdot B \times \nabla \times A + C \cdot A \times \nabla \times B + C \cdot B \cdot \nabla A + C \cdot A \cdot \nabla B + A \cdot B \cdot \nabla C$
34. $\nabla A \cdot B \times C = A \times B\nabla \cdot C - A \times B \cdot \nabla C - A \times C\nabla \cdot B + A \times C \cdot \nabla B + B \times A \cdot \nabla C - C \times A \cdot \nabla B$
$+ B \times C \times \nabla \times A - C \times B \times \nabla \times A + B \cdot C \times \nabla A$
35. $\nabla \cdot A \times B \times C = A \cdot B \cdot \nabla C - A \cdot B\nabla \cdot C - A \cdot C \cdot \nabla B + A \cdot C\nabla \cdot B + B \cdot C \times \nabla \times A$
36. $\nabla \times A \times B \times C = AC \cdot \nabla \times B - AB \cdot \nabla \times C - B \times A \cdot \nabla C + C \times A \cdot \nabla B - B \times C\nabla \cdot A + B \cdot C \times \nabla A$

Some of the most familiar vector differential formulas are derived here for illustration:

$$\nabla A\cdot B = \nabla A_o\cdot B+\nabla A\cdot B_o$$
$$= \nabla A_o\cdot B+\nabla B_o\cdot A \qquad \text{by (5) in Sec. 4.}$$
$$= (A_o\times\nabla\times+A_o\cdot\nabla)B+(B_o\times\nabla\times+B_o\cdot\nabla)A \qquad \text{by (8)}$$
$$= A\times\nabla\times B+A\cdot\nabla B+B\times\nabla\times A+B\cdot\nabla A \qquad \text{remove '}_o\text{', and rearranged}$$

$$\nabla\times A\times B = \nabla\times A_o\times B+\nabla\times A\times B_o$$
$$= \nabla\times A_o\times B-\nabla\times B_o\times A \qquad \text{by (6)}$$
$$= (A_o\nabla\cdot-A_o\cdot\nabla)B-(B_o\nabla\cdot-B_o\cdot\nabla)A \qquad \text{by (9)}$$
$$= A\nabla\cdot B-A\cdot\nabla B-B\nabla\cdot A+B\cdot\nabla A$$

$$\nabla\cdot A\times B = \nabla\cdot A_o\times B-\nabla\cdot B_o\times A \qquad \text{by (6)}$$
$$= -A_o\cdot\nabla\times B+B_o\cdot\nabla\times A \qquad \text{by (7)}$$
$$= -A\cdot\nabla\times B+B\cdot\nabla\times A$$

$$C\cdot\nabla A\times B = C_o\cdot\nabla A_o\times B-C_o\cdot\nabla B_o\times A \qquad \text{by (6)}$$
$$= A_o\times(C_o\cdot\nabla)B-B_o\times(C_o\cdot\nabla)A \qquad \text{by (4) since } (C_o\cdot\nabla)=s$$
$$= A\times C\cdot\nabla B-B\times C\cdot\nabla A$$

Here, a quantity with subscript "o" denotes that the quantity to which it is attached is momentarily being held fixed. It may be freely removed whenever no differential operator is ahead of it.

It follows that

$$\nabla A\cdot B\times C = A\times\nabla\times(B\times C)+A\cdot\nabla(B\times C)+(B\times C)\times\nabla\times A+(B\times C)\cdot\nabla A$$
$$= A\times(B\nabla\cdot C-B\cdot\nabla C-C\nabla\cdot B+C\cdot\nabla B)+(B\times A\cdot\nabla C-C\times A\cdot\nabla B)$$
$$+(B\times C\times-C\times B\times)\nabla\times A+(B\cdot C\times)\nabla A \qquad \text{(by (11) and (12))}$$
$$= A\times B\nabla\cdot C-A\times B\cdot\nabla C-A\times C\nabla\cdot B+A\times C\cdot\nabla B+B\times A\cdot\nabla C-C\times A\cdot\nabla B$$
$$+B\times C\times\nabla\times A-C\times B\times\nabla\times A+B\cdot C\times\nabla A$$

$$\nabla\cdot A\times B\times C = -A\cdot\nabla\times(B\times C)+(B\times C)\cdot\nabla\times A$$
$$= -A\cdot(B\nabla\cdot C-B\cdot\nabla C-C\nabla\cdot B+C\cdot\nabla B)+B\cdot C\times\nabla\times A$$
$$= -A\cdot B\nabla\cdot C+A\cdot B\cdot\nabla C+A\cdot C\nabla\cdot B-A\cdot C\cdot\nabla B+B\cdot C\times\nabla\times A$$

$$\nabla\times A\times B\times C = A\nabla\cdot(B\times C)-A\cdot\nabla(B\times C)-(B\times C)\nabla\cdot A+(B\times C)\cdot\nabla A$$
$$= A(-B\cdot\nabla\times C+C\cdot\nabla\times B)-(B\times A\cdot\nabla C-C\times A\cdot\nabla B)-B\times C\nabla\cdot A+B\cdot C\times\nabla A$$
$$= -AB\cdot\nabla\times C+AC\cdot\nabla\times B-B\times A\cdot\nabla C+C\times A\cdot\nabla B-B\times C\nabla\cdot A+B\cdot C\times\nabla A$$

6 Transformation of Coordinate Basis

Vector functions, position vectors, and unit vectors related to the any two coordinate systems may be expressed as [2]

$$
\begin{aligned}
[\mathbf{A}^v] &= [{}^v\mathrm{T}^u][\mathbf{A}^u] \\
[\mathbf{r}^v] &= [{}^v\mathrm{T}^u][\mathbf{r}^u] \\
[\hat{\mathbf{v}}] &= [{}^v\mathrm{T}^u][\hat{\mathbf{u}}]
\end{aligned}
$$

where $[{}^v\mathrm{T}^u]$ is a coordinate transformation matrix relating u- to v- coordinate systems,

$$[{}^v\mathrm{T}^u] = [{}^u\mathrm{T}^v]^{-1} = [{}^u\mathrm{T}^v]^T, \quad \text{and} \quad \det[{}^v\mathrm{T}^u] = 1.$$

For example,

$$[{}^d\mathrm{T}^c] = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad [{}^s\mathrm{T}^d] = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$[{}^s\mathrm{T}^c] = [{}^s\mathrm{T}^d]\,[{}^d\mathrm{T}^c] = \begin{bmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{bmatrix}$$

Description of coordinate transformation between any two coordinate systems involving both axis rotations and origin relocations may be found extensively in [2].

This transformation of coordinate basis may further extend to the matrix formulation of vector operations [1]:

$$
\boxed{
\begin{aligned}
[\mathbf{A}^v] &= [{}^v\mathbf{T}^u][\mathbf{A}^u] \\
[\mathbf{A}^v\cdot] &= [\mathbf{A}^u\cdot][{}^u\mathbf{T}^v] \\
[\mathbf{A}^v\times] &= [{}^v\mathbf{T}^u][\mathbf{A}^u\times][{}^u\mathbf{T}^v] \\
[\nabla^v] &= [{}^v\mathbf{T}^u][\nabla^u] \\
[\nabla^v\cdot] &= [\nabla^u\cdot][{}^u\mathbf{T}^v] \\
[\nabla^v\times] &= [{}^v\mathbf{T}^u][\nabla^u\times][{}^u\mathbf{T}^v]
\end{aligned}
}
$$

Example: Find $A = B \times C$, where A is in spherical components, and B and C are given, respectively, in Cartesian and cylindrical components. That is,

$$A^s = \hat{\theta} A_\theta + \hat{\phi} A_\phi + \hat{r} A_r, \qquad B^c = \hat{x} B_x + \hat{y} B_y + \hat{z} B_z, \qquad C^d = \hat{\rho} C_\rho + \hat{\phi} C_\phi + \hat{z} C_z$$

And

$$A^s = B^c \times C^d$$

Then,

$$[A^s] = [B^s \times][C^s] = [{}^s\mathrm{T}^c][B^c \times][{}^c\mathrm{T}^s]\,[{}^s\mathrm{T}^d][C^d]$$

or

$$[A^s] = [{}^s\mathrm{T}^c][B^c \times][{}^c\mathrm{T}^d][C^d]$$

Explicitly,

$$\begin{bmatrix} A_\theta \\ A_\phi \\ A_r \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{bmatrix} \begin{bmatrix} 0 & -B_z & B_y \\ B_z & 0 & -B_x \\ -B_y & B_x & 0 \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_\rho \\ C_\phi \\ C_z \end{bmatrix}$$

The desired result is thus obtained by simple straightforward direct matrix multiplication.
In the same manner,

$$D^d = A^s \times B^c \times C^s$$
$$E^s = B^c\; D^d \cdot C^s$$

yields

$$[D^d] = [{}^d\mathrm{T}^s][A^s \times][{}^s\mathrm{T}^c][B^c \times][{}^c\mathrm{T}^s][C^s]$$
$$[E^s] = [{}^s\mathrm{T}^c][B^c][D^d \cdot][{}^d\mathrm{T}^s][C^s]$$

Example: Find the dot-product of two position vectors, $\mathbf{r}(\theta, \phi, r)$ and $\mathbf{r}'(\theta', \phi', r')$. That is,

$$s = \mathbf{r}^s \cdot \mathbf{r}'^{s'}$$

then

$$[s] = [\mathbf{r}^s \cdot][{}^s\mathrm{T}^{s'}][\mathbf{r}'^{s'}]$$
$$= [\mathbf{r}^s \cdot][{}^s\mathrm{T}^c][{}^c\mathrm{T}^{c'}][{}^{c'}\mathrm{T}^{s'}][\mathbf{r}'^{s'}]$$

Or

$$[s] = [0 \quad 0 \quad r] \begin{bmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta'\cos\phi' & -\sin\phi' & \sin\theta'\cos\phi' \\ \cos\theta'\sin\phi' & \cos\phi' & \sin\theta'\sin\phi' \\ -\sin\theta' & 0 & \cos\theta' \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ r' \end{bmatrix}$$

$$= [\, r\sin\theta\cos\phi \quad r\sin\theta\sin\varphi \quad r\cos\theta \,] \begin{bmatrix} r'\sin\theta'\cos\phi' \\ r'\sin\theta'\sin\phi' \\ r'\cos\theta' \end{bmatrix}$$

$$= [r\,r'(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi'))]$$

If the two position vectors are not in the same coordinate frames, then $[{}^c\mathrm{T}^{c'}]$ must be replaced by the coordinate transformation matrix with three Eulerian angles [2].

