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Some results on the convergence of GAOR method

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Abstract. In this paper, we discuss the convergence of GAOR method to solve linear system when the coefficient matrix is a weighted strictly doubly diagonally dominant matrix. Moreover, we show that our results are better than previous ones by using two numerical examples.

Keywords: GAOR method; spectral radius; convergence; weighted doubly diagonally dominant matrix; bound.

1 Introduction

Sometimes we have to solve the following linear system:

$$Hy = f, \tag{1}$$

where

$$H = \begin{pmatrix} I - B_1 & D \\ C & I - B_2 \end{pmatrix}$$

is invertible. For example, in the variance-covariance matrix [1]. A generalized SOR(GSOR) method to solve linear system (1) was proposed in [2], afterwards, a generalized AOR(GAOR) method to solve linear system (1) was established in [3] as follows:

$$y^{(k+1)} = L_{\omega,r} y^{(k)} + \omega k , \tag{2}$$

where

$$L_{\omega,r} = (1 - \omega) I + \omega J + \omega r , \tag{3}$$

$$k = \begin{pmatrix} I & 0 \\ -rC & I \end{pmatrix} f , \tag{4}$$

$$J = \begin{pmatrix} B_1 & -D \\ -C & B_2 \end{pmatrix}, \tag{5}$$

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$$K = \begin{pmatrix} 0 & 0 \\ C(I - B_1) & CD \end{pmatrix} = \begin{pmatrix} 0 \\ C \end{pmatrix} \begin{pmatrix} I - B_1 & D \end{pmatrix}. \tag{6}$$

In [4-6], authors studied the convergence of GAOR method for solving the linear system $Hy = f$. In [4], authors studied the convergence of GAOR method when the coefficient matrices are consistently ordered matrices and gave the regions of convergence. In [5] and [6], authors studied the convergence of GAOR method for strictly diagonally dominant coefficient matrices and gave the regions of convergence. Most linear systems of interest have coefficient matrices that are not strictly diagonally dominant; in this paper we will discuss the convergence of GAOR method when the coefficient matrices are weighted strictly doubly diagonally dominant.

Throughout this paper, we denote the $i-$row sums of the modulus of the entries of J and K by J_i and K_i, the $i-$column sums of the modulus of the entries of J and K by J_i' and K_i', the spectral radius of iterative matrix $L_{\omega,r}$ by $\rho(L_{\omega,r})$, and $N = \{1, 2, \cdots$, $N_1 = \{i \mid K_i = 0, K_i' = 0\}$, $N_2 = N - N_1$, $R_i(A) = \sum_{k \neq i} |a_{ik}|$, $R_i'(A) = \sum_{i \neq k} |a_{ki}|$, $i, k \in N$.

Definition [7] We call matrix $A \in C^{n \times n}$ a strictly diagonally dominant matrix if

$$|a_{ii}| > R_i(A), \forall i \in N,$$

and denote $A \in SD$.

We call matrix $A \in C^{n \times n}$ a weighted strictly doubly diagonally dominant matrix if

$$|a_{ii}||a_{jj}| > \alpha R_i(A) R_j(A) + (1 - \alpha) R_i'(A) R_j'(A), \quad \forall i, j \in N, i \neq j,$$

and denote $A \in SDD(\alpha)$, where $\alpha \in [0,1]$.

Lemma [7] If $A \in SDD(\alpha)$, then A is nonsingular.

In this paper, we study the convergence of GAOR method for solving the linear system $Hy = f$ for weighted strictly doubly diagonally dominant coefficient matrices. Firstly, we obtain upper bound for the spectral radius of the matrix $L_{\omega,r}$ which is the iterative matrix of GAOR iterative method. Moreover, we present one convergence theorem of GAOR method. Finally, we present two numerical examples.

2 Upper bound of the spectral radius of $L_{\omega,r}$

In this section, we obtain upper bound of the spectral radius of the iterative matrix $L_{\omega,r}$.

