On a Basic Analogue of Generalized $H$-function

B.K. Dutta$^{1,*}$, L.K. Arora$^{1,**}$

$^1$ Department of Mathematics, NERIST,
Arunachal Pradesh, India
dutta.bk11@gmail.com $^{*}$
llkarora_13@gmail.com $^{**}$

Abstract. In this paper, we investigate the basic analogue of a new
hypergeometric function, which is a generalization of the basic $I$-function. In
this regard, the application of Riemann-Liouville and Weyl fractional $q$-integral
operator with new hypergeometric function has been discussed. Similar result
obtained by other authors follows as special cases of our findings.

Keywords: Basic analogues of $H$ and $I$-function, Basic hypergeometric
function, Fractional $q$-integral operators.

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1 Introduction

In the past century, many authors have generalized $H$-function. In a recent paper,
Südland et al. [10] have introduced a generalization of Saxena's $I$-function [9], which
is also a generalization of Fox's $H$-function. This function is known as Aleph
function. In their paper, Saxena and Pogany [7] have studied fractional integration
formulae for the Aleph functions.

Südland et al. [11] studied the generalized fractional driftless Fokker-Planck
equation with power law coefficient. As a result a special function was found, which
is a particular case of the Aleph function. The Aleph was defined by means of Mellin-
Barnes type contour integrals as

$$
N[z] := N_{\mu,\nu}^{m,n} \left[ z^{(a_j, A_j)_{m,n}}, \left[ \tau_j^{(a_j, A_j)_{m,n}}, \left( b_j, B_j \right)_{m,n}, \left[ \tau_j^{(b_j, B_j)_{m,n}} \right]_{m+1,\nu} \right] \right] 
= \frac{1}{2\pi i} \int_L \Omega_{\mu,\nu}^{m,n} (\zeta) z^\zeta d\zeta
$$

(1.1)

for all $z \neq 0$, where  $\omega = \sqrt{-1}$ and
The parameters \( u_i, v_i \) are non-negative integers satisfying the inequality \( 0 \leq n \leq u_i, 1 \leq m \leq v_i \) and \( \tau_i > 0; i = 1, \cdots \). The parameters \( A_j, B_j, A_{ji}, B_{ji} \) are positive real numbers and \( a_j, b_j, a_{ji}, b_{ji} \) are complex numbers. The \( L = L_{apex} \) is a suitable contour of the Mellin-Barnes type in the complex \( \zeta \) -plane which runs from \( \gamma - \infty \) to \( \gamma + \infty \) with \( \gamma \in \mathbb{R} \), such that the poles of \( \Gamma(b_j - B_j \zeta) \), \( j = 1, \cdots \) separating from those of \( \Gamma(1 - a_j + A_j \zeta) \), \( j = 1, \cdots \). All the poles of the integrand (1.2) are assumed to be simple, and empty products are interpreted as unity. For the existence conditions, we refer to [8].

We define a basic analogue of this \( \mathbb{N} \)-function in term of Mellin-Barnes type contour integrals in the following manner

\[
\mathbb{N}[z; q] := \mathbb{N}_{u_i, v_i, \tau_i, \rho} \left[ z; q \left\{ (a_j, A_j)_{1,n}, [\tau_i(a_{ji}, A_{ji})]_{n+1,m+1,\rho} \right\} \right.
\]

\[ = \frac{1}{2\pi i} \int_C \Omega_{m,n}^{u_i, v_i, \tau_i, \rho} (\zeta; q) \pi \zeta d\zeta \quad (1.3)
\]

where \( \omega = \sqrt{-1} \),

\[
\Omega_{m,n}^{u_i, v_i, \tau_i, \rho} (\zeta; q) = \prod_{j=1}^{m} G(q^{b_j - B_j \zeta}) \prod_{j=1}^{n} G(q^{1 - a_j + A_j \zeta}) \]

\[ \sum_{i=1}^{\rho} \tau_i \prod_{j=n+1}^{m} G(q^{a_{ji} - A_{ji} \zeta}) \prod_{j=m+1}^{n} G(q^{1 - b_{ji} + B_{ji} \zeta}) \] (1.4)

\[ \times \frac{1}{G(q^{-\zeta}) \sin \pi \zeta}, \]

