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Focal Curves of Bıharmonıc Curves in the $SL_2(\mathbb{R})$

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Abstract. In this paper, we study focal curve of biharmonic curves in the $SL_2(\mathbb{R})$. Finally, we find out their explicit parametric equations.

Keywords: Biharmonic curve, $SL_2(\mathbb{R})$, focal curve.

1 Introduction

The theory of biharmonic functions is an old and rich subject. Biharmonic functions have been studied since 1862 by Maxwell and Airy to describe a mathematical model of elasticity. The theory of polyharmonic functions was developed later on, for example, by E. Almansi, T. Levi-Civita and M. Nicolescu.

As suggested by Eells and Sampson in [6], we can define the bienergy of a map f by

$$E_2(f)=\frac{1}{2}\int_M |\tau(f)|^2 v_g,$$

where $\tau(f)=\operatorname{trace}\nabla df$ is tension field and say that is biharmonic if it is a critical point of the bienergy.

Jiang derived the first and the second variation formula for the bienergy in [8], showing that the Euler--Lagrange equation associated to E_2 is

$$\tau_2(f)=-\mathrm{J}^f(\tau(f))=-\Delta\tau(f)-\operatorname{trace}R^N(df,\tau(f))df=0, \tag{1}$$

where J^f is the Jacobi operator of f. The equation $\tau_2(f)=0$ is called the biharmonic equation. Since J^f is linear, any harmonic map is biharmonic.

This study is organised as follows: Firstly, we obtain focal curve of biharmonic curves in the $SL_2(\mathbb{R})$. Finally, we find out their explicit parametric equations.

2 Preliminaries

We identify $SL_2(\mathbb{R})$ with

$$\mathbb{R}^3_+ = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}$$

endowed with the metric

$$g = ds^2 = (dx + \frac{dy}{z})^2 + \frac{dy^2 + dz^2}{z^2}.$$

The following set of left-invariant vector fields forms an orthonormal basis for $SL_2(\mathbb{R})$

$$\mathbf{e}_1 = \frac{\partial}{\partial x}, \mathbf{e}_2 = z\frac{\partial}{\partial y} - \frac{\partial}{\partial x}, \mathbf{e}_3 = z\frac{\partial}{\partial z}. \tag{2}$$

The characterising properties of g defined by

$$g(\mathbf{e}_1, \mathbf{e}_1) = g(\mathbf{e}_2, \mathbf{e}_2) = g(\mathbf{e}_3, \mathbf{e}_3) = 1, \quad g(\mathbf{e}_1, \mathbf{e}_2) = g(\mathbf{e}_2, \mathbf{e}_3) = g(\mathbf{e}_1, \mathbf{e}_3) = 0.$$

The Riemannian connection ∇ of the metric g is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &- g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]), \end{aligned}$$

which is known as Koszul's formula.
Using the Koszul's formula, we obtain

$$\begin{aligned} &\nabla_{\mathbf{e}_1}\mathbf{e}_1 = 0, \quad \nabla_{\mathbf{e}_1}\mathbf{e}_2 = \frac{1}{2}\mathbf{e}_3, \quad \nabla_{\mathbf{e}_1}\mathbf{e}_3 = -\frac{1}{2}\mathbf{e}_2, \\ &\nabla_{\mathbf{e}_2}\mathbf{e}_1 = \frac{1}{2}\mathbf{e}_3, \quad \nabla_{\mathbf{e}_2}\mathbf{e}_2 = \mathbf{e}_3, \nabla_{\mathbf{e}_2}\mathbf{e}_3 = -\frac{1}{2}\mathbf{e}_1 - \mathbf{e}_2, \\ &\nabla_{\mathbf{e}_3}\mathbf{e}_1 = -\frac{1}{2}\mathbf{e}_2, \quad \nabla_{\mathbf{e}_3}\mathbf{e}_2 = \frac{1}{2}\mathbf{e}_1, \quad \nabla_{\mathbf{e}_3}\mathbf{e}_3 = 0. \end{aligned} \tag{3}$$

Moreover we put

$$R_{ijk} = R(\mathbf{e}_i, \mathbf{e}_j)\mathbf{e}_k, \quad R_{ijkl} = R(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k, \mathbf{e}_l),$$

where the indices i, j, k and l take the values $1,2$ and 3

$$R_{1212} = R_{1313} = \frac{1}{4},\ R_{2323} = -\frac{7}{4}. \tag{4}$$

3 Biharmonic Curves in $\widetilde{SL}_2(\mathbb{R})$

Biharmonic equation for the curve γ reduces to

$$\nabla^3_{\mathbf{T}}\mathbf{T} - R(\mathbf{T}, \nabla_{\mathbf{T}}\mathbf{T})\mathbf{T} = 0, \tag{5}$$

that is, γ is called a biharmonic curve if it is a solution of the equation (5).

