A SIMILARITY TRANSFORMATION FOR THE ROTATION MATRIX

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ABSTRACT. It is shown explicitly a similarity transformation to diagonalize the rotation matrix.

Keywords: Rotation matrix, Similarity transformation, Cayley-Klein parameters.

1. Introduction

The 3-space rotations can be performed with the matrix [1]:

\[
R = \begin{pmatrix}
\frac{1}{2}(\alpha^2 + \alpha^* - \beta^2 - \beta^*) & -\frac{i}{2}(\alpha^2 - \alpha^* + \beta^2 - \beta^*) & -(\alpha\beta + \alpha^*\beta^*) \\
-\frac{i}{2}(\alpha^2 - \alpha^* - \beta^2 + \beta^*) & \frac{1}{2}(\alpha^2 + \alpha^* + \beta^2 + \beta^*) & -i(\alpha\beta - \alpha^*\beta^*) \\
\alpha\beta^* + \alpha^*\beta & i(\alpha^*\beta - \alpha\beta^*) & \alpha\beta^* - \beta^* \alpha^*
\end{pmatrix},
\]

(1)

where \(\alpha\) and \(\beta\) are the Cayley-Klein parameters under the condition:

\[
\alpha\alpha^* + \beta\beta^* = 1,
\]

(2)

which implies the orthogonal character of (1):

\[
R R^T = I, \quad \det R = 1.
\]

(3)

The Euler-Olind Rodrigues coefficients [2],[3] \(a_1, ..., a_4\) defined by:

\[
a_1 = \frac{i}{2}(\beta - \beta^*), \quad a_2 = -\frac{1}{2}(\beta + \beta^*), \quad a_3 = \frac{i}{2}(\alpha - \alpha^*), \quad a_4 = \frac{1}{2}(\alpha + \alpha^*),
\]

(4)
verify:
\[ a_1^2 + a_2^2 + a_3^2 + a_4^2 = 1 \quad , \]
by virtue of (2), so that (1) adopts the form:
\[
R = \begin{bmatrix} 1 - 2(a_1^2 + a_2^2) & 2(a_1a_2 - a_3a_4) & 2(a_1a_3 + a_2a_4) \\ 2(a_1a_2 + a_3a_4) & 1 - 2(a_1^2 + a_2^2) & 2(a_2a_3 - a_4a_4) \\ 2(a_1a_3 - a_2a_4) & 2(a_2a_3 + a_4a_4) & 1 - 2(a_1^2 + a_2^2) \end{bmatrix} ,
\]
(6)

The rotation matrix represents [4] a rotation of the Cartesian axes through a \( \Phi \) angle around the axis defined by the unitary vector \( (l_1, l_2, l_3) \), such that [3]:
\[
a_j = l_j \sin \left( \frac{\Phi}{2} \right) , \quad j = 1, 2, 3 \quad , \quad a_4 = \cos \left( \frac{\Phi}{2} \right) ,
\]
consistent with (5), since:
\[
l_1^2 + l_2^2 + l_3^2 = 1 \quad .
\]
(8)

Then (6) simplifies to:
\[
R = \begin{bmatrix} l_1^2 + \gamma + \cos \Phi & l_1l_2^\gamma - l_1l_3^\gamma - l_2l_3^\gamma + l_2\sin \Phi & l_1l_3^\gamma + l_2\sin \Phi \\ l_1l_2^\gamma + l_2\cos \Phi & l_2^2 + \gamma - l_2l_3^\gamma - l_2l_3\sin \Phi & l_2l_3^\gamma + l_2\sin \Phi \\ l_1l_3^\gamma - l_2\sin \Phi & l_2l_3^\gamma + l_2\sin \Phi & l_3^2 + \gamma + \cos \Phi \end{bmatrix} , \quad \gamma = 1 - \cos \Phi ,
\]
(9)
which is be studied in the present work.

In next Section, the eigenvalue problem of (9) is used to write an \( R \) as a product of three matrices, which in turn explicitly shows the existence of a similarity transformation to diagonalize any rotation matrices.

2. Eigenvalues and eigenvectors of \( R \).

The relation:
\[
R \vec{u} = \lambda \vec{u} .
\]
(10)
expresses the eigenvalue problem for the rotation matrix, such that is quite known [5] that:
\[ \lambda_1 = 1 , \quad \vec{u}_1 = \begin{pmatrix} l_2 \\ -l_1 + il_3 \\ -l_2 - il_1 \end{pmatrix} , \quad \lambda_2 = e^{-\phi} , \quad \lambda_3 = \lambda_2^2 = e^{2\phi} , \]

however, it is very difficult to find explicitly \( \vec{u}_2 \). In fact, the eigenvector \( \vec{u}_2 \) can be any of the following three vectors, proportional among them:

\[ \begin{pmatrix} l_2^2 + l_3^2 \\ -l_1 l_2 - il_3 \\ -l_2 l_3 - il_1 \end{pmatrix} , \quad \begin{pmatrix} -l_1 l_2 + il_3 \\ l_2^2 + l_3^2 \\ -l_2 l_3 + il_1 \end{pmatrix} , \quad \begin{pmatrix} -l_1 l_3 - il_2 \\ -l_2 l_3 - il_1 \\ l_2^2 + l_3^2 \end{pmatrix} , \quad (12) \]

and it is chosen as the most convenient according to the values of \( l_1, l_2, l_3 \), for instance, if \( l_1 = l_3 = 0, l_2 = 1 \), the second vector is the trivial solution \( \vec{u}_2 = \vec{0} \), and in that case it can be used any other vector of (12). Without losing generality, here it will be used:

\[ \vec{u}_2 = \begin{pmatrix} l_2^2 + l_3^2 \\ -l_1 l_2 + il_3 \\ -l_2 l_3 + il_1 \end{pmatrix} , \quad \vec{u}_3 - \vec{u}_2 = \begin{pmatrix} l_2^2 + l_3^2 \\ -l_1 l_2 + il_3 \\ -l_2 l_3 + il_1 \end{pmatrix} . \quad (13) \]

The problem of eigenvalues of a matrix gives algebraic information if it is also solved for the corresponding transpose matrix of (9) [6]:

\[ R^T \vec{\nu} - \vec{\nu} = 0 . \quad (14) \]

then getting the same eigenvalues (11), with:
which together with (8) and (13) imply:

\[ \vec{v}_j \vec{u}_k - \omega_{jk}. \]  

Then it is natural to construct the matrices:

\[ \Lambda = \text{Diag} \left( \lambda_1, \lambda_2, \lambda_3 \right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\phi} & 0 \\ 0 & 0 & e^{i\phi} \end{pmatrix}, \]  

and because the property (16) it results:

\[ \Lambda^T = \Lambda^{-1}. \]  

Notice the existence of $U^{-1}$ because when choosing $\bar{u}_2$ given by (13) it has been supposed that $l_2$ and $l_3$ do not simultaneously cancel, so that $\det U = -2i(l_2^2 + l_3^2) \neq 0$.

General theorems of Linear Algebra [6] lead to the following factorization of the rotation matrix:

$$R = \tilde{U} \Lambda \tilde{V}^T = \tilde{U} \Lambda \tilde{U}^{-1}. \quad (19)$$

which is easily seen through (8), (9) and (17). From (19) it is immediate that:

$$U^{-1} R \tilde{U} = \Lambda = \text{diag}(1, e^{-i\phi}, e^{i\phi}) \quad , \quad (20)$$

that is, the $\tilde{U}$ matrix given explicitly by (17) generates a similarity transformation which diagonalizes $R$, q.e.d.

References