Theorem 1. Let $H \in SDD(\alpha)$, then $\rho(L_{\omega,r})$ satisfies the following inequality:

$$\rho(L_{\omega,r}) \le \max_{\substack{i,j \\ i\neq j}} \Big(|1-\omega| +$$

$$\sqrt{\alpha(|\omega|J_i+|\omega r|K_i)(|\omega|J_j+|\omega r|K_j)+(1-\alpha)(|\omega|J_i'+|\omega r|K_i')(|\omega|J_j'+|\omega r|K_j')}\Big).$$

Proof. Let λ be an arbitrary eigenvalue of the iterative matrix $L_{\omega,r}$, then

$$\det(\lambda I - L_{\omega,r}) = 0, \tag{7}$$

i.e.

$$\det((\lambda+\omega-1)I-\omega J-\omega rK)=0.$$

If $(\lambda+\omega-1)I-\omega J-\omega rK \in SDD(\alpha)$, from Lemma we know that

$$(\lambda+\omega-1)I-\omega J-\omega rK$$

is nonsingular, that is

$$\det((\lambda+\omega-1)I-\omega J-\omega rK)\neq 0,$$

then λ is not an eigenvalue of the iterative matrix $L_{\omega,r}$, we have

$$|\lambda+\omega-1|^2 > \alpha(|\omega|J_i+|\omega r|K_i)(|\omega|J_j+|\omega r|K_j)$$
$$+(1-\alpha)(|\omega|J_i'+|\omega r|K_i')(|\omega|J_j'+|\omega r|K_j'),\ i\neq j.$$

While λ is an eigenvalue of the iterative matrix $L_{\omega,r}$, then there exists at least a couple of $i,j\in N(i\neq j)$, such that

$$|\lambda+\omega-1|^2 \le \alpha(|\omega|J_i+|\omega r|K_i)(|\omega|J_j+|\omega r|K_j)+(1-\alpha)(|\omega|J_i'+|\omega r|K_i')(|\omega|J_j'+|\omega r|K_j').$$

That is

$$|\lambda|^2-2|1-\omega||\lambda|+(1-\omega)^2-\alpha(|\omega|J_i+|\omega r|K_i)(|\omega|J_j+|\omega r|K_j)$$
$$-(1-\alpha)(|\omega|J_i'+|\omega r|K_i')(|\omega|J_j'+|\omega r|K_j')\le 0. \tag{8}$$

It is easy to find that the discriminant of a curve of second order

$$\Delta = 4\alpha\left(|\omega| J_i + |\omega r| K_i\right)\left(|\omega| J_j + |\omega r| K_j\right) + 4(1-\alpha)\left(|\omega| J_i' + |\omega r| K_i'\right)\left(|\omega| J_j' + |\omega r| K_j'\right) \geq 0,$$

then the solution of (8) satisfies

$$|\lambda| \leq |1-\omega| + \sqrt{\alpha\left(|\omega| J_i + |\omega r| K_i\right)\left(|\omega| J_j + |\omega r| K_j\right) + (1-\alpha)\left(|\omega| J_i' + |\omega r| K_i'\right)\left(|\omega| J_j' + |\omega r| K_j'\right)}, i \neq j.$$

So $\rho\left(L_{\omega,r}\right)$ satisfies the following inequality

$$\rho\left(L_{\omega,r}\right) \leq \max_{\substack{i,j \\ i\neq j}}\left(|1-\omega| + \sqrt{\alpha\left(|\omega| J_i + |\omega r| K_i\right)\left(|\omega| J_j + |\omega r| K_j\right) + (1-\alpha)\left(|\omega| J_i' + |\omega r| K_i'\right)\left(|\omega| J_j' + |\omega r| K_j'\right)}\right).$$

3 Convergence of GAOR method

In this section, we investigate the convergence of GAOR method to solve linear system (1). We assume that H is a weighted strictly doubly diagonally dominant coefficient matrix and obtain the regions of convergence of GAOR method.

