Approximate solutions of the nonlinear stochastic differential equations

J. Ahmadi Shali, A. Jodayree Akbarfam and H. Bevrani,
Department of Mathematics and Computer Science, University of Tabriz, Tabriz-Iran
j_ahmadishali@tabrizu.ac.ir, Akbarfam@yahoo.com and bevrani@tabrizu.ac.ir

Abstract. In this note, we study the approximate solutions of nonlinear stochastic differential equations by using the theories and methods of mathematics analysis. An approximate method based on piecewise linearization is developed for the determination of semi-analytical numerical solution of nonlinear stochastic differential equations. Also, linearization methods for initial value problems in stochastic differential equations which have singular points are introduced and applied to a variety of problems arising in engineering, physics, stochastic control and population biology.

Keywords: Stochastic differential equation, Taylor series expansion, linearization method.

1 Introduction

In most dynamical systems which describe processes in engineering, physics and economics, stochastic components and random noise are included. The stochastic aspects of the models are used to capture the uncertainty about the environment with the system is operating and the structure and parameters of the models of physical processes being studied (see,[1],[2] and in the references cited there).

Stochastic differential equations in infinite dimensional spaces are motivated by the development of analysis and the theory of stochastic processes itself such as stochastic partial differential equations and stochastic delay differential equations on the one hand, and by such topics as stochastic control, population biology and turbulence in applications on the other. (See,[3], [4] and in the references cited there) Stochastic modeling has come to play an important role in many branches of science and industry such as Lumped rainfall-runoff model (see [9] and in the references cited there), where more and more people have encountered stochastic differential equations.

Using stochastic differential equations we can successfully model systems that function in the presence of random perturbations. Such systems are among the basic objects of modern control theory. However, the very importance acquired by stochastic differential equations lies, to a large extent, in the strong connections they have with the equations of mathematical physics. It is well known that problems in mathematical physics involve 'damned dimensions', often leading to severe difficulties in solving boundary value problems.

The linearization methods [5],[6] and [7] developed by the author and coworkers were based the linearization of the nonlinear stochastic differential equations, about the previous step; such a linearization cannot be carried out, however, if the there are...
singular points. A piecewise linearization method based on Taylor's series expansion has been developed for nonlinear stochastic differential equations. The method is based on the Taylor series expansion of the nonlinearities with respect to time and the dependent variables and, for single degree-of-freedom problems, provides piecewise analytical solutions in two-point intervals which are continuous everywhere. In addition, the method provides explicit two-point finite difference equations which have an infinite interval and are \( \mathbb{P} \)-stable.

In this note, we study the approximate solutions of nonlinear stochastic differential equations by using the theories and methods of mathematics analysis. An approximate method based on piecewise linearization is developed for the determination of semi-analytical-numerical solution of nonlinear stochastic differential equations. Also, linearization methods for initial value problems in stochastic differential equations which have singular points are introduced and applied to a variety of problems arising in engineering, physics, stochastic control and population biology.

2 Preliminary Notes

Let \( (\Omega, F, \mathbb{P}) \) be a complete probability space with a filtration \( \{F_t\}_{t \geq 0} \) satisfying the usual conditions. We let \( B(t) = (B_1(t), \ldots, B_m(t))^T \), \( t \geq 0 \) be a \( m \)-dimensional Brownian motion defined on the space. Let \( 0 \leq t_0 < T < \infty \) and \( x_0 \) be an \( F_{t_0} \)-measurable \( R^d \)-valued random variable such that \( E |x_0|^2 < \infty \). Let \( f: R^d \times [t_0, T] \to R^d \) and \( g: R^d \times [t_0, T] \to R^{d \times m} \) be both Borel measurable.

