Focal Curves of Biharmonic Curves in the $\mathbb{SL}_2(\mathbb{R})$

Talat Körpinar$^1$ and Essin Turhan$^1$ and J. López-Bonilla$^2$

$^1$Fırat University, Department of Mathematics, 23119, Elazığ, TURKEY
$^2$ESIME-Zacatenco, Instituto Politécnico Nacional, Col. Lindavista, CP 07738, México D.F.
talatkorpinar@gmail.com, essin.turhan@gmail.com, jlopezb@ipn.mx

Abstract. In this paper, we study focal curve of biharmonic curves in the $\mathbb{SL}_2(\mathbb{R})$. Finally, we find out their explicit parametric equations.

Keywords: Biharmonic curve, $\mathbb{SL}_2(\mathbb{R})$, focal curve.

1 Introduction

The theory of biharmonic functions is an old and rich subject. Biharmonic functions have been studied since 1862 by Maxwell and Airy to describe a mathematical model of elasticity. The theory of polyharmonic functions was developed later on, for example, by E. Almansi, T. Levi-Civita and M. Nicolescu.

As suggested by Eells and Sampson in [6], we can define the bienergy of a map $f$ by

$$ E_2(f) = \frac{1}{2} \int_M \tau(f)^2 \, v_g, $$

where $\tau(f) = \text{trace } \nabla df$ is tension field and say that is biharmonic if it is a critical point of the bienergy.

Jiang derived the first and the second variation formula for the bienergy in [8], showing that the Euler--Lagrange equation associated to $E_2$ is

$$ \tau_2(f) = -J' \left( \tau(f) \right) = -\Delta \tau(f) - \text{trace} R^N (df, \tau(f)) df = 0, \tag{1} $$

where $J'$ is the Jacobi operator of $f$. The equation $\tau_2(f) = 0$ is called the biharmonic equation. Since $J'$ is linear, any harmonic map is biharmonic.

This study is organised as follows: Firstly, we obtain focal curve of biharmonic curves in the $\mathbb{SL}_2(\mathbb{R})$. Finally, we find out their explicit parametric equations.
2 Preliminaries

We identify $\text{SL}_2(\mathbb{R})$ with

$$\mathbb{R}_+^3 = \{ (x, y, z) \in \mathbb{R}^3 : z > 0 \}$$

directed with

$$g = ds^2 = (dx + \frac{dy}{z})^2 + \frac{dy^2 + dz^2}{z^2}.$$

The following set of left-invariant vector fields forms an orthonormal basis for $\text{SL}_2(\mathbb{R})$

$$e_1 = \frac{\partial}{\partial x}, e_2 = z \frac{\partial}{\partial y} - \frac{\partial}{\partial x}, e_3 = z \frac{\partial}{\partial z}. \quad (2)$$

The characterising properties of $g$ defined by

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1, \quad g(e_1, e_2) = g(e_2, e_3) = g(e_1, e_3) = 0.$$

The Riemannian connection $\nabla$ of the metric $g$ is given by

\[
2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]),
\]

which is known as Koszul's formula.

Using the Koszul's formula, we obtain

$$\nabla_{e_1} e_1 = 0, \quad \nabla_{e_1} e_2 = \frac{1}{2} e_3, \quad \nabla_{e_1} e_3 = -\frac{1}{2} e_2,$$

$$\nabla_{e_2} e_1 = \frac{1}{2} e_3, \quad \nabla_{e_2} e_2 = e_3, \quad \nabla_{e_2} e_3 = -\frac{1}{2} e_1 - e_2,$$

$$\nabla_{e_3} e_1 = -\frac{1}{2} e_2, \quad \nabla_{e_3} e_2 = \frac{1}{2} e_1, \quad \nabla_{e_3} e_3 = 0. \quad (3)$$

Moreover we put

$$R_{ijk} = R(e_i, e_j)e_k, \quad R_{ijkl} = R(e_i, e_j, e_k, e_l).$$
where the indices $i, j, k$ and $l$ take the values 1, 2 and 3

$$R_{1212} = R_{1313} = \frac{1}{4}, R_{2323} = -\frac{7}{4}. \quad (4)$$

### 3 Biharmonic Curves in $\mathbb{SL}_2(R)$

Biharmonic equation for the curve $\gamma$ reduces to

$$\nabla^2_T T - R(T, \nabla_T T)T = 0, \quad (5)$$

that is, $\gamma$ is called a biharmonic curve if it is a solution of the equation $(5)$.