It is very convenient to derive useful formulas as a whole, instead of one-by-one.

Let derive $\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}$ in terms of $\frac{\partial}{\partial \theta},\frac{\partial}{\partial \phi},\frac{\partial}{\partial r}$ in spherical coordinates.

Since

$$[\nabla^c]=[{}^c\mathrm{T}^s][\nabla^s]$$

Explicitly, we have

$$\begin{bmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z}\end{bmatrix}=\begin{bmatrix}\cos\theta\cos\phi & -\sin\phi & \sin\theta\cos\phi\\ \cos\theta\sin\phi & \cos\phi & \sin\theta\sin\phi\\ -\sin\theta & 0 & \cos\theta\end{bmatrix}\begin{bmatrix}\frac{1}{r}\frac{\partial}{\partial\theta}\\ \frac{1}{r}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\\ \frac{\partial}{\partial r}\end{bmatrix}$$
$$=\begin{bmatrix}\frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta}-\frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}+\sin\theta\cos\phi\frac{\partial}{\partial r}\\ \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta}+\frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}+\sin\theta\sin\phi\frac{\partial}{\partial r}\\ -\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\quad+\cos\theta\frac{\partial}{\partial r}\end{bmatrix}$$

The same results can be obtained, a little more laboriously, by either of the following:

$$[\nabla^c\cdot]=[\nabla^s\cdot][{}^s\mathrm{T}^c]$$

$$[\nabla^c\times]=[{}^c\mathrm{T}^s][\nabla^s\times][{}^s\mathrm{T}^c]$$

The formulas $\frac{\partial^2}{\partial x^2},\frac{\partial^2}{\partial y^2},\frac{\partial^2}{\partial z^2}$ and $\frac{\partial^2}{\partial x\partial y},\frac{\partial^2}{\partial y\partial z},\frac{\partial^2}{\partial z\partial x}$ may also be found in a similar manner:

$$[\nabla^c\nabla^c\cdot]=[{}^c\mathrm{T}^s][\nabla^s\nabla^s\cdot][{}^s\mathrm{T}^c]$$

$$[\nabla^c\times\nabla^c\times]=[{}^c\mathrm{T}^s][\nabla^s\times\nabla^s\times][{}^s\mathrm{T}^c]$$

Detail derivations are given in Appendix B.

7 Application in Electromagnetic Fields

It is well known that the Maxwell's equations [6] in the isotropic medium with time harmonic $e^{j\omega t}$,

$$\begin{aligned}\nabla \times \mathbf{H}(\mathbf{r}) &= j\omega\varepsilon \mathbf{E}(\mathbf{r}) + \mathbf{J}(\mathbf{r}) \\ \nabla \times \mathbf{E}(\mathbf{r}) &= -j\omega\mu \mathbf{H}(\mathbf{r})\end{aligned}$$

the desired solutions, electric field $\mathbf{E}(\mathbf{r})$ and magnetic field $\mathbf{H}(\mathbf{r})$, can be determined:

$$\begin{aligned}\mathbf{E}(\mathbf{r}) &= -j\omega\mu \mathbf{A}(\mathbf{r}) + \frac{1}{j\omega\varepsilon}\nabla\nabla\cdot\mathbf{A}(\mathbf{r}) \\ \mathbf{H}(\mathbf{r}) &= \nabla \times \mathbf{A}(\mathbf{r})\end{aligned}$$

where $\mathbf{A}(\mathbf{r})$ is a vector potential due to the given current distribution $\mathbf{J}(\mathbf{r})$,

$$\begin{aligned}\mathbf{A}(\mathbf{r}) &= \int_V g(|\,\mathbf{r}-\mathbf{r}'\,|)\mathbf{J}(\mathbf{r}')dv' \\ g(r) &= \frac{e^{-jkr}}{4\pi r}\end{aligned}$$

Generally, as in antenna applications, the resulting fields are most likely expressed in spherical components. If the vector potential is given in Cartesian components, the desired fields may then be found through the transformation of coordinate systems. Employing the matrix formulation of vector operations, the magnetic field $\mathbf{H}(\mathbf{r})$ can be written as

$$[\,\mathbf{H}(\mathbf{r})]^s = [\nabla^s \times][\,^s\mathbf{T}^c\,][\,\mathbf{A}(\mathbf{r})]^c$$

Explicitly,

$$\begin{bmatrix} H_\theta(\theta,\phi,r) \\ H_\phi(\theta,\phi,r) \\ H_r(\theta,\phi,r)\end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{r}\frac{\partial}{\partial r}r & \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \\ \frac{1}{r}\frac{\partial}{\partial r}r & 0 & -\frac{1}{r}\frac{\partial}{\partial\theta} \\ -\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} & \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\sin\theta & 0\end{bmatrix}$$
$$\begin{bmatrix}\cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta\end{bmatrix}\begin{bmatrix}A_x(\theta,\phi,r) \\ A_y(\theta,\phi,r) \\ A_z(\theta,\phi,r)\end{bmatrix}$$

The operator is clearly separated from the operand, compared to those of the conventional complicated expression.

After some manipulation, it may further be simplified as (see Appendix E)

$$[\mathbf{H}(\mathbf{r})]^s = [\nabla^s \times]_M [{}^s\mathrm{T}^c]_O [\mathbf{A}(\mathbf{r})]^c$$

Explicitly,

$$\begin{bmatrix} H_\theta(\theta,\phi,r) \\ H_\phi(\theta,\phi,r) \\ H_r(\theta,\phi,r) \end{bmatrix} = \begin{bmatrix} 0 & -\frac{\partial}{\partial r} & \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \\ \frac{\partial}{\partial r} & 0 & -\frac{1}{r}\frac{\partial}{\partial\theta} \\ -\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} & \frac{1}{r}\frac{\partial}{\partial\theta} & 0 \end{bmatrix}$$
$$\begin{bmatrix} \cos\theta_o\cos\phi_o & \cos\theta_o\sin\phi_o & -\sin\theta_o \\ -\sin\phi_o & \cos\phi_o & 0 \\ \sin\theta_o\cos\phi_o & \sin\theta_o\sin\phi_o & \cos\theta_o \end{bmatrix} \begin{bmatrix} A_x(\theta,\phi,r) \\ A_y(\theta,\phi,r) \\ A_z(\theta,\phi,r) \end{bmatrix}$$

Here the "modified" differential operator $[\nabla^s \times]_M$ is derived from the "formal" operator $[\nabla^s \times]$; and the coordinate transformation matrix $[{}^s\mathrm{T}^c]_O$ is exactly identical to $[{}^s\mathrm{T}^c]$ but treated as a constant matrix. The subscript "*o*" may be freely removed whenever no differential operators are ahead of it. As a result, the computation is significantly simplified.

From the above two expressions for $[\mathbf{H}(\mathbf{r})]^s$, an identity $[\nabla^s\times][{}^s\mathrm{T}^c] = [\nabla^s\times]_M[{}^s\mathrm{T}^c]_O$ can be established between the formal and modified differential operators. Other identities, including double differential operations, such as $[\nabla^s\nabla^s\cdot][{}^s\mathrm{T}^c] = [\nabla^s\nabla^s\cdot]_M[{}^s\mathrm{T}^c]_O$, are also useful and listed in Appendix A [7].

Before finding the field $\mathbf{H}(\mathbf{r})$ directly related to the given current distribution $\mathbf{J}(\mathbf{r})$, we need to consider a very special case that the vector potential $\mathbf{A}(\mathbf{r})$ is a function of r only, but not of θ and ϕ.

$$\mathbf{A}(\mathbf{r}) = \mathbf{N}_O g(r) = (N_{ox}\hat{x} + N_{oy}\hat{y} + N_{oz}\hat{z})\frac{e^{-jkr}}{4\pi r}$$

where $\mathbf{N}_O$ is a constant vector. The field $[\mathbf{H}(\mathbf{r})]^s$ may then be expressed without differential operations after performing only the r-differential operations on the modified formulas

$$[\mathbf{H}(\mathbf{r})]^s = -jk\,g(r)(1+\frac{1}{jkr})[\hat{\mathbf{r}}\times]^s[{}^s\mathrm{T}^c]\,[\mathbf{N}_o]^c$$

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And then in the usual vector expression,

$$\mathbf{H}(\mathbf{r}) = -jk\,g(r)(1+\frac{1}{jkr})\,\hat{\mathbf{r}}\times\mathbf{N}_o$$

Applying this procedure, the desired magnetic field $\mathbf{H}(\mathbf{r})$ directly related to $\mathbf{J}(\mathbf{r})$ can be obtained:

$$\begin{aligned}\mathbf{H}(\mathbf{r}) &= -jk\int_V (\frac{1}{-jk}\nabla\times)g(R)\mathbf{J}(\mathbf{r}')\mathrm{d}v' \\ &= -jk\int_V g(R)\left[(1+\frac{1}{jkR})\hat{\mathbf{R}}\times\right]\mathbf{J}(\mathbf{r}')\mathrm{d}v'\end{aligned}$$

And the desired electric field $\mathbf{E}(\mathbf{r})$ may be obtained in the similar manner (Appendix E):

$$\begin{aligned}\mathbf{E}(\mathbf{r}) &= -jk\eta\int_V (1+\frac{1}{k^2}\nabla\nabla\cdot)g(R)\,\mathbf{J}(\mathbf{r}')\mathrm{d}v' \\ &= -jk\eta\int_V g(R)\left[(1+\frac{1}{jkR}+\frac{1}{(jkR)^2})(1-\hat{\mathbf{R}}\hat{\mathbf{R}}\cdot)+(-\frac{2}{jkR}-\frac{2}{(jkR)^2})\hat{\mathbf{R}}\hat{\mathbf{R}}\cdot\right]\mathbf{J}(\mathbf{r}')\mathrm{d}v'\end{aligned}$$

where

$$\mathbf{R} = \hat{\mathbf{R}}R = \mathbf{r}-\mathbf{r}', \qquad R = |\,\mathbf{r}-\mathbf{r}'\,|$$

and $k=\omega\sqrt{\mu\varepsilon}$ and $\eta=\sqrt{\mu/\varepsilon}$.