Consider the \( d \)-dimensional nonlinear stochastic differential equation of \( \text{Itô} \) type

\[
\frac{dx(t)}{dt} = f(x(t), t)dt + g(x(t), t)dB(t), \quad t \in [t_0, T],
\]

with initial condition \( x(t_0) = x_0 \). By the definition of stochastic differential, this equation is equivalent to the following stochastic integral equation

\[
x(t) = x_0 + \int_{t_0}^t f(x(s), s)ds + \int_{t_0}^t g(x(s), s)dB(s), \quad t \in [t_0, T].
\]

**Definition 2.1** An \( R^d \)-valued stochastic process \( \{x(t)\}_{t \in [t_0, T]} \) is called a solution of equation (1) if it has the following properties:

1. \( \{x(t)\} \) is continuous and \( F_t \)-adapted;
2. \( \{f(x(t), t)\} \in L^2([t_0, T]; R^d) \) and \( \{g(x(t), t)\} \in L^2([t_0, T]; R^{d \times m}) \);
3. Equation (2) holds for every \( t \in [t_0, T] \) with probability 1.
A solution \( \{x(t)\} \) is said to be unique if any other solution \( \{\tilde{x}(t)\} \) is indistinguishable from \( \{x(t)\} \), that is,

\[
P = \{x(t) = \tilde{x}(t) \text{ for all } t_0 \leq t \leq T\} = 1,
\]

where \( L^1(R_+; R^d) \) denote the family of all \( R^d \)-valued measurable \( \{F_t\} \)-adapted processes \( f = \{f(t)\}_{t \geq 0} \) such that

\[
\int_0^T |f(t)| \, dt < \infty \quad \text{a.s. for every } T > 0,
\]

and \( L^2(R_+; R^{d \times m}) \) denote the family of all \( d \times m \)-matrix-valued measurable \( \{F_t\} \)-adapted processes \( f = \{f(t)\}_{t \geq 0} \) such that

\[
\int_0^T |f(t)|^2 \, dt < \infty \quad \text{a.s. for every } T > 0,
\]

**Theorem 2.1** Assume that there exist two positive constants \( \overline{K} \) and \( K \) such that

1. (Lipschitz condition) for all \( x, y \in R^d \) and \( t \in [t_0, T] \)

\[
\max \{|f(x,t) - f(y,t)|^2, |g(x,t) - g(y,t)|^2\} \leq \overline{K} |x - y|^2,
\]

2. (Linear growth condition) for all \( (x,t) \in R^d \times [t_0, T] \)

\[
\max \{|f(x,t)|^2, |g(x,t)|^2\} \leq K(1+|x|^2).
\]

Then there exist a unique solution \( \{x(t)\} \) to equation (1) and the solution belongs to \( M^2([t_0, T]; R^d) \) where \( M^2(R_+; R^d) \) denote the family of all processes \( f \in L^2(R_+; R^{d \times m}) \) such that

\[
E\left[\int_0^T |f(t)|^2 \, dt\right] < \infty \quad \text{for every } T > 0.
\]

**Proof:** For proof see [3]. □
3 Main results

The purpose of this section is to develop a SDE for a lumped rainfall-runoff model and apply it to a watershed and we present our method for solving the nonlinear SDE.

3.1 Lumped rainfall-runoff model

A watershed can be divided into \( n \) subwatersheds in series according to geographical properties. For each subwatershed, the mass balance equation is

\[
\frac{ds}{dt} = I - x
\]

(3)

where \( S(\text{mm}) \) is the storage of a subwatershed, \( I(\frac{\text{mm}}{\text{h}}) \) the input to the subwatershed, and \( x(\frac{\text{mm}}{\text{h}}) \) is the output from the subwatershed. The storage-release equation is assumed to be

\[
S = Kx
\]

(4)

where \( K(\text{h}) \) is the storage constant of the watershed. Substituting equation (3) into (4) leads to

\[
\frac{dx}{dt} = \frac{1}{K} (I - x)
\]

(5)

Using the control area of stream flow stations (the stream flow stations are located at the outlet of watersheds) as a guide, the area of the watershed is divided into \( n \) subwatersheds. The storage-release equations for the \( n \) subwatersheds are

\[
\frac{dx_i}{dt} = \frac{1}{K_i} (I_i - x_i), \quad i = 1, 2, \ldots, n.
\]

(6)