Let us consider biharmonicity of curves in $\mathbb{SL}_2(R)$. Let $\{T, N, B\}$ be the Frenet frame field along $\gamma$. Then, the Frenet frame satisfies the following Frenet--Serret equations:

$$\nabla_T T = \kappa N, \quad \nabla_T N = -\kappa T + \tau B, \quad \nabla_T B = -\tau N, \quad (6)$$

where $\kappa$ is the curvature of $\gamma$ and $\tau$ its torsion and

$$g(T, T) = 1, g(N, N) = 1, g(B, B) = 1, \quad g(T, N) = g(T, B) = g(N, B) = 0.$$

With respect to the orthonormal basis $\{e_1, e_2, e_3\}$, we can write

$$T = T_1 e_1 + T_2 e_2 + T_3 e_3, \quad N = N_1 e_1 + N_2 e_2 + N_3 e_3, \quad B = T \times N = B_1 e_1 + B_2 e_2 + B_3 e_3. \quad (7)$$

**Theorem 3.1.** $\gamma : I \to \mathbb{SL}_2(R)$ is a biharmonic curve if and only if

$$\kappa = \text{constant} \neq 0, \quad \kappa^2 + \tau^2 = -\frac{1}{4} + \frac{15}{4} B_1^2, \quad \tau' = 2 N_1 B_1. \quad (8)$$

**Proof.** Using $(5)$ and Frenet formulas $(6)$, we have $(8)$.  

**Theorem 3.2.** $(9)$ Let $\gamma : I \to \mathbb{SL}_2(R)$ be a unit speed non-geodesic biharmonic curve. Then, the parametric equations of $\gamma$ are
\[
x(s) = \frac{1}{\kappa} \sin \varphi \sin [\kappa s + C] + \frac{1}{\kappa} \sin \varphi \cos [\kappa s + C] + \varphi_2,
\]
\[
y(s) = \frac{1}{\kappa^2 + \cos^2 \varphi} \sin \varphi \varphi_1 e^{\cos \varphi} (-\kappa \cos [\kappa s + C] + \cos \varphi \sin [\kappa s + C]),
\]
\[
z(s) = \varphi_1 e^{\cos \varphi},
\]
where \( \kappa, \ C, \ \varphi_1, \ \varphi_2 \) are constants of integration.

4 Focal Curve of Biharmonic Curves in \( \text{SL}_2(\mathbb{R}) \)

Denoting the focal curve by \( \varphi_\gamma \), we can write
\[
\varphi_\gamma(s) = (\gamma + c_1 N + c_2 B)(s),
\]  
where the coefficients \( c_1, \ c_2 \) are smooth functions of the parameter of the curve \( \gamma \), called the first and second focal curvatures of \( \gamma \), respectively. Further, the focal curvatures \( c_1, \ c_2 \) are defined by
\[
c_1 = \frac{1}{\kappa}, \ c_2 = \frac{c_1}{\tau}, \_kappa \neq 0, \tau \neq 0.
\]

**Lemma 4.1.** Let \( \gamma: I \rightarrow \text{SL}_2(\mathbb{R}) \) be a unit speed biharmonic curve and \( \varphi_\gamma \) its focal curve on \( \text{SL}_2(\mathbb{R}) \). Then,
\[
c_1 = \frac{1}{\kappa} = \text{constant and } c_2 = 0.
\]