Let us consider biharmonicity of curves in $\widetilde{SL}_2(\mathbb{R})$. Let $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame field along γ. Then, the Frenet frame satisfies the following Frenet--Serret equations:

$$\nabla_{\mathbf{T}}\mathbf{T} = \kappa\mathbf{N}, \qquad \nabla_{\mathbf{T}}\mathbf{N} = -\kappa\mathbf{T} + \tau\mathbf{B}, \qquad \nabla_{\mathbf{T}}\mathbf{B} = -\tau\mathbf{N}, \tag{6}$$

where κ is the curvature of γ and τ its torsion and

$$g(\mathbf{T},\mathbf{T}) = 1, g(\mathbf{N},\mathbf{N}) = 1, g(\mathbf{B},\mathbf{B}) = 1, \qquad g(\mathbf{T},\mathbf{N}) = g(\mathbf{T},\mathbf{B}) = g(\mathbf{N},\mathbf{B}) = 0.$$

With respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, we can write

$$\begin{gathered}\mathbf{T} = T_1\mathbf{e}_1 + T_2\mathbf{e}_2 + T_3\mathbf{e}_3, \quad \mathbf{N} = N_1\mathbf{e}_1 + N_2\mathbf{e}_2 + N_3\mathbf{e}_3, \\ \mathbf{B} = \mathbf{T} \times \mathbf{N} = B_1\mathbf{e}_1 + B_2\mathbf{e}_2 + B_3\mathbf{e}_3.\end{gathered} \tag{7}$$

Theorem 3.1. $\gamma : I \to \widetilde{SL}_2(\mathbb{R})$ *is a biharmonic curve if and only if*

$$\kappa = \text{constant} \neq 0, \quad \kappa^2 + \tau^2 = -\frac{1}{4} + \frac{15}{4}B_1^2, \quad \tau' = 2N_1B_1. \tag{8}$$

Proof. Using (5) and Frenet formulas (6), we have (8).

Theorem 3.2. ([9]) *Let* $\gamma : I \to \widetilde{SL}_2(\mathbb{R})$ *be a unit speed non-geodesic biharmonic curve. Then, the parametric equations of* γ *are*

$$x(s)=\frac{1}{\aleph}\sin\varphi\sin[\aleph s+C]+\frac{1}{\aleph}\sin\varphi\cos[\aleph s+C]+\wp_2, \tag{9}$$

$$y(s)=\frac{1}{\aleph^2+\cos^2\varphi}\sin\varphi\,\wp_1 e^{\cos\varphi s}(-\aleph\cos[\aleph s+C]+\cos\varphi\sin[\aleph s+C]),$$

$$z(s)=\wp_1 e^{\cos\varphi s},$$

where $\aleph$, C, $\wp_1$, $\wp_2$ *are constants of integration*.

4 Focal Curve of Biharmonic Curves in $\widetilde{SL}_2(\mathbb{R})$

Denoting the focal curve by $\wp_\gamma$, we can write

$$\wp_\gamma(s)=(\gamma+c_1\mathbf{N}+c_2\mathbf{B})(s), \tag{10}$$

where the coefficients c_1, c_2 are smooth functions of the parameter of the curve γ, called the first and second focal curvatures of γ, respectively. Further, the focal curvatures c_1, c_2 are defined by

$$c_1=\frac{1}{\kappa}, c_2=\frac{c_1'}{\tau}, \kappa\neq 0, \tau\neq 0. \tag{11}$$

Lemma 4.1. *Let* $\gamma: I\to\widetilde{SL}_2(\mathbb{R})$ *be a unit speed biharmonic curve and* $\wp_\gamma$ *its focal curve on* $\widetilde{SL}_2(\mathbb{R})$. *Then,*

$$c_1=\frac{1}{\kappa}=\text{constant and } c_2=0. \tag{12}$$

Proof. Using (7) and (11), we get (12).

Lemma 4.2. *Let* $\gamma: I\to\widetilde{SL}_2(\mathbb{R})$ *be a unit speed biharmonic curve and* $\wp_\gamma$ *its focal curve on* $\widetilde{SL}_2(\mathbb{R})$. *Then,*

$$\wp_\gamma(s)=(\gamma+c_1\mathbf{N})(s). \tag{13}$$

Lemma 4.3. *Let* $\gamma : I \to \widetilde{\mathrm{SL}}_2(\mathrm{R})$ *be a unit speed non-geodesic biharmonic curve. Then, the position vector of* γ *is*

$$\begin{aligned}\gamma(s) = [&\frac{1}{\aleph}\sin\varphi\sin[\aleph s + C] + \frac{1}{\aleph}\sin\varphi\cos[\aleph s + C] + \wp_2 \\ &+ [\frac{1}{(\aleph^2 + \cos^2\varphi)}\sin\varphi(-\aleph\cos[\aleph s + C] + \cos\varphi\sin[\aleph s + C])]]\mathbf{e}_1 \\ &[\frac{1}{(\aleph^2 + \cos^2\varphi)}\sin\varphi(-\aleph\cos[\aleph s + C] + \cos\varphi\sin[\aleph s + C])]\mathbf{e}_2 + \mathbf{e}_3,\end{aligned} \tag{14}$$

where $\aleph, C, \wp_2$ *are constants of integration.*

Proof. Assume that γ is a non-geodesic biharmonic curve $\widetilde{\mathrm{SL}}_2(\mathrm{R})$.
Using (2), yields

$$\frac{\partial}{\partial x} = \mathbf{e}_1, \frac{\partial}{\partial y} = \frac{1}{z}(\mathbf{e}_2 + \mathbf{e}_1), \frac{\partial}{\partial z} = \frac{1}{z}\mathbf{e}_3. \tag{15}$$

Substituting (15) to (9), we have (14) as desired.