The input of the first subwatershed is the rainfall excess, \( P_1 \), only, which represents the amount of rainfall after infiltration. For the down-gradient subwatersheds, the input will be the rainfall excess, \( P_i \), plus the direct runoff, \( x_i \), from the upstream subwatershed. For \( n \) subwatersheds, the input can be written mathematically as
When equation (7) is substituted into (6), we obtain the following equation:

\[
\frac{dx_i}{dt} = \frac{1}{K_i} (P_i + x_{i-1} - x_i), \quad i = 1, 2, \ldots, n, \quad \xi = 0,
\]

(8)

This equation represents a series lumped rainfall-runoff model of a watershed. The rainfall excess input to a model always has some degree of uncertainty because of the temporal and spatial variability of rainfall, infiltration, and measurement errors. Accordingly, the rainfall excess of each subwatershed is expressed as follows:

\[
P_i = \overline{P} + P'_i, \quad i = 1, 2, \ldots, n,
\]

(9)

where \( \overline{P} \) represented the deterministic components, which may be time-dependent, and \( P'_i \) represents the fluctuating components. Substituting equation (9) into equation (8) yields:

\[
\frac{dx_i}{dt} = \frac{1}{K_i} (\overline{P} + x_{i-1} - x_i) + \frac{P'_i}{K_i}, \quad i = 1, 2, \ldots, n, \quad \xi = 0,
\]

(10)

The stochastic components are assumed white-noise Gaussian processes with zero means and a delta-correlated structure as described by Unny [8]:

\[
E[W(t)] = 0,
\]

(11)

\[
E[W(t)W^T(t - \tau)] = D\delta(\tau),
\]

(12)

where \( W(t) = [P'_1, \ldots, P'_n]^T \), \( \delta(\tau) \) is the Dirac delta function and \( D = \text{diag}([\sigma^2_{P'_1}, \ldots, \sigma^2_{P'_n}] \) in which the squared terms \( \sigma^2_{P'_1}, \ldots, \sigma^2_{P'_n} \) represent the variances of \( P'_1, \ldots, P'_n \).

As the white Gaussian process is not an ordinary function of time \( t \), the second term on the right side of the SDE (10) is not mean square Riemann integrable. However, the white Gaussian process can be described as the formal derivative in time of a Brownian motion process \( B \) with independent increments \( dB \) in time \( dt \) characterized by:

\[
E[dB(t)] = [0],
\]

(13)
Thus, equation (10) can be rewritten as

\[
dx = f(x, t)dt + g(x, t)dB(t) \tag{16}
\]

where

\[
dx = [dx_1, ..., dx_n]^T, \quad f(x, t) = [(P_n - x)/K_n, ..., (P_n + x_{n-1} - x_n)/K_n]^T
\]

and

\[
g(x, t) = diag[1/K_n, ..., 1/K_n]^T, \quad dB(t) = [P'_n, ..., P'_n]^T dt.
\]

Equation (16) is a nonlinear SDE and its solution, \(x(t)\), is a time-dependent Markov diffusion process. Subsequently, \(x(t)\) is fully determined if the joint probability density function of the random variable \(x(t)\) is defined for all finite sets of \(t\). This is, however, a very ambitious goal, which is difficult to achieve in most cases.

Consider an arbitrary function \(\phi(x(t), t)\) of \(x(t)\) and \(t\), whose partial derivatives \(\frac{\partial^2 \phi}{\partial x_j \partial x_k}\) and \(\frac{\partial \phi}{\partial t}\) are continuous and bonded over any finite interval of \(x(t)\) and \(t\). If we use \(\delta\) as a finite forward increment operator over the time increment \(\delta t\), we have

\[
\delta \phi(x(t), t) = \phi(x + \delta x, t + \delta t) - \phi(x, t). \tag{17}
\]

The Taylor series expansion of \(\delta \phi\) then gives

\[
\delta \phi = \sum_{j=1}^{n} \delta x_j \frac{\partial \phi}{\partial x_j} + \frac{1}{2} \sum_{i,j=1}^{n} \delta x_i \delta x_j \frac{\partial^2 \phi}{\partial x_i \partial x_j} + \delta t \frac{\partial \phi}{\partial t} + o(\delta x \delta x^T) + o(\delta t). \tag{18}
\]