In summary, the desired electric and magnetic fields can be found by

$$\begin{aligned}\mathbf{E}(\mathbf{r}) &= -jk\eta\int_V \bar{\mathbf{G}}_\nabla(\mathbf{r})g(|\,\mathbf{r}-\mathbf{r}'\,|)\mathbf{J}(\mathbf{r}')\mathrm{d}v' \\ \mathbf{H}(\mathbf{r}) &= -jk\int_V \bar{\mathbf{K}}_\nabla(\mathbf{r})g(|\,\mathbf{r}-\mathbf{r}'\,|)\mathbf{J}(\mathbf{r}')\mathrm{d}v' \\ \bar{\mathbf{G}}_\nabla(\mathbf{r}) &= 1+\frac{1}{k^2}\nabla\nabla\cdot \\ \bar{\mathbf{K}}_\nabla(\mathbf{r}) &= \frac{1}{-jk}\nabla\times\end{aligned}$$

Or

$$\begin{aligned}\mathbf{E}(\mathbf{r}) &= -jk\eta\int_V g(R)\bar{\mathbf{G}}(\mathbf{R})\mathbf{J}(\mathbf{r}')\mathrm{d}v' \\ \mathbf{H}(\mathbf{r}) &= -jk\int_V g(R)\bar{\mathbf{K}}(\mathbf{R})\mathbf{J}(\mathbf{r}')\mathrm{d}v' \\ \bar{\mathbf{G}}(\mathbf{r}) &= \left(1+\frac{1}{jkr}+\frac{1}{(jkr)^2}\right)(1-\hat{\mathbf{r}}\,\hat{\mathbf{r}}\cdot) + \left(-\frac{2}{jkr}-\frac{2}{(jkr)^2}\right)\hat{\mathbf{r}}\,\hat{\mathbf{r}}\cdot \\ \bar{\mathbf{K}}(\mathbf{r}) &= \left(1+\frac{1}{jkr}\right)\hat{\mathbf{r}}\times\end{aligned}$$

Therefore, the solutions of Maxwell's equations are determined by the integration of those vector functions that are derived either through the complicated vector differential operations involving gradient, divergent, and curl operations, or by the simple vector algebraic operations involving only dot-product and cross-product operations. The later formulation is found to be especially convenient for numerical computation involving mixed coordinate systems. The similar results have been also found in some other approach [9].

The desired solutions can further be explicitly written in matrix form such that both the current sources and the resulting fields are expressed in terms of any coordinate system. To show this formulation, we express the fields in spherical coordinate system $s:(\theta, \phi, r)$, and the current source in Cartesian coordinate system $c':(x',y',z')$. The fields may then be written in matrix form as

$$
\begin{aligned}
[\mathbf{E}(\mathbf{r})]^{s} &= -jk\eta\int_{V} g(R)[{}^{s}\mathrm{T}^{s_R}][\bar{\mathbf{G}}(\mathbf{R})]^{s_R}[{}^{s_R}\mathrm{T}^{c'}]\ [\mathbf{J}(\mathbf{r}')]^{c'}\mathrm{d}v^{c'} \\
[\mathbf{H}(\mathbf{r})]^{s} &= -jk\int_{V} g(R)[{}^{s}\mathrm{T}^{s_R}]\ [\bar{\mathbf{K}}(\mathbf{R})]^{s_R}[{}^{s_R}\mathrm{T}^{c'}]\ [\mathbf{J}(\mathbf{r}')]^{c'}\}\mathrm{d}v^{c'}
\end{aligned}
$$

where $s_R:(\theta_R,\phi_R,R)$ is the spherical coordinates with radial vector $\mathbf{R}=\mathbf{r}-\mathbf{r}'$, and

$$
\begin{aligned}
R &= \sqrt{(r\sin\theta\cos\phi - x')^2 + (r\sin\theta\sin\phi - y')^2 + (r\cos\theta - z')^2} \\
\theta_R &= \cos^{-1}\left(\frac{r\cos\theta - z'}{R}\right) \\
\phi_R &= \tan^{-1}\left(\frac{r\sin\theta\sin\phi - y'}{r\sin\theta\cos\phi - x'}\right)
\end{aligned}
$$

If the electromagnetic field in the far region is of primary interest, then for $r >> r'$, we have $\mathbf{R}\to\mathbf{r}$, $\hat{\mathbf{R}}\to\hat{\mathbf{r}}$, $\theta_R\to\theta$, $\phi_R\to\phi$, $R\to r$ for the amplitude factor, and $R \to r-\hat{\mathbf{r}}\cdot\mathbf{r}'$ for the phase factor. The fields are then simply

$$
\begin{aligned}
\mathbf{E}(\mathbf{r}) &= -jk\eta g(r)(1-\hat{\mathbf{r}}\hat{\mathbf{r}}\cdot)\int_V \mathbf{J}(\mathbf{r}')\, e^{jk\hat{\mathbf{r}}\cdot\mathbf{r}'}dv' \\
\mathbf{H}(\mathbf{r}) &= -jkg(r)\,\hat{\mathbf{r}}\times\int_V \mathbf{J}(\mathbf{r}')\, e^{jk\hat{\mathbf{r}}\cdot\mathbf{r}'}dv'
\end{aligned}
$$

Explicitly in matrix form:

$$
\begin{bmatrix} E_\theta(\theta,\phi,r) \\ E_\phi(\theta,\phi,r) \\ E_r(\theta,\phi,r) \end{bmatrix} = -jk\eta\frac{e^{-jkr}}{4\pi r}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\begin{bmatrix} J_{x'}(x',y',z') \\ J_{y'}(x',y',z') \\ J_{z'}(x',y',z') \end{bmatrix} e^{jk[(x'\cos\phi+y'\sin\phi)\sin\theta+z'\cos\theta]}dx'dy'dz'
$$

$$
\begin{bmatrix} H_\theta(\theta,\phi,r) \\ H_\phi(\theta,\phi,r) \\ H_r(\theta,\phi,r) \end{bmatrix} = -jk\frac{e^{-jkr}}{4\pi r}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\begin{bmatrix} J_{x'}(x',y',z') \\ J_{y'}(x',y',z') \\ J_{z'}(x',y',z') \end{bmatrix} e^{jk[(x'\cos\phi+y'\sin\phi)\sin\theta+z'\cos\theta]}dx'dy'dz'
$$

The radiation field may thus be determined when the current distribution is given.

8 Conclusion

An efficient technique "Vector operations Transform into Matrix operations" has been presented in detail. The objective of this technique is to simplify the evaluation of vector algebraic and differential operations in applications related to vector problems. The technique is especially useful when there are mixed coordinate basis involved in the vector operations.

It is interesting to note that in many literatures, the electric field $\mathbf{E}(\mathbf{r})$ is expressed in dyadic form as [9]

$$\begin{aligned}\mathbf{E}(\mathbf{r}) &= -jk\eta\int_V (\bar{1} + \frac{1}{k^2}\nabla\nabla)g(R)\cdot\mathbf{J}(\mathbf{r}')\mathrm{d}v' \\ &= -jk\eta\int_V g(R)\left[(1+\frac{1}{jkR}+\frac{1}{(jkR)^2})(\bar{1}-\hat{\mathbf{R}}\hat{\mathbf{R}}) + (-\frac{2}{jkR}-\frac{2}{(jkR)^2})\hat{\mathbf{R}}\hat{\mathbf{R}}\right]\cdot\mathbf{J}(\mathbf{r}')\mathrm{d}v'\end{aligned}$$

and for the far field, $r >> r'$

$$\mathbf{E}(\mathbf{r}) = -jk\eta g(r)(1-\hat{\mathbf{r}}\hat{\mathbf{r}})\cdot\int_V \mathbf{J}(\mathbf{r}')\, e^{jk\hat{\mathbf{r}}\cdot\mathbf{r}'}dv'$$

The dyadic form seems to be identical to the vector form described in the previous section. Both will certainly yield the same exact results. However, in the vector form the differential operations can directly carry out in spherical coordinates by VTM approach, while in the dyadic form the differential operations must perform in Cartesian coordinates and then return back to spherical coordinates, which is fairly laborious in computation.

Applying VTM technique may easily resolve some problems, which appear to be impossible or difficult to accomplish. It seems to be a very easy task for us to derive some of vector differential formulas, such as $\nabla\times\mathbf{A}\times\mathbf{B}\times\mathbf{C}$, $\nabla\cdot\mathbf{A}\times\mathbf{B}\times\mathbf{C}$, and $\nabla\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}$. However, simply applying the existing well-known formulas of $\nabla\times\mathbf{A}\times\mathbf{B}$, $\nabla\cdot\mathbf{A}\times\mathbf{B}$, and $\nabla\mathbf{A}\cdot\mathbf{B}$ will fail to achieve the results, unless having a formula of $\mathrm{C}\cdot\nabla\mathbf{A}\times\mathbf{B}$ to work with. Perhaps, this is the reason why these formulas appear rarely in any textbooks and literatures in vector analysis.

No claim is made that the VTM technique presented is superior to those conventional evaluations appeared in most of vector analysis textbooks. It is not intended to replace the existing well known establishment in vector problems, but rather to aid readers in overcoming the difficulties in resolving some applications related to vector operations.

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Appendix A. Modified operators related to formal operators

The identities, relating the formal operators to the modified operators, are listed below:

$$[\,\nabla^s\cdot\,]\,[{}^s\mathrm{T}^c] = [\,\nabla^s\cdot\,]_M[{}^s\mathrm{T}^c]_O$$
$$[\,\nabla^s\times\,]\,[{}^s\mathrm{T}^c] = [\,\nabla^s\times\,]_M[{}^s\mathrm{T}^c]_O$$
$$[\,\nabla^s\nabla^s\cdot\,]\,[{}^s\mathrm{T}^c] = [\,\nabla^s\nabla^s\cdot\,]_M[{}^s\mathrm{T}^c]_O$$
$$[\,\nabla^s\times\nabla^s\times\,]\,[{}^s\mathrm{T}^c] = [\,\nabla^s\times\nabla^s\times\,]_M[{}^s\mathrm{T}^c]_O$$
$$[\nabla^s\nabla^s\cdot\,-\nabla^s\times\nabla^s\times\,]\,[{}^s\mathrm{T}^c] = [\nabla^s\nabla^s\cdot\,-\,\nabla^s\times\nabla^s\times\,]_M[{}^s\mathrm{T}^c]_O$$

where $[{}^s\mathrm{T}^c]$ is the transformation matrix from Cartesian to spherical coordinates. While both $[{}^s\mathrm{T}^c]_O$ and $[{}^s\mathrm{T}^c]$ are identical, the matrix $[{}^s\mathrm{T}^c]_O$ should be treated as a constant matrix.