and

\[ G(q^\delta) = \left\{ \prod_{j=0}^{\infty} (1 - q^{\delta+j}) \right\}^{-1} = \frac{1}{(q^\delta; q)_\infty} \quad (1.5) \]
The parameters \( u_i, v_i \) are non-negative integers satisfying the inequality 
\( 0 \leq n \leq u_i, 1 \leq m \leq v_i \) and \( \tau_i > 0; i = 1, \cdots \); \( r \) is finite and \( A_j, B_j, A_{ji}, B_{ji} \) are positive real numbers and \( a_j, b_j, a_{ji}, b_{ji} \) are complex numbers. The \( C = C_{a_{ji}} \) is a suitable contour of Mellin-Barnes type in the complex \( \zeta \)-plane, which runs from \( \gamma - \omega \infty \) to \( \gamma + \omega \infty \) with \( \gamma \in \mathbb{R} \), such that the poles of \( G(q^{1/2} \zeta) \), \( j = 1, \cdots \) separating from those of \( G(q^{1/2} \zeta) \), \( j = 1, \cdots \). All the poles of the integrand \( (1.4) \) are assumed to be simple and empty products are interpreted as unity. The integral converges if \( \Im(\zeta) \log(z) - \log \sin \pi \zeta < 0 \), for large value of \( |\zeta| \) on the contour, that is if \( |\arg(z) - \sigma_2 \sigma_1^{-1} \log | z || < \pi \), where \( 0 < \sigma_1 < 1 \), \( \log q = -\sigma = - (\sigma_1 + \omega \sigma_2) \), \( \sigma \), \( \sigma_1 \), \( \sigma_2 \) are definite quantities, \( \sigma_1 \) and \( \sigma_2 \) being real.

When all \( \tau_i = 1; (1.3) \) yields the \( q \)-analogous of the \( I \)-function due to Saxena and Kumar [6].

Again, when \( r = 1, u_i = u, v_i = v \) and \( c_i = 1; (1.3) \) yields the \( q \)-analogous of the \( H \)-function due to Saxena et al. [7].

### 2 Preliminary Notes

In this section, we first recall some definitions and fundamental facts of basic analogue of special function and integral operator.

The fractional \( q \)-calculus is the \( q \)-extension of the ordinary calculus. Agarwal [1] introduced, the \( q \)-analogue of the Riemann-Liouville fractional integral operator in the following form:

\[
I^\alpha_q f(x) := \frac{1}{\Gamma_q(\alpha)} \int_0^x (x - tq)^{\alpha-1} f(t) dt; q, \quad (2.1)
\]

and Al-Salam [2] introduced, the \( q \)-analogue of the Weyl fractional integral operator in the following form:

\[
K^\mu_q f(x) := \frac{q^{-\mu(\mu-1)/2}}{\Gamma_q(\mu)} \int_x^\infty (t-x)^{\mu-1} f(tq^{-1}) dt; q, \quad (2.2)
\]

where \( \Re(\alpha) > 0, \Re(\mu) > 0, \) \( q < 1 \). So that

\[
I^0_q f(x) = f(x) = K^0_q f(x). \quad (2.3)
\]
The theory of $q$-calculus for a real parameter $q \in \mathbb{R}$, a $q$-real number $[a]_q$, was introduced (see [4]) as:

$$[a]_q := \frac{1-q^a}{1-q}, \quad a \in \mathbb{N}$$

(2.4)

The $q$-analog of the Pochhammer's symbol ($q$-shift factorial) is defined by:

$$(a; q)_0 = 1; \quad (a; q)_n = \prod_{j=0}^{n-1} (1-aq^j) = \frac{(a; q)_{\infty}}{(a^n; q)_n},$$

(2.5)

where $a \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}$.