Referring back to the general equation (1), we have shown that

\[
E[\delta x_j(t) | x] = f_j(x, t)\delta t + o(\delta t) \tag{19}
\]
Substituting equation (20) and (19) into equation (19) and taking the conditional expectation of equation (19) yields

\[ E[\partial_i (t) \xi_j (t) \mid x] = 2(gDg^T)_{ij} \partial_i + o(\partial_i) \]

(20)

The expectation \( E[\partial \phi \mid x] \) is regarded here as a random variable. In view of the above, the expectation of equation (21) gives

\[ E[\partial \phi] = E[E[\partial \phi \mid x]] \]

\[ = \sum_{j=1}^{n} E[f_j (x, t)(\partial \phi \mid x)} \partial_i + \sum_{i,j=1}^{n} (gDg^T)_{ij} \frac{\partial^2 \phi}{\partial x_i \partial x_j} \partial_i + \frac{\partial \phi}{\partial t} \partial_i + o(\partial_i). \]

(21)

Finally, upon dividing equation (22) by \( \partial_i \), and taking the limit when \( \partial_i \to 0 \), we obtain the following ordinary differential equation

\[ \frac{dE[\phi]}{dt} = \sum_{j=1}^{n} E[f_j (x, t)(\partial \phi \mid x) \partial_i] + \sum_{i,j=1}^{n} E[(gDg^T)_{ij} \frac{\partial^2 \phi}{\partial x_i \partial x_j}] + E[\frac{\partial \phi}{\partial t}] \].

(23)

Now, we will need to analyze the solution of the nonlinear SDE (16). To do this, we will study the stochastic differential equations in the general form.

### 3.2 Stochastic differential equations and analytical solutions

Let \((\Omega, F, P)\) be a complete probability space with a filtration \(\{F_t\}_{t\geq 0}\) satisfying the usual conditions. We let \(B(t) = (B_1(t), \ldots, B_m(t))^T, t \geq 0\) be a \(m\)-dimensional Brownian motion defined on the space. Let \(0 \leq t_0 < T < \infty\) and \(x_0\) be an \(F_{t_0}\)-measurable \(R^d\)-valued random variable such that \(E \mid x_0 \mid^2 < \infty\). Let \(f : R^d \times [t_0, T] \to R^d \) and \(g : R^d \times [t_0, T] \to R^{d \times m}\) be both Borel measurable. Consider the \(d\)-dimensional nonlinear stochastic differential equation of \(Ito\) type.
\[ dx(t) = f(x(t), t)dt + g(x(t), t)dB(t), \quad t \in [t_0, T]. \]  

(24)

with initial condition \( x(t_0) = x_0 \). Where

\[
\begin{align*}
x &: [t_0, T] \to \mathbb{R}^d, \\
f &: \mathbb{R}^d \times [t_0, T] \to \mathbb{R}^d, \\
x(t) &= (x_1(t), \ldots, x_d(t))^T, \\
f(x, t) &= (f_1(x, t), \ldots, f_d(x, t))^T, \\
x_i &: [t_0, T] \to \mathbb{R}, \\
f_i &: \mathbb{R}^d \times [t_0, T] \to \mathbb{R}, \\
i &= 1, 2, \ldots, d,
\end{align*}
\]

also,

\[
g : \mathbb{R}^d \times [t_0, T] \to \mathbb{R}^{d \times m} 
\]

\[
g(x, t) = 
\begin{bmatrix}
g_{11}(x, t) & \cdots & g_{1m}(x, t) \\
\vdots & \ddots & \vdots \\
g_{n1}(x, t) & \cdots & g_{nm}(x, t)
\end{bmatrix}
\]

and

\[
g_{0i}(x, t) : \mathbb{R}^d \times [t_0, T] \to \mathbb{R},
\]

are given functions. By substituting this relation in equation (24), we have

\[
\begin{bmatrix}
dx_1(t) \\
\vdots \\
dx_d(t)
\end{bmatrix}
= 
\begin{bmatrix}
f_1(x, t) \\
\vdots \\
f_d(x, t)
\end{bmatrix} dt + 
\begin{bmatrix}
g_{11}(x, t) & \cdots & g_{1m}(x, t) \\
\vdots & \ddots & \vdots \\
g_{n1}(x, t) & \cdots & g_{nm}(x, t)
\end{bmatrix}
\begin{bmatrix}
\int dB_1(t) \\
\int dB_2(t) \\
\int dB_m(t)
\end{bmatrix}
\]