The formal vector differential operators in spherical coordinates are already given in Sec. 3. The modified vector differential operators are listed below for reference:

$$\left[\nabla^s\right]_M = \begin{bmatrix} \frac{1}{r}\frac{\partial}{\partial\theta} \\ \frac{1}{r}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} \\ \frac{\partial}{\partial r} \end{bmatrix}$$

$$\left[\nabla^s\cdot\right]_M = \begin{bmatrix} \frac{1}{r}\frac{\partial}{\partial\theta} & \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} & \frac{\partial}{\partial r} \end{bmatrix}$$

$$\left[\nabla^s\times\right]_M = \begin{bmatrix} 0 & -\frac{\partial}{\partial r} & \frac{1}{r}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} \\ \frac{\partial}{\partial r} & 0 & -\frac{1}{r}\frac{\partial}{\partial\theta} \\ -\frac{1}{r}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} & \frac{1}{r}\frac{\partial}{\partial\theta} & 0 \end{bmatrix}$$

$$\left[\nabla^s\,\nabla^s\cdot\right]_M = \begin{bmatrix} \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{1}{r}\frac{\partial}{\partial r} & \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial^2}{\partial\theta\partial\phi}-\frac{1}{r^2}\frac{\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\phi} & \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}-\frac{1}{r^2}\frac{\partial}{\partial\theta} \\ \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial^2}{\partial\theta\partial\phi}-\frac{1}{r^2}\frac{\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\phi} & \frac{1}{r^2}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}+\frac{1}{r^2}\frac{\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\theta}+\frac{1}{r}\frac{\partial}{\partial r} & \frac{1}{r}\frac{1}{\sin\theta}\frac{\partial^2}{\partial r\partial\phi}-\frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} \\ \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}-\frac{1}{r^2}\frac{\partial}{\partial\theta} & \frac{1}{r}\frac{1}{\sin\theta}\frac{\partial^2}{\partial r\partial\phi}-\frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} & \frac{\partial^2}{\partial r^2} \end{bmatrix}$$

$$\left[\nabla^s\times\nabla^s\times\right]_M = \begin{bmatrix} -\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}-\frac{1}{r^2}\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} & \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial^2}{\partial\theta\partial\phi}-\frac{1}{r^2}\frac{\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\phi} & \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}-\frac{1}{r^2}\frac{\partial}{\partial\theta} \\ \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial^2}{\partial\theta\partial\phi}-\frac{1}{r^2}\frac{\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\phi} & -\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} & \frac{1}{r}\frac{1}{\sin\theta}\frac{\partial^2}{\partial r\partial\phi}-\frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} \\ \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}-\frac{1}{r^2}\frac{\partial}{\partial\theta} & \frac{1}{r}\frac{1}{\sin\theta}\frac{\partial^2}{\partial r\partial\phi}-\frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} & -\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}-\frac{1}{r^2}\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}-\frac{1}{r^2}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}-\frac{2}{r}\frac{\partial}{\partial r} \end{bmatrix}$$

$$\left[\nabla^s\,\nabla^s\cdot-\nabla^s\times\nabla^s\times\right]_M = \begin{bmatrix} \Delta^s & 0 & 0 \\ 0 & \Delta^s & 0 \\ 0 & 0 & \Delta^s \end{bmatrix}$$

$$\Delta^s = \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{1}{r^2}\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}+\frac{1}{r^2}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}+\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}$$

Appendix B. Partial differential operators in term of spherical coordinates

B.1 Find $\frac{\partial}{\partial \mathrm{x}}, \frac{\partial}{\partial \mathrm{y}}, \frac{\partial}{\partial \mathrm{z}}$ in terms of spherical coordinates

It can be obtained by applying either one of the following three identities:

1. $[\ \nabla^c\ ] = [{}^c\mathrm{T}^s][\ \nabla^s\ ]$

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\phi & -\sin\phi & \sin\theta\cos\phi \\ \cos\theta\sin\phi & \cos\phi & \sin\theta\sin\phi \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \frac{1}{r}\frac{\partial}{\partial\theta} \\ \frac{1}{r}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} \\ \frac{\partial}{\partial r} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi} + \sin\theta\cos\phi\frac{\partial}{\partial r} \\ \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi} + \sin\theta\sin\phi\frac{\partial}{\partial r} \\ -\frac{\sin\theta}{r}\frac{\partial}{\partial\theta} + \cos\theta\frac{\partial}{\partial r} \end{bmatrix}$$

$$= \begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix}$$

2. $[\ \nabla^c\cdot\ ] = [\ \nabla^s\cdot\ ][{}^s\mathrm{T}^c]$

$= [\ \nabla^s\cdot\ ]_M[{}^s\mathrm{T}^c]_O$ (Pure matrix multiplication)

$$\begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{1}{r}\frac{\partial}{\partial\theta} & \frac{1}{r}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} & \frac{\partial}{\partial r} \end{bmatrix} \begin{bmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{bmatrix}_O$$

$$= \begin{bmatrix} D_x & D_y & D_z \end{bmatrix}$$

3. $[\ \nabla^c\times\ ] = [{}^c\mathrm{T}^s][\ \nabla^s\times\ ][{}^s\mathrm{T}^c]$

$= [{}^c\mathrm{T}^s][\ \nabla^s\times\ ]_M[{}^s\mathrm{T}^c]_O$ (Pure matrix multiplication)

$$\begin{bmatrix} 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\phi & -\sin\phi & \sin\theta\cos\phi \\ \cos\theta\sin\phi & \cos\phi & \sin\theta\sin\phi \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 0 & -\frac{\partial}{\partial r} & \frac{1}{r}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} \\ \frac{\partial}{\partial r} & 0 & -\frac{1}{r}\frac{\partial}{\partial\theta} \\ -\frac{1}{r}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} & \frac{1}{r}\frac{\partial}{\partial\theta} & 0 \end{bmatrix} \begin{bmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{bmatrix}_O$$

$$= \begin{bmatrix} 0 & -D_z & D_y \\ D_z & 0 & -D_x \\ -D_y & D_x & 0 \end{bmatrix}$$

B.2 Find $\frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial z^2}$ and $\frac{\partial^2}{\partial x\partial y}, \frac{\partial^2}{\partial y\partial z}, \frac{\partial^2}{\partial z\partial x}$, in spherical coordinates

It can be obtained by applying either one of the following two identities:

4. $[\nabla^c\nabla^c\cdot] = [{}^c\mathrm{T}^s][\nabla^s\nabla^s\cdot][{}^s\mathrm{T}^c]$

$$= [{}^c\mathrm{T}^s][\nabla^s\nabla^s\cdot]_M[{}^s\mathrm{T}^c]_O \quad \text{(Pure matrix multiplication)}$$

$$\begin{bmatrix} \frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial x\partial y} & \frac{\partial^2}{\partial x\partial z} \\ \frac{\partial^2}{\partial y\partial x} & \frac{\partial^2}{\partial y^2} & \frac{\partial^2}{\partial y\partial z} \\ \frac{\partial^2}{\partial z\partial x} & \frac{\partial^2}{\partial z\partial y} & \frac{\partial^2}{\partial z^2} \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\phi & -\sin\phi & \sin\theta\cos\phi \\ \cos\theta\sin\phi & \cos\phi & \sin\theta\sin\phi \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{1}{r}\frac{\partial}{\partial r} & \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial^2}{\partial\theta\partial\phi}-\frac{1}{r^2}\frac{\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\phi} & \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}-\frac{1}{r^2}\frac{\partial}{\partial\theta} \\ \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial^2}{\partial\theta\partial\phi}-\frac{1}{r^2}\frac{\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\phi} & \frac{1}{r^2}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}+\frac{1}{r^2}\frac{\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\theta}+\frac{1}{r}\frac{\partial}{\partial r} & \frac{1}{r}\frac{1}{\sin\theta}\frac{\partial^2}{\partial r\partial\phi}-\frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} \\ \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}-\frac{1}{r^2}\frac{\partial}{\partial\theta} & \frac{1}{r}\frac{1}{\sin\theta}\frac{\partial^2}{\partial r\partial\phi}-\frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} & \frac{\partial^2}{\partial r^2} \end{bmatrix} \begin{bmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{bmatrix}_O$$

$$= \begin{bmatrix} D_{xx} & D_{xy} & D_{xz} \\ D_{yx} & D_{yy} & D_{yz} \\ D_{zx} & D_{zy} & D_{zz} \end{bmatrix}$$

5. $[\nabla^c\times\nabla^c\times] = [{}^c\mathrm{T}^s][\nabla^s\times\nabla^s\times][{}^s\mathrm{T}^c]$

$$= [{}^c\mathrm{T}^s][\nabla^s\times\nabla^s\times]_M[{}^s\mathrm{T}^c]_O \quad \text{(Pure matrix multiplication)}$$

$$\begin{bmatrix} -\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial z^2} & \frac{\partial^2}{\partial x\partial y} & \frac{\partial^2}{\partial x\partial z} \\ \frac{\partial^2}{\partial y\partial x} & -\frac{\partial^2}{\partial z^2}-\frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial y\partial z} \\ \frac{\partial^2}{\partial z\partial x} & \frac{\partial^2}{\partial z\partial y} & -\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2} \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\phi & -\sin\phi & \sin\theta\cos\phi \\ \cos\theta\sin\phi & \cos\phi & \sin\theta\sin\phi \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} -\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}-\frac{1}{r^2}\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} & \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial^2}{\partial\theta\partial\phi}-\frac{1}{r^2}\frac{\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\phi} & \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}-\frac{1}{r^2}\frac{\partial}{\partial\theta} \\ \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial^2}{\partial\theta\partial\phi}-\frac{1}{r^2}\frac{\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\phi} & -\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} & \frac{1}{r}\frac{1}{\sin\theta}\frac{\partial^2}{\partial r\partial\phi}-\frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} \\ \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}-\frac{1}{r^2}\frac{\partial}{\partial\theta} & \frac{1}{r}\frac{1}{\sin\theta}\frac{\partial^2}{\partial r\partial\phi}-\frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} & -\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}-\frac{1}{r^2}\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}-\frac{1}{r^2}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}-\frac{2}{r}\frac{\partial}{\partial r} \end{bmatrix} \begin{bmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{bmatrix}_O$$

$$= \begin{bmatrix} -D_{yy}-D_{zz} & D_{xy} & D_{xz} \\ D_{yx} & -D_{zz}-D_{xx} & D_{yz} \\ D_{zx} & D_{zy} & -D_{xx}-D_{yy} \end{bmatrix}$$