The $q$-gamma function (cf. Gasper and Rahman [4]) is defined as follows:

$$\Gamma_q(b) = \frac{(q; q)_x}{(q^b; q)_x (1-q)^{-b-1}}, \quad (b \neq 0, -1, -2, \ldots)$$

(2.6)

and

$$(x-y)_v = x^v \prod_{n=0}^{\infty} \left[ \frac{1-(y/x)q^n}{1-(y/x)q^{v+n}} \right].$$

(2.7)

The $q$-binomial summation theorem is given by

$$\Phi_q[a; -; z] = \sum_{n=0}^{\infty} \frac{(a; q)_n z^n}{(q; q)_n} = \frac{(az; q)_\infty}{(z; q)_\infty}, \quad |z| < 1.$$  

(2.8)

The basic integration (cf. Gasper and Rahman [4]), is defined as:

$$\int_0^x f(t) dt(t; q) = x(1-q) \sum_{k=0}^{\infty} q^{-k} f(xq^{-k}),$$

(2.9)

$$\int_0^\infty f(t) dt(t; q) = x(1-q) \sum_{k=1}^{\infty} q^{-k} f(xq^{-k}).$$

(2.10)

For $\Re(\alpha) > 0$, using (2.6), (2.7), (2.9) in the $q$-analogue of the Riemann-Liouville fractional integral operator (2.1) can be expressed as:

$$\Gamma_q^\alpha f(x) = \frac{x^\alpha (1-q)}{\Gamma_q(\alpha)} \sum_{k=0}^{\infty} q^k (1-q^k) \alpha f(xq^k)$$

(2.11)

Similarly, for $\Re(\mu) > 0$ and using (2.6), (2.7), (2.10) in the $q$-analogue of the Weyl fractional integral operator (2.2) can be expressed as:
For the basic concept of $q$-calculus we refer the reader to [3].

3 Main Results

In this section, we will establish fractional $q$-integral formula for the basic analogous of Aleph function.

Theorem 3.1. Let $\Re(\alpha) > 0$, $|q|<1$ and $I_q^\alpha$ be the $q$-analogue of the Riemann-Liouville fractional integral operator (2.1), then following result holds:

$$I_q^\alpha \left\{ t^{\alpha-1} \sum_{n=0}^{\infty} \left[ \frac{\eta t^\lambda}{q} (a_j, A_j)_{n,m} \right] \right\} = \begin{cases} t^{\alpha+\rho-1} (1-q)^\alpha \sum_{n=0}^{\infty} \left[ \frac{\eta t^\lambda}{q} \right] \left( (a_j, A_j)_{n+1,m}, \right) \\ \left[ \tau_t (b_j, B_j)_{m+1,n} \right]^{(1-\rho-\alpha, \lambda)}, \end{cases} \quad \text{for } \lambda \geq 0 \quad (3.1)$$

$$t^{\alpha+\rho-1} (1-q)^\alpha \sum_{n=0}^{\infty} \left[ \frac{\eta t^\lambda}{q} \right] \left( (a_j, A_j)_{n+1,m}, \right) \\ \left[ \tau_t (b_j, B_j)_{m+1,n} \right]^{(\rho, -\lambda)}, \quad \text{for } \lambda < 0$$

where $\Re[\zeta-\log \eta-\log \sin \pi \zeta] < 0$ and $\rho > 0$.

Proof. For the sake of convenience, we denote the left side of (3.1) by $S$ and applying the equation (2.11) and (1.3), we obtain
S := t^{α}(1−q^{α})\sum_{k=0}^{∞}q^{k}\frac{(q^{α};q)_{k}}{(q;q)_{k}}(tq^{k})^{p−1}\frac{1}{2πiσ}∫_{C}Ω_{u,v,τ,ρ}(ζ;q)\pi(η^{2}−q^{2})^{ζ}dζ \tag{3.2}

Under the conditions stated above, we can interchange the order of summation and integration. Applying q-binomial theorem (2.8) expression (3.2) reduces to

\begin{align*}
S &= \frac{t^{α+ρ−1}(1−q^{α})}{2πiσ}∫_{C}Ω_{u,v,τ,ρ}(ζ;q)\pi(η^{2})^{ζ}1\Phi_{0}[q^{α};−;q^{α+ρ}]dζ \\
&= \frac{t^{α+ρ−1}(1−q^{α})}{2πiσ}∫_{C}Ω_{u,v,τ,ρ}(ζ;q)\pi(η^{2})^{ζ}\frac{G(q^{α+ρ})}{G(q^{α+ρ+2})}−dζ. \tag{3.3}
\end{align*}

Depending on the sign of \( λ \), (3.3) can be expressed as the right hand side of (3.1).