Theorem 2. If $H \in SDD(\alpha)$, $\forall i,j \in N, i \neq j$, $\alpha J_i J_j + (1-\alpha) J_i' J_j' < 1$, then GAOR method convergences if ω, r satisfy either

(I) $0 < \omega \leq 1$,

$$|r| < \min\left\{\min_{\substack{i,j \\ i\neq j}} \frac{1-\alpha J_i J_j - (1-\alpha) J_i' J_j'}{\alpha\left(J_i K_j + J_j K_i\right) + (1-\alpha)\left(J_i' K_j' + J_j' K_i'\right)}, \min_{\substack{i,j \\ i\neq j}} \frac{-\left[\alpha\left(J_i K_j + J_j K_i\right) + (1-\alpha)\left(J_i' K_j' + J_j' K_i'\right)\right] + \sqrt{\Delta_1}}{2\left[\alpha K_i K_j + (1-\alpha) K_i' K_j'\right]}\right\}$$

or

$$\text{(II)}\ 1 < \omega < \min\left\{2, \min_{\substack{i,j \\ i\neq j}} \frac{2 - 2\sqrt{\alpha J_i J_j + (1-\alpha) J_i' J_j'}}{1 - \alpha J_i J_j - (1-\alpha) J_i' J_j'}\right\},$$

$$|r| < \min\left\{ \min_{\substack{i,j\\ i\neq j}} \frac{(2-\omega)^2 - \alpha\omega^2 J_i J_j - (1-\alpha)\omega^2 J_i' J_j'}{\omega^2\left[\alpha\left(J_i K_j + J_j K_i\right) + (1-\alpha)\left(J_i' K_j' + J_j' K_i'\right)\right]},\right.$$

$$\left. \min_{\substack{i,j\\ i\neq j}} \frac{-\omega^2\left[\alpha\left(J_i K_j + J_j K_i\right) + (1-\alpha)\left(J_i' K_j' + J_j' K_i'\right)\right] + \sqrt{\Delta_2}}{2\omega^2\left[\alpha K_i K_j + (1-\alpha) K_i' K_j'\right]} \right\},$$

where

$$\Delta_1 = \left[\alpha\left(J_i K_j + J_j K_i\right) + (1-\alpha)\left(J_i' K_j' + J_j' K_i'\right)\right]^2$$
$$+4\left[\alpha K_i K_j + (1-\alpha) K_i' K_j'\right]\left[1 - \alpha J_i J_j - (1-\alpha) J_i' J_j'\right],$$

$$\Delta_2 = \left\{\omega^2\left[\alpha\left(J_i K_j + J_j K_i\right) + (1-\alpha)\left(J_i' K_j' + J_j' K_i'\right)\right]\right\}^2$$
$$+4\omega^2\left[\alpha K_i K_j + (1-\alpha) K_i' K_j'\right]\left\{\left[1 - \alpha J_i J_j - (1-\alpha) J_i' J_j'\right]\omega^2 + 4(1-\omega)\right\}.$$

Proof. Because $H \in SDD(\alpha)$, then GAOR method convergences if

$$\max_{\substack{i,j\\ i\neq j}}\left(|1-\omega| + \right.$$
$$\left.\sqrt{\alpha\left(|\omega| J_i + |\omega r| K_i\right)\left(|\omega| J_j + |\omega r| K_j\right) + (1-\alpha)\left(|\omega| J_i' + |\omega r| K_i'\right)\left(|\omega| J_j' + |\omega r| K_j'\right)}\right) < 1$$

That is

$$|1-\omega| +$$
$$\max_{\substack{i,j\\ i\neq j}}\left(\sqrt{\alpha\left(|\omega| J_i + |\omega r| K_i\right)\left(|\omega| J_j + |\omega r| K_j\right) + (1-\alpha)\left(|\omega| J_i' + |\omega r| K_i'\right)\left(|\omega| J_j' + |\omega r| K_j'\right)}\right) < 1.$$

Firstly, when $0 < \omega \le 1$, we have

$$\omega^2 > \alpha\left(\omega^2 J_i J_j + \omega^2 |r| J_i K_j + \omega^2 |r| J_j K_i + \omega^2 |r|^2 K_i K_j\right)$$
$$+(1-\alpha)\left(\omega^2 J_i' J_j' + \omega^2 |r| J_i' K_j' + \omega^2 |r| J_j' K_i' + \omega^2 |r|^2 K_i' K_j'\right).$$

That is

$$|r|^2\left[\alpha K_i K_j + (1-\alpha) K_i' K_j'\right] + |r|\left[\alpha\left(J_i K_j + J_j K_i\right) + (1-\alpha)\left(J_i' K_j' + J_j' K_i'\right)\right]$$
$$+\alpha J_i J_j + (1-\alpha) J_i' J_j' - 1 < 0. \quad (9)$$

Then, we have the following conditions:

(1) when $i, j \in N_1$, then $K_i = K_j = 0, K_i' = K_j' = 0$. From

$$\alpha J_i J_j + (1-\alpha) J_i' J_j' < 1,$$

We have $|r| < +\infty$.