Appendix C. Time derivative of unit vectors and vector functions

C.1. Time derivatives of unit vectors $(\dot{\hat\theta},\dot{\hat\phi},\dot{\hat r})$ and $(\ddot{\hat\theta},\ddot{\hat\phi},\ddot{\hat r})$ in terms of $(\hat\theta,\hat\phi,\hat r)$

$$\begin{bmatrix}\hat\theta\\ \hat\phi\\ \hat r\end{bmatrix}=\begin{bmatrix}\cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta\\ -\sin\phi & \cos\phi & 0\\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta\end{bmatrix}\begin{bmatrix}\hat x\\ \hat y\\ \hat z\end{bmatrix}$$

$$\begin{bmatrix}\dot{\hat\theta}\\ \dot{\hat\phi}\\ \dot{\hat r}\end{bmatrix}=\left\{\dot\theta\begin{bmatrix}-\sin\theta\cos\phi & -\sin\theta\sin\phi & -\cos\theta\\ 0 & 0 & 0\\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta\end{bmatrix}+\dot\phi\begin{bmatrix}-\cos\theta\sin\phi & \cos\theta\cos\phi & 0\\ -\cos\phi & -\sin\phi & 0\\ -\sin\theta\sin\phi & \sin\theta\cos\phi & 0\end{bmatrix}\right\}\begin{bmatrix}\cos\theta\cos\phi & -\sin\phi & \sin\theta\cos\phi\\ \cos\theta\sin\phi & \cos\phi & \sin\theta\sin\phi\\ -\sin\theta & 0 & \cos\theta\end{bmatrix}\begin{bmatrix}\hat\theta\\ \hat\phi\\ \hat r\end{bmatrix}$$

$$=\left\{\dot\theta\begin{bmatrix}0 & 0 & -1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{bmatrix}+\dot\phi\begin{bmatrix}0 & \cos\theta & 0\\ -\cos\theta & 0 & -\sin\theta\\ 0 & \sin\theta & 0\end{bmatrix}\right\}\begin{bmatrix}\hat\theta\\ \hat\phi\\ \hat r\end{bmatrix}$$

$$=\begin{bmatrix}0 & \cos\theta\,\dot\phi & -\dot\theta\\ -\cos\theta\,\dot\phi & 0 & -\sin\theta\,\dot\phi\\ \dot\theta & \sin\theta\,\dot\phi & 0\end{bmatrix}\begin{bmatrix}\hat\theta\\ \hat\phi\\ \hat r\end{bmatrix}$$

$$\begin{bmatrix}\ddot{\hat\theta}\\ \ddot{\hat\phi}\\ \ddot{\hat r}\end{bmatrix}=\left\{\ddot\theta\begin{bmatrix}0 & 0 & -1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{bmatrix}+\ddot\phi\begin{bmatrix}0 & \cos\theta & 0\\ -\cos\theta & 0 & -\sin\theta\\ 0 & \sin\theta & 0\end{bmatrix}+\dot\theta\,\dot\phi\begin{bmatrix}0 & -\sin\theta & 0\\ \sin\theta & 0 & -\cos\theta\\ 0 & \cos\theta & 0\end{bmatrix}\right\}\begin{bmatrix}\hat\theta\\ \hat\phi\\ \hat r\end{bmatrix}$$

$$+\left\{\dot\theta\begin{bmatrix}0 & 0 & -1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{bmatrix}+\dot\phi\begin{bmatrix}0 & \cos\theta & 0\\ -\cos\theta & 0 & -\sin\theta\\ 0 & \sin\theta & 0\end{bmatrix}\right\}\begin{bmatrix}\dot{\hat\theta}\\ \dot{\hat\phi}\\ \dot{\hat r}\end{bmatrix}$$

$$=\left\{\ddot\theta\begin{bmatrix}0 & 0 & -1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{bmatrix}+\ddot\phi\begin{bmatrix}0 & \cos\theta & 0\\ -\cos\theta & 0 & -\sin\theta\\ 0 & \sin\theta & 0\end{bmatrix}+\dot\theta\dot\phi\begin{bmatrix}0 & -2\sin\theta & 0\\ 0 & 0 & 0\\ 0 & 2\cos\theta & 0\end{bmatrix}\right.$$

$$\left.+\dot\theta^2\begin{bmatrix}-1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{bmatrix}+\dot\phi^2\begin{bmatrix}-\cos^2\theta & 0 & -\cos\theta\sin\theta\\ 0 & -1 & 0\\ -\cos\theta\sin\theta & 0 & -\sin^2\theta\end{bmatrix}\right\}\begin{bmatrix}\hat\theta\\ \hat\phi\\ \hat r\end{bmatrix}$$

$$=\begin{bmatrix}-\dot\theta^2-\cos^2\theta\,\dot\phi^2 & \cos\theta\,\ddot\phi-2\sin\theta\,\dot\theta\dot\phi & -\ddot\theta-\cos\theta\sin\theta\,\dot\phi^2\\ -\cos\theta\,\ddot\phi & -\dot\phi^2 & -\sin\theta\,\ddot\phi\\ \ddot\theta-\cos\theta\sin\theta\,\dot\phi^2 & \sin\theta\,\ddot\phi+2\cos\theta\,\dot\theta\dot\phi & -\dot\theta^2-\sin^2\theta\,\dot\phi^2\end{bmatrix}\begin{bmatrix}\hat\theta\\ \hat\phi\\ \hat r\end{bmatrix}$$

C.2 Time derivatives of vector functions $\dot{\mathbf{F}}(\theta,\phi,r)$ and $\ddot{\mathbf{F}}(\theta,\phi,r)$ where $\mathbf{F}(\theta,\phi,r)=\mathbf{F}_\theta+\mathbf{F}_\phi+\mathbf{F}_r=F_\theta\hat{\theta}+F_\phi\hat{\phi}+F_r\hat{r}$

$$\begin{bmatrix}\mathbf{F}_\theta\\ \mathbf{F}_\phi\\ \mathbf{F}_r\end{bmatrix}=\begin{bmatrix}F_\theta & & \\ & F_\phi & \\ & & F_r\end{bmatrix}\begin{bmatrix}\hat{\theta}\\ \hat{\phi}\\ \hat{r}\end{bmatrix}$$

$$\begin{bmatrix}\dot{\mathbf{F}}_\theta\\ \dot{\mathbf{F}}_\phi\\ \dot{\mathbf{F}}_r\end{bmatrix}=\begin{bmatrix}\dot{F}_\theta & & \\ & \dot{F}_\phi & \\ & & \dot{F}_r\end{bmatrix}\begin{bmatrix}\hat{\theta}\\ \hat{\phi}\\ \hat{r}\end{bmatrix}+\begin{bmatrix}F_\theta & & \\ & F_\phi & \\ & & F_r\end{bmatrix}\begin{bmatrix}\dot{\hat{\theta}}\\ \dot{\hat{\phi}}\\ \dot{\hat{r}}\end{bmatrix}$$

$$=\left\{\begin{bmatrix}\dot{F}_\theta & & \\ & \dot{F}_\phi & \\ & & \dot{F}_r\end{bmatrix}+\begin{bmatrix}F_\theta & & \\ & F_\phi & \\ & & F_r\end{bmatrix}\left(\dot{\theta}\begin{bmatrix}0 & 0 & -1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{bmatrix}+\dot{\phi}\begin{bmatrix}0 & \cos\theta & 0\\ -\cos\theta & 0 & -\sin\theta\\ 0 & \sin\theta & 0\end{bmatrix}\right)\right\}\begin{bmatrix}\hat{\theta}\\ \hat{\phi}\\ \hat{r}\end{bmatrix}$$

$$=\begin{bmatrix}\dot{F}_\theta & F_\theta\cos\theta\,\dot{\phi} & -F_\theta\,\dot{\theta}\\ -F_\phi\cos\theta\,\dot{\phi} & \dot{F}_\phi & -F_\phi\sin\theta\,\dot{\phi}\\ F_r\,\dot{\theta} & F_r\sin\theta\,\dot{\phi} & \dot{F}_r\end{bmatrix}\begin{bmatrix}\hat{\theta}\\ \hat{\phi}\\ \hat{r}\end{bmatrix}$$

$$\begin{bmatrix}\ddot{\mathbf{F}}_\theta\\ \ddot{\mathbf{F}}_\phi\\ \ddot{\mathbf{F}}_r\end{bmatrix}=\begin{bmatrix}\ddot{F}_\theta & & \\ & \ddot{F}_\phi & \\ & & \ddot{F}_r\end{bmatrix}\begin{bmatrix}\hat{\theta}\\ \hat{\phi}\\ \hat{r}\end{bmatrix}+2\begin{bmatrix}\dot{F}_\theta & & \\ & \dot{F}_\phi & \\ & & \dot{F}_r\end{bmatrix}\begin{bmatrix}\dot{\hat{\theta}}\\ \dot{\hat{\phi}}\\ \dot{\hat{r}}\end{bmatrix}+\begin{bmatrix}F_\theta & & \\ & F_\phi & \\ & & F_r\end{bmatrix}\begin{bmatrix}\ddot{\hat{\theta}}\\ \ddot{\hat{\phi}}\\ \ddot{\hat{r}}\end{bmatrix}$$