In the general form, we have

\[
dx_i(t) = f_i(x(t), t)dt + \sum_{j=0}^{m} g_{ij}(x(t), t)dB_j(t), \quad i = 0, 1, \ldots, d
\]

(25)
with initial condition \( x(t_0) = x_0 \), where \( f_i \in C^{2,1}(R^d \times [t_0, T], R) \), 
\( g_{ij} \in C^{2,1}(R^d \times [t_0, T], R) \), \( i = 0, 1, 2, \ldots, d \), \( j = 1, 2, \ldots, m \) and \( C^{2,1}(R^d \times R_+; R) \) 
do not the family of all real-valued functions defined on \( R^d \times R_+ \) such that they are 
continuously twice differentiable in \( x \) and once in \( t \).

Now, we are interested to find the semi-analytical-numerical solution for the equation 
(25), in domain \([t_0, T]\). Consider the interval \([t_0, T]\), and divide it into a series of 
subintervals \([t_n, t_{n+1}]\) such \( t_n = t_0 + nh \), \( h = t_{n+1} - t_n \), \( n = 0, 1, \ldots \). In each subinterval 
\( f_i(x, t) \) and \( g_{ij}(x, t) \) may be linearized as follows:

### 3.2.1 Regular Points

If \( f_i(x, t) \) and \( g_{ij}(x, t) \) are regular and differentiable, \( f_i \) and \( g_{ij} \) may be 
approximated by the first three terms of its classical Taylor series expansion around 
\((x_n, t_n)\), in the following form:

\[
f_i(x, t) = f_{i,j}(t) + f_{2,j}(t)x
\]
(26)

\[
g_{ij}(x, t) = g_{1,ij}(t) + g_{2,ij}(t)x
\]
(27)

where \( x_n = (x_{1,n}, x_{2,n}, \ldots, x_{d,n})^T \) and

\[
f_{i,j}(t) = f_i(x_n, t_n) + \frac{\partial f_i(x_n, t_n)}{\partial t} |_{(x_n, t_n)} (t - t_n) - \nabla f_i(x_n, t_n)|_{(x_n, t_n)} x_n,
\]

\[
f_{2,j}(t) = \nabla f_i(x_n, t_n)|_{(x_n, t_n)} x_n,
\]

\[
g_{1,ij}(t) = g_{ij}(x_n, t_n) + \frac{\partial g_{ij}(x_n, t_n)}{\partial t} |_{(x_n, t_n)} (t - t_n) - \nabla g_{ij}(x_n, t_n)|_{(x_n, t_n)} x_n,
\]

\[
g_{2,ij}(t) = \nabla g_{ij}(x_n, t_n)|_{(x_n, t_n)} x_n.
\]

By substitution equations (26) and (27) in equation (25), we obtain

\[
dx_i(t) = (f_{i,1}(t) + f_{i,2}(t)x(t)) + \sum_{j=0}^{m} (g_{1,ij}(t) + g_{2,ij}(t)x(t)) dB_j(t), \quad i = 0, 1, 2, \ldots, d.
\]

In the general form, by using (28), for \( i = 1, 2, \ldots, d \) the matrix form of equation (29) as 
follows:
or briefly, we have

$$dx(t) = (f^*(t) + F^*(t)x(t))dt + \sum_{j=1}^{\infty} [g_j^*(t) + G_j^*(t)x(t)]dB_j(t)$$