(2) when $i \in N_1, j \in N_2$ or $i \in N_2, j \in N_1$, then

$$|r|\left[\alpha\left(J_i K_j + J_j K_i\right) + (1-\alpha)\left(J_i' K_j' + J_j' K_i'\right)\right] + \alpha J_i J_j + (1-\alpha) J_i' J_j' - 1 < 0.$$

From $\alpha J_i J_j + (1-\alpha) J_i' J_j' < 1$, we have

$$|r| < \min_{\substack{i,j \\ i\neq j}} \frac{1 - \alpha J_i J_j - (1-\alpha) J_i' J_j'}{\alpha\left(J_i K_j + J_j K_i\right) + (1-\alpha)\left(J_i' K_j' + J_j' K_i'\right)}.$$

(3) when $i, j \in N_2$, it is easy to prove that the discriminant of a curve of second order $\Delta_1 > 0$, and then

$$|r| < \min_{\substack{i,j \\ i\neq j}} \frac{-\left[\alpha\left(J_i K_j + J_j K_i\right) + (1-\alpha)\left(J_i' K_j' + J_j' K_i'\right)\right] + \sqrt{\Delta_1}}{2\left[\alpha K_i K_j + (1-\alpha) K_i' K_j'\right]}.$$

So, when $0 < \omega \le 1$, we have

$$|r| < \min\left\{ \min_{\substack{i,j \\ i\neq j}} \frac{1 - \alpha J_i J_j - (1-\alpha) J_i' J_j'}{\alpha\left(J_i K_j + J_j K_i\right) + (1-\alpha)\left(J_i' K_j' + J_j' K_i'\right)}, \right.$$
$$\left. \min_{\substack{i,j \\ i\neq j}} \frac{-\left[\alpha\left(J_i K_j + J_j K_i\right) + (1-\alpha)\left(J_i' K_j' + J_j' K_i'\right)\right] + \sqrt{\Delta_1}}{2\left[\alpha K_i K_j + (1-\alpha) K_i' K_j'\right]} \right\}.$$

Secondly, when $1 < \omega < 2$, we have

$$4 - 4\omega + \omega^2 > \alpha\left(\omega^2 J_i J_j + \omega^2 |r| J_i K_j + \omega^2 |r| J_j K_i + \omega^2 |r|^2 K_i K_j\right)$$
$$+ (1-\alpha)\left(\omega^2 J_i' J_j' + \omega^2 |r| J_i' K_j' + \omega^2 |r| J_j' K_i' + \omega^2 |r|^2 K_i' K_j'\right).$$

That is

$$\omega^2\left[\alpha K_i K_j + (1-\alpha) K_i' K_j'\right]|r|^2 + \omega^2\left[\alpha\left(J_i K_j + J_j K_i\right) + (1-\alpha)\left(J_i' K_j' + J_j' K_i'\right)\right]|r|$$
$$+ \omega^2\left[\alpha J_i J_j + (1-\alpha) J_i' J_j'\right] - (2-\omega)^2 < 0. \qquad (10)$$

Then we have the following conditions:

(1) when $i, j \in N_1$, then $K_i = K_j = 0, K_i' = K_j' = 0$. From (10), we have

$$\omega^2\left[1 - \alpha J_i J_j - (1-\alpha) J_i' J_j'\right] - 4\omega + 4 > 0.$$

From $\alpha J_i J_j + (1-\alpha) J_i' J_j' < 1$, we have

$$1 < \omega < \min_{\substack{i,j \\ i \neq j}} \frac{2 - 2\sqrt{\alpha J_i J_j + (1-\alpha) J_i' J_j'}}{1 - \alpha J_i J_j - (1-\alpha) J_i' J_j'}, |r| < +\infty .$$