$$=\left\{\begin{bmatrix}\ddot{F}_\theta & & \\ & \ddot{F}_\phi & \\ & & \ddot{F}_r\end{bmatrix}+\begin{bmatrix}\dot{F}_\theta & & \\ & \dot{F}_\phi & \\ & & \dot{F}_r\end{bmatrix}\left(\dot{\theta}\begin{bmatrix}0 & 0 & -2\\ 0 & 0 & 0\\ 2 & 0 & 0\end{bmatrix}+\dot{\phi}\begin{bmatrix}0 & 2\cos\theta & 0\\ -2\cos\theta & 0 & -2\sin\theta\\ 0 & 2\sin\theta & 0\end{bmatrix}\right)\right.$$

$$+\begin{bmatrix}F_\theta & & \\ & F_\phi & \\ & & F_r\end{bmatrix}\left(\ddot{\theta}\begin{bmatrix}0 & 0 & -1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{bmatrix}+\ddot{\phi}\begin{bmatrix}0 & \cos\theta & 0\\ -\cos\theta & 0 & -\sin\theta\\ 0 & \sin\theta & 0\end{bmatrix}+\dot{\theta}\dot{\phi}\begin{bmatrix}0 & -2\sin\theta & 0\\ 0 & 0 & 0\\ 0 & 2\cos\theta & 0\end{bmatrix}\right.$$

$$\left.\left.+\dot{\theta}^2\begin{bmatrix}-1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{bmatrix}+\dot{\phi}^2\begin{bmatrix}-\cos^2\theta & 0 & -\cos\theta\sin\theta\\ 0 & -1 & 0\\ -\cos\theta\sin\theta & 0 & -\sin^2\theta\end{bmatrix}\right)\right\}\begin{bmatrix}\hat{\theta}\\ \hat{\phi}\\ \hat{r}\end{bmatrix}$$

$$=\begin{bmatrix}\ddot{F}_\theta-F_\theta\dot{\theta}^2-F_\theta\cos^2\theta\,\dot{\phi}^2 & 2\dot{F}_\theta\cos\theta\,\dot{\phi}+F_\theta\cos\theta\,\ddot{\phi}-2F_\theta\sin\theta\,\dot{\theta}\dot{\phi} & -2\dot{F}_\theta\,\dot{\theta}-F_\theta\,\ddot{\theta}-F_\theta\cos\theta\sin\theta\,\dot{\phi}^2\\ -2\dot{F}_\phi\cos\theta\,\dot{\phi}-F_\phi\cos\theta\,\ddot{\phi} & \ddot{F}_\phi-F_\phi\,\dot{\phi}^2 & -2\dot{F}_\phi\sin\theta\,\dot{\phi}-F_\phi\sin\theta\,\ddot{\phi}\\ 2\dot{F}_r\,\dot{\theta}+F_r\,\ddot{\theta}-F_r\cos\theta\sin\theta\,\dot{\phi}^2 & 2\dot{F}_r\sin\theta\,\dot{\phi}+F_r\sin\theta\,\ddot{\phi}+2F_r\cos\theta\,\dot{\theta}\dot{\phi} & \ddot{F}_r-F_r\,\dot{\theta}^2-F_r\sin^2\theta\,\dot{\phi}^2\end{bmatrix}\begin{bmatrix}\hat{\theta}\\ \hat{\phi}\\ \hat{r}\end{bmatrix}$$

C.3. Differentiation of unit vectors in spherical coordinates

$$\nabla\times\hat{\theta}=\frac{1}{r}\hat{\phi} \qquad \nabla\cdot\hat{\theta}=\frac{1}{r}\frac{\cos\theta}{\sin\theta}$$

$$\nabla\times\hat{\phi}=-\frac{1}{r}\hat{\theta}+\frac{1}{r}\frac{\cos\theta}{\sin\theta}\hat{\phi} \qquad \nabla\cdot\hat{\phi}=0$$

$$\nabla\times\hat{r}=0 \qquad \nabla\cdot\hat{r}=\frac{2}{r}$$

$$\nabla\times\nabla\times\hat{\theta}=\frac{1}{r^2}\frac{1}{\sin^2\theta}\hat{r} \qquad \nabla\nabla\cdot\hat{\theta}=-\frac{1}{r^2}\frac{1}{\sin^2\theta}\hat{\theta}-\frac{1}{r^2}\frac{\cos\theta}{\sin\theta}\hat{r}$$

$$\nabla\times\nabla\times\hat{\phi}=\frac{1}{r^2}\frac{1}{\sin^2\theta}\hat{\phi} \qquad \nabla\nabla\cdot\hat{\phi}=0$$

$$\nabla\times\nabla\times\hat{r}=0 \qquad \nabla\nabla\cdot\hat{r}=-\frac{2}{r^2}\hat{r}$$

$$(\nabla\nabla\cdot-\nabla\times\nabla\times)\,\hat{\theta}=-\frac{1}{r^2}\frac{1}{\sin^2\theta}\hat{\theta}-\frac{2}{r^2}\frac{\cos\theta}{\sin\theta}\hat{r}$$

$$(\nabla\nabla\cdot-\nabla\times\nabla\times)\hat{\phi}=-\frac{1}{r^2}\frac{1}{\sin^2\theta}\hat{\phi}$$

$$(\nabla\nabla\cdot-\nabla\times\nabla\times)\hat{r}=-\frac{2}{r^2}\hat{r}$$

$$\frac{\partial}{\partial\theta}\hat{\theta}=-\hat{r} \qquad \frac{\partial}{\partial\phi}\hat{\theta}=\cos\theta\hat{\phi} \qquad \frac{\partial}{\partial r}\hat{\theta}=0$$

$$\frac{\partial}{\partial\theta}\hat{\phi}=0 \qquad \frac{\partial}{\partial\phi}\hat{\phi}=-\cos\theta\hat{\theta}-\sin\theta\hat{r} \qquad \frac{\partial}{\partial r}\hat{\phi}=0$$

$$\frac{\partial}{\partial\theta}\hat{r}=\hat{\theta} \qquad \frac{\partial}{\partial\phi}\hat{r}=\sin\theta\hat{\phi} \qquad \frac{\partial}{\partial r}\hat{r}=0$$

$$\frac{\partial^2}{\partial\theta^2}\hat{\theta}=-\hat{\theta} \qquad \frac{\partial^2}{\partial\phi^2}\hat{\theta}=-\cos^2\theta\hat{\theta}-\cos\theta\sin\theta\hat{r} \qquad \frac{\partial^2}{\partial r^2}\hat{\theta}=0$$

$$\frac{\partial^2}{\partial\theta^2}\hat{\phi}=0 \qquad \frac{\partial^2}{\partial\phi^2}\hat{\phi}=-\hat{\phi} \qquad \frac{\partial^2}{\partial r^2}\hat{\phi}=0$$

$$\frac{\partial^2}{\partial\theta^2}\hat{r}=-\hat{r} \qquad \frac{\partial^2}{\partial\phi^2}\hat{r}=-\cos\theta\sin\theta\hat{\theta}-\sin^2\theta\hat{r} \qquad \frac{\partial^2}{\partial r^2}\hat{r}=0$$

$$\frac{\partial^2}{\partial\theta\partial\phi}\hat{\theta}=-\sin\theta\hat{\phi} \qquad \frac{\partial^2}{\partial r\partial\theta}\hat{\theta}=0 \qquad \frac{\partial^2}{\partial r\partial\phi}\hat{\theta}=0$$

$$\frac{\partial^2}{\partial\theta\partial\phi}\hat{\phi}=0 \qquad \frac{\partial^2}{\partial r\partial\theta}\hat{\phi}=0 \qquad \frac{\partial^2}{\partial r\partial\phi}\hat{\phi}=0$$

$$\frac{\partial^2}{\partial\theta\partial\phi}\hat{r}=\cos\theta\hat{\phi} \qquad \frac{\partial^2}{\partial r\partial\theta}\hat{r}=0 \qquad \frac{\partial^2}{\partial r\partial\phi}\hat{r}=0$$

Examples for derivation:

$$[\nabla^S \times][\hat{\theta}\,\hat{\phi}\,\hat{r}]^S = \begin{bmatrix} 0 & -\frac{\partial}{\partial r}-\frac{1}{r} & \frac{1}{r}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} \\ \frac{\partial}{\partial r}+\frac{1}{r} & 0 & -\frac{1}{r}\frac{\partial}{\partial\theta} \\ -\frac{1}{r}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} & \frac{1}{r}\frac{\partial}{\partial\theta}+\frac{1}{r}\frac{\cos\theta}{\sin\theta} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{r} & 0 \\ -\frac{1}{r} & 0 & 0 \\ 0 & \frac{1}{r}\frac{\cos\theta}{\sin\theta} & 0 \end{bmatrix}$$

$$[\nabla^S \cdot][\hat{\theta}\,\hat{\phi}\,\hat{r}]^S = \begin{bmatrix} \frac{1}{r}\frac{\partial}{\partial\theta}+\frac{1}{r}\frac{\cos\theta}{\sin\theta} & \frac{1}{r}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} & \frac{\partial}{\partial r}+\frac{2}{r} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{r}\frac{\cos\theta}{\sin\theta} & 0 & \frac{2}{r} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial}{\partial\theta}\hat{\theta} \\ \frac{\partial}{\partial\theta}\hat{\phi} \\ \frac{\partial}{\partial\theta}\hat{r} \end{bmatrix} = \frac{\partial}{\partial\theta}\begin{bmatrix} \hat{\theta} \\ \hat{\phi} \\ \hat{r} \end{bmatrix} = \frac{\partial}{\partial\theta}\begin{bmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{bmatrix}\begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix}$$

$$= \begin{bmatrix} -\sin\theta\cos\phi & -\sin\theta\sin\phi & -\cos\theta \\ 0 & 0 & 0 \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \end{bmatrix} \cdot \begin{bmatrix} \cos\theta\cos\phi & -\sin\phi & \sin\theta\cos\phi \\ \cos\theta\sin\phi & \cos\phi & \sin\theta\sin\phi \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}\begin{bmatrix} \hat{\theta} \\ \hat{\phi} \\ \hat{r} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} \hat{\theta} \\ \hat{\phi} \\ \hat{r} \end{bmatrix} = \begin{bmatrix} -\hat{r} \\ 0 \\ \hat{\theta} \end{bmatrix}$$