**Proof.** Using (7) and (11), we get (12).

**Lemma 4.2.** Let \( \gamma: I \rightarrow \text{SL}_2(\mathbb{R}) \) be a unit speed biharmonic curve and \( \varphi_\gamma \) its focal curve on \( \text{SL}_2(\mathbb{R}) \). Then,
\[
\varphi_\gamma(s) = (\gamma + c_1 N)(s).
\]
Lemma 4.3. Let \( \gamma : I \to \text{SL}_2(\mathbb{R}) \) be a unit speed non-geodesic biharmonic curve. Then, the position vector of \( \gamma \) is

\[
\gamma(s) = \frac{1}{\kappa} \sin \varphi \sin [N s + C] + \frac{1}{\kappa} \sin \varphi \cos [N s + C] + \varphi_2
\]

\[
+ \left[ \frac{1}{(N^2 + \cos^2 \varphi)} \sin \varphi (-N \cos [N s + C] + \cos \varphi \sin [N s + C]) \right] e_1
\]

\[
- \frac{c_1}{\kappa} \sin \sin [N s + C] e_1
\]

and \( \varphi_1 \) is \( \gamma \)’s focal curve on \( \text{SL}_2(\mathbb{R}) \).

Proof. Assume that \( \gamma \) is a non-geodesic biharmonic curve \( \text{SL}_2(\mathbb{R}) \). Using (2), yields

\[
\frac{\partial}{\partial x} = e_1, \quad \frac{\partial}{\partial y} = \frac{1}{z} (e_2 + e_1), \quad \frac{\partial}{\partial z} = \frac{1}{z} e_3.
\]

Substituting (15) to (9), we have (14) as desired.

Theorem 4.4. Let \( \gamma : I \to \text{SL}_2(\mathbb{R}) \) be a unit speed non-geodesic biharmonic curve and \( \varphi_1 \) its focal curve on \( \text{SL}_2(\mathbb{R}) \). Then,

\[
\varphi_1(s) = \frac{1}{\kappa} \sin \varphi \sin [N s + C] + \frac{1}{\kappa} \sin \varphi \cos [N s + C] + \varphi_2
\]

\[
+ \left[ \frac{1}{(N^2 + \cos^2 \varphi)} \sin \varphi (-N \cos [N s + C] + \cos \varphi \sin [N s + C]) \right] e_1
\]

\[
- \frac{c_1}{\kappa} \sin \sin [N s + C] e_1
\]

\[
+ \left[ \frac{1}{(N^2 + \cos^2 \varphi)} \sin \varphi (-N \cos [N s + C] + \cos \varphi \sin [N s + C]) \right] e_2
\]

\[
+ \frac{c_1}{\kappa} (N \sin \varphi \cos [N s + C] - \sin^2 \varphi \cos [N s + C] \sin [N s + C] - \cos \varphi \sin \varphi \cos [N s + C]) e_3
\]
\[ + \left(1 + \frac{C_1}{\kappa} \left(\sin^2 \varphi \cos[Ns + C] \sin[Ns + C] + \sin^2 \varphi \sin^2[Ns + C]\right)\right)e_3, \]

where \( N, C, \varphi_1, \varphi_2 \) are constants of integration.

**Proof.** We assume that \( \gamma: I \rightarrow SL_2(\mathbb{R}) \) be a unit speed biharmonic curve. Using Lemma 4.1, we get

\[ T = \sin \varphi \cos[Ns + C]e_1 + \sin \varphi \sin[Ns + C]e_2 + \cos \varphi e_3. \]

Using first equation of the system (6) and (4), we have

\[ \nabla_1 T = (T'_1) e_1 + (T'_2 - T'_1 T_2 - T_1 T'_3) e_2 + (T'_3 + T_1 T'_2 + T_2^2) e_3. \]

By the use of Frenet formulas and above equation, we get

\[
N = -\frac{N}{\kappa} \sin \sin[Ns + C] e_1 \\
+ \frac{1}{\kappa} (N \sin \varphi \cos[Ns + C] - \sin^2 \varphi \cos[Ns + C] \sin[Ns + C] - \cos \varphi \sin \varphi \cos[Ns + C] e_2 \\
+ \frac{1}{\kappa} \left(\sin^2 \varphi \cos[Ns + C] \sin[Ns + C] + \sin^2 \varphi \sin^2[Ns + C]\right) e_3.
\]

Combining (17) and (11), we obtain (16). This concludes the proof of Theorem. We can use Mathematica in above Theorems 3.3 - 4.2, yields
Fig. 1. Mathematical’s result in Theorems 3.3 – 4.2.

References


12. Sato, I.: On a structure similar to the almost contact structure, Tensor, (N.S.), 30, 219--224 (1976)