(2) when $i \in N_1, j \in N_2$ or $i \in N_2, j \in N_1$, then $K_i = K_i' = 0$ or $K_j = K_j' = 0$. From (10), we have

$$(2-\omega)^2 > \alpha\omega^2 J_i J_j + (1-\alpha) J_i' J_j'$$
$$+\omega^2 |r| \left[\alpha \left(J_i K_j + J_j K_i \right) + (1-\alpha) \left(J_i' K_j' + J_j' K_i' \right) \right],$$

so

$$|r| < \min_{\substack{i,j \\ i \neq j}} \frac{(2-\omega)^2 - \alpha\omega^2 J_i J_j - (1-\alpha)\omega^2 J_i' J_j'}{\omega^2 \left[\alpha \left(J_i K_j + J_j K_i \right) + (1-\alpha) \left(J_i' K_j' + J_j' K_i' \right) \right]} .$$

(3) when $i, j \in N_2$, then $K_i > 0, K_j > 0, K_i' > 0, K_j' > 0$. From

$$1 < \omega < \frac{2 - 2\sqrt{\alpha J_i J_j + (1-\alpha) J_i' J_j'}}{1 - \alpha J_i J_j - (1-\alpha) J_i' J_j'}, \ 1 - \alpha J_i J_j - (1-\alpha) J_i' J_j' > 0,$$

we know that

$$\left[1 - \alpha J_i J_j - (1-\alpha) J_i' J_j' \right] \omega^2 + 4(1-\omega) > 0 .$$

It is easy to find that the discriminant of a curve of second order of (10)

$$\Delta_2 = \left\{ \omega^2 \left[\alpha \left(J_i K_j + J_j K_i \right) + (1-\alpha) \left(J_i' K_j' + J_j' K_i' \right) \right] \right\}^2$$
$$+4\omega^2 \left[\alpha K_i K_j + (1-\alpha) K_i' K_j' \right] \left\{ \left[1 - \alpha J_i J_j - (1-\alpha) J_i' J_j' \right] \omega^2 + 4(1-\omega) \right\} > 0 .$$

From (10), we have

$$|r| < \min_{\substack{i,j \\ i \neq j}} \frac{-\omega^2 \left[\alpha \left(J_i K_j + J_j K_i \right) + (1-\alpha) \left(J_i' K_j' + J_j' K_i' \right) \right] + \sqrt{\Delta_2}}{2\omega^2 \left[\alpha K_i K_j + (1-\alpha) K_i' K_j' \right]} .$$

So, when $1 < \omega < \min_{\substack{i,j \\ i \neq j}} \left\{ 2, \frac{2 - 2\sqrt{\alpha J_i J_j + (1-\alpha) J_i' J_j'}}{1 - \alpha J_i J_j - (1-\alpha) J_i' J_j'} \right\}$, we have

$$|r| < \min\left\{ \min_{\substack{i,j \\ i\neq j}} \frac{(2-\omega)^2 - \alpha\omega^2 J_i J_j - (1-\alpha)\omega^2 J_i' J_j'}{\omega^2\left[\alpha\left(J_i K_j + J_j K_i\right) + (1-\alpha)\left(J_i' K_j' + J_j' K_i'\right)\right]}, \right.$$

$$\left. \min_{\substack{i,j \\ i\neq j}} \frac{-\omega^2\left[\alpha\left(J_i K_j + J_j K_i\right) + (1-\alpha)\left(J_i' K_j' + J_j' K_i'\right)\right] + \sqrt{\Delta_2}}{2\omega^2\left[\alpha K_i K_j + (1-\alpha) K_i' K_j'\right]} \right\}.$$

3 Examples

In this section, we give two numerical examples to show that our results are better than previous ones.

Example 1 Let

$$H = \begin{pmatrix} 1 & \frac{1}{3} & \frac{1}{6} & 0 \\ 0 & 1 & \frac{7}{6} & \frac{5}{6} \\ \frac{1}{3} & 0 & 1 & \frac{1}{6} \\ 0 & \frac{1}{6} & \frac{1}{3} & 1 \end{pmatrix} = \begin{pmatrix} I - B_1 & D \\ C & I - B_2 \end{pmatrix},$$

where

$$I - B_1 = \begin{pmatrix} 1 & \frac{1}{3} \\ 0 & 1 \end{pmatrix}, \quad I - B_2 = \begin{pmatrix} 1 & \frac{1}{6} \\ \frac{1}{3} & 1 \end{pmatrix}.$$