Appendix D. Vector differential operations to vector algebraic operations

Vector differential operations can convert into pure vector algebraic operations when the operands involve only functions of $\mathbf{R}$, where $\mathbf{R} = \mathbf{r} - \mathbf{r}'$.

$$\nabla \cdot \mathbf{R} = 3$$
$$\nabla \times \mathbf{R} = 0$$
$$\nabla(\mathbf{R} \cdot \mathbf{J}) = \mathbf{J}$$
$$\nabla \cdot (\mathbf{R} \times \mathbf{J}) = 0$$
$$\nabla \times (\mathbf{R} \times \mathbf{J}) = -2\mathbf{J}$$
$$\mathbf{J} \cdot \nabla \mathbf{R} = \mathbf{J}$$
$$\nabla \cdot \hat{\mathbf{R}} = \frac{2}{R}$$
$$\nabla \times \hat{\mathbf{R}} = 0$$
$$\nabla(\hat{\mathbf{R}} \cdot \mathbf{J}) = \frac{1}{R}(1 - \hat{\mathbf{R}}\hat{\mathbf{R}}\cdot)\mathbf{J}$$
$$\nabla \cdot (\hat{\mathbf{R}} \times \mathbf{J}) = 0$$
$$\nabla \times (\hat{\mathbf{R}} \times \mathbf{J}) = -\frac{1}{R}(1 + \hat{\mathbf{R}}\hat{\mathbf{R}}\cdot)\mathbf{J}$$
$$\mathbf{J} \cdot \nabla \hat{\mathbf{R}} = \frac{1}{R}(1 - \hat{\mathbf{R}}\hat{\mathbf{R}}\cdot)\mathbf{J}$$
$$\nabla R = \hat{\mathbf{R}}$$
$$\nabla \cdot R\mathbf{J} = \hat{\mathbf{R}} \cdot \mathbf{J}$$
$$\nabla \times R\mathbf{J} = \hat{\mathbf{R}} \times \mathbf{J}$$
$$\nabla \cdot f(R)\mathbf{J} = \hat{\mathbf{R}} \cdot f'(R)\,\mathbf{J}$$
$$\nabla \times f(R)\mathbf{J} = \hat{\mathbf{R}} \times f'(R)\,\mathbf{J}$$
$$\nabla f(R) = \hat{\mathbf{R}} f'(R)$$
$$\nabla \cdot \mathbf{F}(R) = \hat{\mathbf{R}} \cdot \mathbf{F}'(R)$$
$$\nabla \times \mathbf{F}(R) = \hat{\mathbf{R}} \times \mathbf{F}'(R)$$
$$\nabla(\mathbf{F}(R) \cdot \mathbf{J}) = \hat{\mathbf{R}}\mathbf{F}'(R) \cdot \mathbf{J}$$

$$\nabla\times\nabla f(R)=0$$
$$\nabla\cdot\nabla f(R)=\frac{2}{R}f'(R)+f''(R)$$
$$\nabla\cdot\nabla\times f(R)\mathbf{J}=0$$
$$\nabla\nabla\cdot f(R)\mathbf{J}=\Big((1-\hat{\mathbf{R}}\hat{\mathbf{R}}\cdot)\frac{1}{R}f'(R)+\hat{\mathbf{R}}\hat{\mathbf{R}}\cdot f''(R)\Big)\mathbf{J}$$
$$\nabla\times\nabla\times f(R)\mathbf{J}=-\Big((1+\hat{\mathbf{R}}\hat{\mathbf{R}}\cdot)\frac{1}{R}f'(R)+(1-\hat{\mathbf{R}}\hat{\mathbf{R}}\cdot)f''(R)\Big)\mathbf{J}$$
$$(\nabla\nabla\cdot-\nabla\times\nabla\times)f(R)\mathbf{J}=\Big(\frac{2}{R}f'(R)+f''(R)\Big)\mathbf{J}$$
$$\nabla\times\nabla(\mathbf{F}(R)\cdot\mathbf{J})=0$$
$$\nabla\cdot\nabla(\mathbf{F}(R)\cdot\mathbf{J})=\Big(\frac{2}{R}\mathbf{F}'(R)\cdot+\mathbf{F}''(R)\cdot\Big)\mathbf{J}$$
$$\nabla\cdot\nabla\times\mathbf{F}(R)=0$$
$$\nabla\nabla\cdot\mathbf{F}(R)=(1-\hat{\mathbf{R}}\hat{\mathbf{R}}\cdot)\frac{1}{R}\mathbf{F}'(R)+\hat{\mathbf{R}}\hat{\mathbf{R}}\cdot\mathbf{F}''(R)$$
$$\nabla\times\nabla\times\mathbf{F}(R)=-(1+\hat{\mathbf{R}}\hat{\mathbf{R}}\cdot)\frac{1}{R}\mathbf{F}'(R)-(1-\hat{\mathbf{R}}\hat{\mathbf{R}}\cdot)\mathbf{F}''(R)$$
$$(\nabla\nabla\cdot-\nabla\times\nabla\times)\mathbf{F}(R)=\frac{2}{R}\mathbf{F}'(R)+\mathbf{F}''(R)$$

Here

$$\mathbf{R}=\mathbf{r}-\mathbf{r}'=(x-x')\hat{x}+(y-y')\hat{y}+(z-z')\hat{z}$$
$$R=|\mathbf{r}-\mathbf{r}'|=[(x-x')^2+(y-y')^2+(z-z')^2]^{1/2}$$
$$\hat{\mathbf{R}}=\frac{\mathbf{R}}{R}=\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}=\hat{x}\frac{\partial R}{\partial x}+\hat{y}\frac{\partial R}{\partial y}+\hat{z}\frac{\partial R}{\partial z}$$

and the vector differential operator ∇ operates upon **r**, but not **r'**, and **J** is treated as a constant vector under ∇. Also $\mathbf{F}(R)$ and $f(R)$ are vector and scalar functions of R, respectively. It is noted that the all derived formulas are independent of coordinate systems.

Appendix E. Derivation of some equations in Sec. 7 of main text

E.1 Derivation of $[\mathbf{H}(\mathbf{r})]^S = [\nabla\times]^S_M [{}^S\mathrm{T}^C]_O [\mathbf{A}(\mathbf{r})]^C$ from $[\mathbf{H}(\mathbf{r})]^S = [\nabla\times]^S [{}^S\mathrm{T}^C][\mathbf{A}(\mathbf{r})]^C$

In $$[\mathbf{H}(\mathbf{r})]^S = [\nabla\times]^S [{}^S\mathrm{T}^C][\mathbf{A}(\mathbf{r})]^C$$

Let

$$[\nabla\times]^S = [\partial\times]^S + [C\times]^S = \begin{bmatrix} 0 & -\frac{\partial}{\partial r} & \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \\ \frac{\partial}{\partial r} & 0 & -\frac{1}{r}\frac{\partial}{\partial\theta} \\ -\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} & \frac{1}{r}\frac{\partial}{\partial\theta} & 0 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{1}{r} & 0 \\ \frac{1}{r} & 0 & 0 \\ 0 & \frac{\cos\theta}{r\sin\theta} & 0 \end{bmatrix}$$

Then,

$$\begin{aligned}[\mathbf{H}(\mathbf{r})]^S &= [\nabla\times]^S [{}^S\mathrm{T}^C][\mathbf{A}(\mathbf{r})]^C = ([\partial\times]^S + [C\times]^S)[{}^S\mathrm{T}^C][\mathbf{A}(\mathbf{r})]^C \\ &= [\partial\times]^S [{}^S\mathrm{T}^C][\mathbf{A}(\mathbf{r})]^C + [C\times]^S [{}^S\mathrm{T}^C][\mathbf{A}(\mathbf{r})]^C \\ &= [\partial\times]^S [{}^S\mathrm{T}^C]_O [\mathbf{A}(\mathbf{r})]^C + [\partial\times]^S [{}^S\mathrm{T}^C][\mathbf{A}(\mathbf{r})]^C_O + [C\times]^S [{}^S\mathrm{T}^C][\mathbf{A}(\mathbf{r})]^C\end{aligned}$$

Since

$$[\partial\times]^S[{}^S\mathrm{T}^C] = \begin{bmatrix} 0 & -\frac{\partial}{\partial r} & \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \\ \frac{\partial}{\partial r} & 0 & -\frac{1}{r}\frac{\partial}{\partial\theta} \\ -\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} & \frac{1}{r}\frac{\partial}{\partial\theta} & 0 \end{bmatrix} \begin{bmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{bmatrix} = \begin{bmatrix} -\frac{1}{r}\sin\phi & \frac{1}{r}\cos\phi & 0 \\ -\frac{\cos\theta}{r}\cos\phi & -\frac{\cos\theta}{r}\sin\phi & \frac{\sin\theta}{r} \\ \frac{\cos\theta}{r\sin\theta}\sin\phi & -\frac{\cos\theta}{r\sin\theta}\cos\phi & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} 0 & -\frac{1}{r} & 0 \\ \frac{1}{r} & 0 & 0 \\ 0 & \frac{\cos\theta}{r\sin\theta} & 0 \end{bmatrix} \begin{bmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{bmatrix} = -[C\times]^S[{}^S\mathrm{T}^C]$$

We have

$$[\mathbf{H}(\mathbf{r})]^S = [\partial\times]^S [{}^S\mathrm{T}^C]_O [\mathbf{A}(\mathbf{r})]^C$$

That is,

$$[\mathbf{H}(\mathbf{r})]^S = [\nabla\times]^S_M [{}^S\mathrm{T}^C]_O [\mathbf{A}(\mathbf{r})]^C$$

E. 2 Derivation of

$$\mathbf{H}(\mathbf{r}) = -jk\int_V \bar{\mathbf{K}}_\nabla(\mathbf{r})g(R)\mathbf{J}(\mathbf{r}')dv' = -jk\int_V g(R)\bar{\mathbf{K}}(\mathbf{R})\mathbf{J}(\mathbf{r}')dv' \quad \textbf{and}$$

$$\mathbf{E}(\mathbf{r}) = -jk\eta\int_V \bar{\mathbf{G}}_\nabla(\mathbf{r})g(R)\mathbf{J}(\mathbf{r}')dv' - jk\eta\int_V g(R)\bar{\mathbf{G}}(\mathbf{R})\mathbf{J}(\mathbf{r}')dv'$$