Theorem 4.4. *Let* $\gamma : I \to \widetilde{\mathrm{SL}}_2(\mathrm{R})$ *be a unit speed non-geodesic biharmonic curve and* $\wp_\gamma$ *its focal curve on* $\widetilde{\mathrm{SL}}_2(\mathrm{R})$*. Then,*

$$\begin{aligned}\wp_\gamma(s) = [&\frac{1}{\aleph}\sin\varphi\sin[\aleph s + C] + \frac{1}{\aleph}\sin\varphi\cos[\aleph s + C] + \wp_2 \\ &+ [\frac{1}{(\aleph^2 + \cos^2\varphi)}\sin\varphi(-\aleph\cos[\aleph s + C] + \cos\varphi\sin[\aleph s + C])] \\ &- \frac{c_1\aleph}{\kappa}\sin \quad \sin[\aleph s + C]]\mathbf{e}_1 \\ &+ [\frac{1}{(\aleph^2 + \cos^2\varphi)}\sin\varphi(-\aleph\cos[\aleph s + C] + \cos\varphi\sin[\aleph s + C]) \\ &+ \frac{c_1}{\kappa}(\aleph\sin\varphi\cos[\aleph s + C] - \sin^2\varphi\cos[\aleph s + C]\sin[\aleph s + C] \\ &- \cos\varphi\sin\varphi\cos[\aleph s + C])]\mathbf{e}_2\end{aligned} \tag{16}$$

$$+(1+\frac{c_1}{\kappa}(\sin^2\varphi\cos[\aleph s+C]\sin[\aleph s+C]+\sin^2\varphi\sin^2[\aleph s+C]))\mathbf{e}_3,$$

where $\aleph,\ C,\ \wp_1,\ \wp_2$ *are constants of integration*.

Proof. We assume that $\gamma: I\to \mathrm{SL}_2(\mathrm{R})$ be a unit speed biharmonic curve.
Using Lemma 4.1, we get

$$\mathbf{T}=\sin\varphi\cos[\aleph s+C]\mathbf{e}_1+\sin\varphi\sin[\aleph s+C]\mathbf{e}_2+\cos\varphi\mathbf{e}_3.$$

Using first equation of the system (6) and (4), we have

$$\nabla_{\mathbf{T}}\mathbf{T}=(T_1^{'})\mathbf{e}_1+(T_2^{'}-T_1T_2-T_1T_3)\mathbf{e}_2+(T_3^{'}+T_1T_2+T_2^2)\mathbf{e}_3.$$

By the use of Frenet formulas and above equation, we get

$$\begin{aligned}\mathbf{N}=&-\frac{\aleph}{\kappa}\sin\quad\sin[\aleph s+C]\mathbf{e}_1\\&+\frac{1}{\kappa}(\aleph\sin\varphi\cos[\aleph s+C]-\sin^2\varphi\cos[\aleph s+C]\sin[\aleph s+C]\\&-\cos\varphi\sin\varphi\cos[\aleph s+C])\mathbf{e}_2\\&+\frac{1}{\kappa}(\sin^2\varphi\cos[\aleph s+C]\sin[\aleph s+C]+\sin^2\varphi\sin^2[\aleph s+C])\mathbf{e}_3.\end{aligned}\qquad(17)$$

Combining (17) and (11), we obtain (16). This concludes the proof of Theorem.
We can use Mathematica in above Theorems 3.3 - 4.2, yields

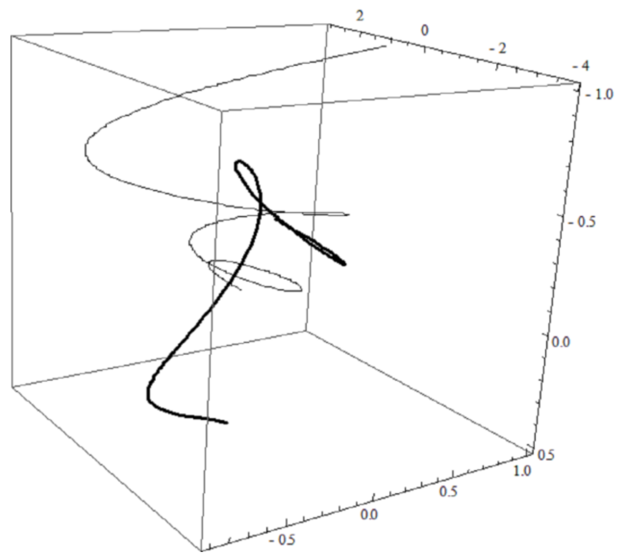

Fig. 1. Mathematical's result in Theorems 3.3 – 4.2.

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