**Theorem 3.2.** Let \( \Re(μ) > 0 \), \( |q| < 1 \) and \( K_{q}^{α} \) be the \( q \)-analogue of the Weyl fractional integral operator (2.2), then following result holds:

\begin{align*}
K_{q}^{α} &\int_{m+1,n}^{m,n}\left[ \left\{ (a_{j},A_{j})_{1,α},[τ]_{j}(a_{ji},A_{ji})j_{m+1,n,ij,ρ}\right\} \right] \\
&= \frac{t^{μ−p−1}(1−q^{α})}{2πiσ}q^{−μ(ρ−1)−μ(μ+1)/2}\left[ (a_{j},J_{j})_{1,α},[τ]_{j}(a_{ji},J_{ji})j_{m+1,n,ij,ρ}\right],\text{for } λ \geq 0 \\
&\int_{m+1,n}^{m,n}\left[ \left\{ (b_{j},B_{j})_{1,α},[τ]_{j}(b_{ji},B_{ji})j_{m+1,n,ij,ρ}\right\} \right],\text{for } λ < 0 \tag{3.4}
\end{align*}

where \( \Re[ζ−log η−log sin πζ] < 0 \) and \( ρ > 0 \).

**Proof.** For the sake of convenience, we denote the left hand side of (3.4) by \( U \).

Applying the equation (2.12) and (1.3), we have

\begin{align*}
U := t^{μ}(1−q^{α})^{α−μ(μ+1)/2}\sum_{k=0}^{∞}q^{−k\frac{2k}{2}}\frac{(q^{2};q)_{k}}{(q;q)_{k}}(tq^{−k})^{p−1} \\
&\times \frac{1}{2πiσ}∫_{C}Ω_{u,v,τ,ρ}(ζ;q)\pi(η^{2}q^{−k})^{ζ}dζ \tag{3.5}
\end{align*}

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Under the conditions stated above, we can interchange the order of summation and integration. Applying \( q \)-binomial theorem (2.8) the expression (3.5) reduces to

\[
U = \frac{(1-q)^{\frac{\mu}{\rho-1}}}{2\pi i} \int_{\gamma} \Omega_{\mu,n,\tau,\rho}(\xi;q) \pi(\eta^{\dag}q^{-\lambda})^\xi \\
\times \Phi_{t}[q^{\mu};-;q^{1-\rho-\mu-\xi}]d\xi
\]

\[
= \frac{(1-q)^{\frac{\mu}{\rho-1}}}{2\pi i} \int_{\gamma} \Omega_{\mu,n,\tau,\rho}(\xi;q) \pi(\eta^{\dag}q^{-\lambda})^\xi \\
\times \frac{G(q^{1-\rho-\mu-\xi})}{G(q^{1-\rho-\mu})}d\xi,
\]

Depending on the sign of \( \lambda \), the expression (3.6) can be reduced to the right hand side of (3.4).

4 Special Cases

The \( q \)-extension of Aleph function defined by (1.3) in terms of Mellin-Barnes type of basic integrals is most general in nature, which includes a number of basic analogues of special functions. In this section we discuss only the case involving \( I_{q}(\cdot) \)-function and \( H_{q}(\cdot) \)-function.

4.1 Special Case of Theorem 3.1

If we set \( \tau_{i} = 1, \; i = 1, \cdots \) in Theorem 3.1 we obtain the following formula for \( I_{q}(\cdot) \)-function.

**Corollary 4.1** Let \( \Re(\alpha) > 0, |q| < 1 \), then following result holds:
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\[
I_{q}^{\alpha} \left\{ t^{\rho-1} I_{u,v}^{m,n} \left[ \eta^\lambda; q \left( (a_j, A_j)_{l,m}, (b_j, B_j)_{l,m} \right) \right] \right\} =
\left\{ \begin{array}{l}
\left( 1 - q \right)^{a} I_{u,v}^{m,n+1} \left[ \eta^\lambda; q \left( (a_j, A_j)_{l,m}, (b_j, B_j)_{l,m} \right), (\rho - \lambda), (\rho - \lambda), (\rho - \lambda) \right]
\end{array} \right., \text{for } \lambda \geq 0, (4.1)
\]

where \( \Re[\zeta \log \eta - \log \sin \pi \zeta] < 0 \) and \( \rho > 0 \).