Obviously, $H \notin SD$, so we can't use the results of paper [2] and paper [6]. But $H \in SDD\left(\frac{1}{2}\right)$ and

$$J = \begin{pmatrix} 0 & -\frac{1}{3} & -\frac{1}{6} & 0 \\ 0 & 0 & -\frac{7}{6} & -\frac{5}{6} \\ -\frac{1}{3} & 0 & 0 & -\frac{1}{6} \\ 0 & -\frac{1}{6} & -\frac{1}{3} & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{9} & \frac{1}{18} & 0 \\ 0 & \frac{1}{6} & \frac{7}{36} & \frac{5}{36} \end{pmatrix},$$

$J_1 = J_3 = J_4 = \frac{1}{2}, J_2 = 2, \quad J_1' = \frac{1}{3}, J_2' = \frac{1}{2}, J_3' = \frac{5}{3}, J_4' = 1,$

$K_1 = K_2 = 0, \; K_3 = K_4 = \frac{1}{2}, \quad K_1' = \frac{1}{3}, K_2' = \frac{5}{18}, K_3' = \frac{1}{4}, K_4' = \frac{5}{36}.$

So

$N_1 = \varnothing$, $N_2 = \{1,2,3,4\}$, $\frac{1}{2}J_iJ_j + \frac{1}{2}J_i'J_j' < 1, i, j \in N, i \neq j$.

By Theorem 2, we obtain the following regions of convergence:

$$\text{(I)}\quad 0 < \omega \le 1,\ |r| < \frac{\sqrt{55765}}{15} - \frac{47}{3}, \text{ or}$$

$$\text{(II)}\quad 1 < \omega < 48 - 48\sqrt{\frac{23}{24}},\quad |r| < \sqrt{\frac{576}{5}\left(\frac{1}{\omega^2} - \frac{1}{\omega}\right) + \frac{11153}{45}} - \frac{47}{3}.$$

Example 2 (Example 1 of [6]) Let

$$H = \begin{pmatrix} 1 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 1 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 1 \end{pmatrix} = \begin{pmatrix} I - B_1 & D \\ C & I - B_2 \end{pmatrix},$$

where

$$I - B_1 = \begin{pmatrix} 1 & \frac{1}{3} \\ \frac{1}{3} & 1 \end{pmatrix},\quad I - B_2 = (1).$$

It is easy to know that $H \in SDD\left(\frac{1}{2}\right)$ and

$$J = \begin{pmatrix} 0 & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & 0 & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix},\quad K = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{4}{9} & \frac{4}{9} & \frac{2}{9} \end{pmatrix},$$

$J_1 = J_2 = J_3 = \frac{2}{3}$, $K_1 = K_2 = 0$, $K_3 = \frac{10}{9}$, $J_1' = J_2' = J_3' = \frac{2}{3}$,

$K_1' = K_2' = \frac{4}{9}$, $K_3' = \frac{2}{9}$.

So $N_1 = \varnothing, N_2 = \{1,2,3\}$, $\frac{1}{2}J_iJ_j + \frac{1}{2}J_i'J_j' < 1, i, j \in N, i \neq j$.

By Theorem 2, we obtain the following regions of convergence: .

$$\text{(I)}\quad 0 < \omega \le 1 \text{ and } |r| < \frac{3}{2}\left(\sqrt{21} - 4\right), \text{ or}$$

$$\text{(II)}\quad 1 < \omega < \frac{6}{5} \text{ and } |r| < 9\sqrt{\left(\frac{1}{\omega} - \frac{1}{2}\right)^2 + \frac{1}{3}} - 6.$$

In addition, $H \in SD$.

By Theorem 6 of paper [6], we obtain the following regions of convergence:

(I) $0 \le r \le 1$ and $0 < \omega \le 1$, or

(II) $0 \le r < 0.3$ and $0 < \omega < 1.2$, or

(III) $-0.3 < r < 0$ and $0 < \omega < \dfrac{18}{15-10r}$.

By Theorem 3 of paper [5], we obtain the following region of convergence:

$0 \le r \le \omega$ and $0 < \omega < \dfrac{18}{15+10r}$.

Comparing with the results of Example 1 of paper [6], we know that the regions of convergence got by Theorem 2 in this paper are larger than ones of paper [6] and paper [5].

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