Consider $\mathbf{H}(\mathbf{r})$ due to $\mathbf{J}(\mathbf{r})$:

$$\mathbf{H}(\mathbf{r}) = -jk\int_V \bar{\mathbf{K}}_\nabla(\mathbf{r})g(|\mathbf{r}-\mathbf{r}'|)\mathbf{J}(\mathbf{r}')dv', \qquad \bar{\mathbf{K}}_\nabla(\mathbf{r}) \equiv \frac{1}{-jk}\nabla\times$$

$$= -jk\int_V \frac{1}{-jk}\nabla\times g(R)\mathbf{J}(\mathbf{r}')dv'$$

$$[\mathbf{H}(\mathbf{r})]^S = -jk\int_V [{}^S\mathrm{T}^{S_R}]\frac{1}{-jk}[\nabla^{S_R}\times][{}^{S_R}\mathrm{T}^{C_R}][{}^{C_R}\mathrm{T}^{C'}]g(R)[\mathbf{J}(\mathbf{r}')]^{C'}dv'$$

$$= -jk\int_V [{}^S\mathrm{T}^{S_R}]\frac{1}{-jk}[\nabla^{S_R}\times]_M[{}^{S_R}\mathrm{T}^{C_R}]_O[{}^{C_R}\mathrm{T}^{C'}]g(R)[\mathbf{J}(\mathbf{r}')]^{C'}dv'$$

Since

$$\frac{1}{-jk}[\nabla^{S_R}\times]_M\ g(R) = \frac{1}{-jk}\begin{bmatrix} 0 & -\frac{\partial}{\partial R} & \frac{1}{R\sin\theta_R}\frac{\partial}{\partial\phi_R} \\ \frac{\partial}{\partial R} & 0 & -\frac{1}{R}\frac{\partial}{\partial\theta_R} \\ -\frac{1}{R\sin\theta_R}\frac{\partial}{\partial\phi_R} & \frac{1}{R}\frac{\partial}{\partial\theta_R} & 0 \end{bmatrix}\frac{e^{-jkR}}{4\pi R}$$

$$= \frac{e^{-jkR}}{4\pi R}\begin{bmatrix} 0 & -(1+\frac{1}{jkR}) & 0 \\ (1+\frac{1}{jkR}) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= g(R)(1+\frac{1}{jkR})[\hat{\mathbf{R}}\times]^{S_R} = g(R)[\bar{\mathbf{K}}(\mathbf{R})]^{S_R}$$

then

$$[\mathbf{H}(\mathbf{r})]^S = -jk\int_V g(R)[{}^S\mathrm{T}^{S_R}][\bar{\mathbf{K}}(\mathbf{R})]^{S_R}[{}^{S_R}\mathrm{T}^{C_R}][{}^{C_R}\mathrm{T}^{C'}][\mathbf{J}(\mathbf{r}')]^{C'}dv'$$

$$\mathbf{H}(\mathbf{r}) = -jk\int_V g(R)\bar{\mathbf{K}}(\mathbf{R})\mathbf{J}(\mathbf{r}')dv', \qquad \bar{\mathbf{K}}(\mathbf{R}) = (1+\frac{1}{jkR})\hat{\mathbf{R}}\times$$

Next, consider $\mathbf{E}(\mathbf{r})$ due to $\mathbf{J}(\mathbf{r})$:

$$\mathbf{E}(\mathbf{r}) = -jk\eta \int_V \bar{\mathbf{G}}_\nabla(\mathbf{r}) g(|\mathbf{r}-\mathbf{r}'|)\mathbf{J}(\mathbf{r}')dv', \qquad \bar{\mathbf{G}}_\nabla(\mathbf{r}) \equiv 1 + \frac{1}{k^2}\nabla\nabla\cdot$$

$$= -jk\eta \int_V (1+\frac{1}{k^2}\nabla\nabla\cdot) g(R)\mathbf{J}(\mathbf{r}')dv'$$

$$[\mathbf{E}(\mathbf{r})]^S = \int_V [{}^S\mathrm{T}^{S_R}]([1]+\frac{1}{k^2}[\nabla\nabla\cdot]^{S_R})[{}^{S_R}\mathrm{T}^{C_R}][{}^{C_R}\mathrm{T}^{C'}]g(R)[\mathbf{J}(\mathbf{r}')]^{C'}dv'$$

$$= \int_V [{}^S\mathrm{T}^{S_R}]([1]+\frac{1}{k^2}[\nabla\nabla\cdot]^{S_R})_M[{}^{S_R}\mathrm{T}^{C_R}]_O[{}^{C_R}\mathrm{T}^{C'}]g(R)[\mathbf{J}(\mathbf{r}')]^{C'}dv'$$

Since

$$([1]+\frac{1}{k^2}[\nabla\nabla\cdot]^{S_R})_M\, g(R) = \left(\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} + \frac{1}{k^2}\begin{bmatrix}\$+\frac{1}{R}\frac{\partial}{\partial R} & \$ & \$\\ \$ & \$+\frac{1}{R}\frac{\partial}{\partial R} & \$\\ \$ & \$ & \frac{\partial^2}{\partial R^2}\end{bmatrix}\right)\frac{e^{-jkR}}{4\pi R^2}$$

$$= \frac{e^{-jkR}}{4\pi R^2}\begin{bmatrix}1+\frac{1}{jkR}+\frac{1}{(jkR)^2} & 0 & 0\\ 0 & 1+\frac{1}{jkR}+\frac{1}{(jkR)^2} & 0\\ 0 & 0 & \frac{-2}{jkR}+\frac{-2}{(jkR)^2}\end{bmatrix}$$

$$= g(R)\left((1+\frac{1}{jkR}+\frac{1}{(jkR)^2})[1-\hat{R}\hat{R}\cdot]^{S_R} + (\frac{-2}{jkR}+\frac{-2}{(jkR)^2})[\hat{R}\hat{R}\cdot]^{S_R}\right)$$

$$= g(R)[\bar{\mathbf{G}}(\mathbf{R})]^{S_R}$$

then

$$[\mathbf{E}(\mathbf{r})]^S = -jk\eta\int_V g(R)[{}^S\mathrm{T}^{S_R}][\bar{\mathbf{G}}(\mathbf{R})]^{S_R}[{}^{S_R}\mathrm{T}^{C_R}][{}^{C_R}\mathrm{T}^{C'}][\mathbf{J}(\mathbf{r}')]^{C'}dv'$$

$$\mathbf{E}(\mathbf{r}) = -jk\eta\int_V g(R)\bar{\mathbf{G}}(\mathbf{R})\mathbf{J}(\mathbf{r}')dv', \quad \bar{\mathbf{G}}(\mathbf{R}) = (1+\frac{1}{jkR}+\frac{1}{(jkR)^2})(1-\hat{R}\hat{R}\cdot)+(\frac{-2}{jkR}+\frac{-2}{(jkR)^2})\,\hat{R}\hat{R}\cdot$$

E. 3 Alternative derivation of

$$\mathbf{H}(\mathbf{r}) = -jk\int_V \bar{\mathbf{K}}_\nabla(\mathbf{r})g(R)\mathbf{J}(\mathbf{r}')dv' = -jk\int_V g(R)\bar{\mathbf{K}}(\mathbf{R})\mathbf{J}(\mathbf{r}')dv' \quad \textbf{and}$$

$$\mathbf{E}(\mathbf{r}) = -jk\eta\int_V \bar{\mathbf{G}}_\nabla(\mathbf{r})g(R)\mathbf{J}(\mathbf{r}')dv' - jk\eta\int_V g(R)\bar{\mathbf{G}}(\mathbf{R})\mathbf{J}(\mathbf{r}')dv'$$

From some formulas in App. D

$$\begin{aligned}\bar{\mathbf{K}}_\nabla(\mathbf{r})g(R)\mathbf{J}(\mathbf{r}') &= \frac{1}{-jk}\nabla\times g(R)\mathbf{J}(\mathbf{r}') \\ &= \frac{1}{-jk}\hat{\mathbf{R}}\times g'(R)\mathbf{J}(\mathbf{r}') \\ \bar{\mathbf{G}}_\nabla(\mathbf{r})g(R)\mathbf{J}(\mathbf{r}') &= (1+\frac{1}{k^2}\nabla\nabla\cdot)g(R)\mathbf{J}(\mathbf{r}') \\ &= g(R)\mathbf{J}(\mathbf{r}') + \frac{1}{k^2}g'(R)\frac{1}{R}(\mathbf{1}-\hat{\mathbf{R}}\hat{\mathbf{R}}\cdot)\mathbf{J}(\mathbf{r}') + \frac{1}{k^2}g''(R)\hat{\mathbf{R}}\hat{\mathbf{R}}\cdot\mathbf{J}(\mathbf{r}')\end{aligned}$$

Since

$$\begin{aligned}g(R) &= \frac{e^{-jkR}}{4\pi r} \\ g'(R) &= -jk(1+\frac{1}{jkR})g(R) \\ g''(R) &= -k^2(1+\frac{2}{jkR}+\frac{2}{(jkR)^2})g(R)\end{aligned}$$

then

$$\begin{aligned}\bar{\mathbf{K}}_\nabla(\mathbf{R})g(R)\mathbf{J}(\mathbf{r}') &= g(R)(1+\frac{1}{jkR})\hat{\mathbf{R}}\times\mathbf{J}(\mathbf{r}') \\ &= g(R)\bar{\mathbf{K}}(\mathbf{R})\mathbf{J}(\mathbf{r}')\end{aligned}$$

$$\begin{aligned}\bar{\mathbf{G}}_\nabla(\mathbf{R})g(R)\mathbf{J}(\mathbf{r}') &= g(R)\Big((1+\frac{1}{jkR}+\frac{1}{(jkR)^2})(\mathbf{1}-\hat{\mathbf{R}}\hat{\mathbf{R}}\cdot)+(\frac{-2}{jkR}+\frac{-2}{(jkR)^2})\hat{\mathbf{R}}\hat{\mathbf{R}}\cdot\Big)\mathbf{J}(\mathbf{r}') \\ &= g(R)\bar{\mathbf{G}}(\mathbf{R})\mathbf{J}(\mathbf{r}')\end{aligned}$$

This completes the alternative derivation.