If we set \( r = 1, \tau = 1, u_1 = u, v_1 = v, a_j = a_j, b_j = b_j, A_j = A_j, B_j = B_j \) in Theorem 3.1 we obtain the following formula for \( H_q(\cdot) \)-function.

**Corollary 4.2** Let \( \Re(\alpha) > 0, |q| < 1 \), then following result holds:

\[
I_{q}^{\alpha} \left\{ t^{\rho-1} H_{u,v}^{m,n} \left[ \eta^\lambda; q \left( (a_j, A_j)_{l,u}, (b_j, B_j)_{l,v} \right) \right] \right\} =
\left\{ \begin{array}{l}
\left( 1 - q \right)^{a} H_{u,v}^{m,n+1} \left[ \eta^\lambda; q \left( (a_j, A_j)_{l,u}, (b_j, B_j)_{l,v} \right), (1 - \rho - \alpha, \lambda), (1 - \rho - \alpha, \lambda), (1 - \rho - \alpha, \lambda) \right]
\end{array} \right., \text{for } \lambda \geq 0, (4.2)
\]

where \( \Re[\zeta \log \eta - \log \sin \pi \zeta] < 0 \) and \( \rho > 0 \).

When \( \rho = 1 \) the Corollary 4.2 reduces to the main result due to Kalla et al. [5].

**4.2 Special Case of Theorem 3.2**
If we set \( \tau_i = 1, i = 1, \ldots \) in Theorem 3.2 we obtain the following formula for \( I_{q} (\cdot) \)-function.

**Corollary 4.3** Let \( \Re(\mu) > 0 \), \( |q| < 1 \), then following result holds:

\[
K_q^{\mu} \left[ t^{\rho-1} I_{a_1}^{m_{u,v,r}} \left[ \eta^\lambda; q \left( (a_j, A_j)_{1,u}, (a_{j_1}, A_{j_1})_{n+1,u,r} \right) \right] \right] \\
= t^{\rho+\mu-1} (1-q)^{\alpha} q^{-\mu(\rho-1) - \mu(\mu+1)/2} \times H_{m_{u,v,r}}^{1+1} \left[ \eta^\lambda; q \left( (a_j, A_j)_{1,u}, (1-\rho, \lambda), (b_j, B_j)_{1,v} \right) \right], \text{ for } \lambda \geq 0, \quad (4.3)
\]

where \( \Re[\zeta \log \eta - \log \sin n\pi \zeta] < 0 \) and \( \rho > 0 \).

If we set \( r = 1, \tau_i = u_i, v_i = v, a_{j_1} = a_j, b_{j_1} = b_j, A_{j_1} = A_j, B_{j_1} = B_j \) in Theorem 3.2 we obtain the following formula for \( H_{q} (\cdot) \)-function.

**Corollary 4.4** Let \( \Re(\mu) > 0 \), \( |q| < 1 \), then following result holds:

\[
K_q^{\mu} \left[ t^{\rho-1} H_{a_1}^{m_{u,v,r}} \left[ \eta^\lambda; q \left( (a_j, A_j)_{1,u} \right) \right] \right] \\
= t^{\rho+\mu-1} (1-q)^{\alpha} q^{-\mu(\rho-1) - \mu(\mu+1)/2} \times H_{m_{u,v,r}}^{1+1} \left[ \eta^\lambda; q \left( (a_j, A_j)_{1,u}, (1-\rho, \lambda), (b_j, B_j)_{1,v} \right) \right], \text{ for } \lambda \geq 0, \quad (4.4)
\]

where \( \Re[\zeta \log \eta - \log \sin n\pi \zeta] < 0 \) and \( \rho > 0 \).
When $\rho = 1$, $\lambda = 1$ the Corollary 4.4 reduces to the result due to Yadav and Purohit [12].

5 Conclusion

Since most of the special function can be expressed in term of the $q$-extension of Aleph function defined by (1.3). It is useful to make tables for the $q$-extension of Riemann-Liouville and Weyl fractional integral operators. In this regard we refer to Kalla et al. in [5] [eq. no. 3.1-3.11, table 1, p. 320], Yadav and Purohit in [12] [eq. no. 1-40, table 1, p. 241].

References