(30)

on $[t_0, T]$, where $F^*(\cdot)$, $G^*(\cdot)$ are $d \times d$-matrix-valued functions, $f^*(\cdot)$, $g^*(\cdot)$ are $R^d$-valued functions and as before, $B(t) = (B_1(t), ..., B_m(t))^T$ is an $m$-dimensional Brownian motion and

$$F^*(t) = \begin{bmatrix}
\frac{\partial f_1(x_n, t_n)}{\partial x_1} & \frac{\partial f_1(x_n, t_n)}{\partial x_d} \\
\vdots & \vdots \\
\frac{\partial f_d(x_n, t_n)}{\partial x_1} & \frac{\partial f_d(x_n, t_n)}{\partial x_d}
\end{bmatrix}_{d \times d}, \quad f^*(t) = \begin{bmatrix}
f_{11}(t) \\
\vdots \\
f_{1d}(t)
\end{bmatrix}_{d \times 1}, \quad (31)$$
It is clear that, equation (30) is a linear stochastic differential equation.

### 3.2.2 Analytical solution

In this section, our aim is to get an explicit expression for the analytical solution of the equations (30).

Throughout this section we shall assume that $F^*, f^*, G^*$ and $g^*$ are all Borel-measurable and bounded. For every initial value $x(t_0) = x_0$, $n = 0, 1, 2, \ldots$ which is $F_{t_0}$-measurable and belongs to $L^2(\Omega; R^d)$.

Now, consider the homogeneous linear stochastic differential equation

$$dx(t) = F^*(t)x(t)dt + \sum_{j=0}^{d} G_j^*(t)x(t)dB_j(t)$$

on $[t_0, T]$. As assumed,

$$F^*(t) = (F_{rt}(t))_{d \times d}, \quad G_j^*(t) = (G_{rj}(t))_{d \times d}$$

are all Borel-measurable and bounded. For every $l = 1, 2, \ldots, d$, let $e_l$ be the unit column-vector in the $x_l$-direction, i.e.

$$e_l = (0, \ldots, \underbrace{1}_{l}, 0, \ldots)\text{T}.$$
\[ \Phi(t) = (\Phi_1(t), \ldots, \Phi_d(t))^T = (\Phi_j(t))_{d \times d}. \]

We call \( \Phi(t) \) the fundamental matrix of equation (33). It is useful to note that \( \Phi(t_0) \) the \( d \times d \) identity matrix and

\[ d\Phi(t) = F^*(t)\Phi(t)dt + \sum_{j=1}^{m} G_j^*(t)\Phi(t)dB_j(t). \]  
(34)

**Lemma 3.1** Given the initial value \( x(t_0) \), the unique solution of equation (33) is

\[ x(t) = \Phi(t)x_0. \]

**Proof:** For proof see [3]. \( \Box \)

Let us now turn to the general \( d \)-dimensional linear stochastic differential equation (30). Equation (33) is called the corresponding homogeneous equation of system (30). In this section we shall establish a useful formula, which represents the unique solution of equation (30) in terms of the fundamental matrix of the corresponding homogeneous equation (33).

**Theorem 3.2** The unique solution of equation (30) can be expressed as

\[ x(t) = \Phi(t)x_n + \int_{t_n}^{t} \Phi^{-1}(s)[f^*(s) - \sum_{j=1}^{m} G_j^*(s)g_j^*(s)]ds + \]
\[ + \sum_{j=1}^{m} \int_{t_n}^{t} \Phi^{-1}(s)g_j^*(s)dB_j(s), \]

where \( \Phi(t) \) is the fundamental matrix of equation (33).

**Proof:** Set

\[ \epsilon(t) = x_n + \int_{t_n}^{t} \Phi^{-1}(s)[f^*(s) - \sum_{j=1}^{m} G_j^*(s)g_j^*(s)]ds + \]
\[ + \sum_{j=1}^{m} \int_{t_n}^{t} \Phi^{-1}(s)g_j^*(s)dB_j(s). \]

Then \( \epsilon(t) \) has the differential
\[
d\xi(t) = \Phi^{-1}(t)\{f^*(t) - \sum_{j=1}^{m} G_j^*(t)g_j^*(t)\}dt + \sum_{j=1}^{m} \Phi^{-1}(t)g_j^*(s)dB_j(t).
\]  

(36)

Let

\[
\eta(t) = \Phi(t)\xi(t).
\]

(37)

Clearly, \(\eta(t_n) = x_n\). Moreover, by Itô’s formula

\[
d\eta(t) = d\Phi(t)\xi(t) + \Phi(t)d\xi(t) + d\Phi(t)d\xi(t).
\]

Substituting (34) and (36) into it and using the formal multiplication table

\[
d\eta(t) = 0, \quad dB_j dt = 0, \quad dB_j dB_j = \delta_{ij}
\]

we derive that

\[
d\eta(t) = F^*(t)\eta(t)dt + \sum_{j=1}^{m} G_j^*(t)\eta(t)dB_j(t)
\]

\[
+ [f^*(t) - \sum_{j=1}^{m} G_j^*(t)g_j^*(s)]dt + \sum_{j=1}^{m} g_j^*(t)dB_j(t)
\]

\[
+ F^*(t)\Phi(t)dt + \sum_{j=1}^{m} G_j^*(t)\Phi(t)dB_j(t)
\]

\[
\times (\Phi^{-1}(t)f^*(t)dt + \sum_{j=1}^{m} \Phi^{-1}(t)g_j^*(t)dB_j(t)
\]

\[
- \sum_{j=1}^{m} \Phi^{-1}(t)G_j^*(t)g_j^*(t)dt)
\]

\[
= (F^*(t)\eta(t) + f^*(t))dt + \sum_{j=1}^{m} G_j^*(t)\eta(t) + g_j^*(t)dB_j(t).
\]

In other words, we have shown that \(\eta(t)\) is a solution to equation (30) satisfying the initial condition \(\eta(t_n) = x_n\). On the other hand, equation (30) has only one solution \(x_n\). So we must have that \(x(t) = \eta(t)\), which is the required formula (35).

The proof is complete. \(\Box\)

Since we assume that \(x_n \in L^2(\Omega; \mathbb{R}^d)\), the first and second moments of the solution of equation (30) exist and are finite. The following theorem shows that one can obtain
first and second moments by solving the corresponding linear ordinary differential equations.

**Theorem 3.3** For the solution of equation (30), we have:

1. \( m(t) := Ex(t) \) is the unique solution of equation

\[
m(t) = F^*(t)m(t) + f^*(t), \quad t \in [t_n, T]
\]

with initial value \( m(t_n) := Ex_n \).

2. \( P(t) := E(x(t)x^T(t)) \) is the unique nonnegative-definite symmetric solution of the equation

\[
\dot{P}(t) = F^*(t)P(t) + P(t)F^{*T}(t) + f^*(t)m^T(t) \\
+ m(t)f^{*T}(t) + \sum_{j=1}^{m} [G_j^*(t)P(t)G_j^{*T}(t) + G_j^*(t)m(t)g_j^{*T}(t)] \\
+ g_j^*(t)m^T(t)G_j^{*T}(t) + g_j^*(t)g_j^{*T}(t), \quad t \in [t_n, T],
\]

with initial value \( P(t_n) := E(x_n x_n^T) \). Note that (39) represents a system of \( \frac{d(d+1)}{2} \) linear equations.

**Proof:**

1. Note that

\[
x(t) = x(t_n) + \int_{t_n}^{t} (F^*(s)x(s) + f^*(s))ds + \sum_{j=1}^{m} \int_{t_n}^{t} (G_j^*(s)x(s) + g_j^*(s))dB_j(s).
\]

Taking the expectation on both sides yields

\[
m(t) = m(t_n) + \int_{t_n}^{t} (F^*(s)m(s) + f^*(s))ds,
\]

which is the integral form of equation (38). So the conclusion of part (1) follows.

2. By \( \text{Itô formal} \),
\[ d[x(t)x^T(t)] = dx(t)x^T(t) + x(t)dx^T(t) + \sum_{j=1}^{m} [G_j^*(t)x(t) + g_j^*(t)]T dt \]
\[ = [F^*(t)x(t)x^T(t) + f^*(t)x^T(t) + x(t)x^T(t)F^*T(t)] dt \]
\[ + \sum_{j=1}^{m} [G_j^*(t)x(t)x^T(t)G_j^T(t) + g_j^*(t)x^T(t)G_j^T(t) + G_j^*(t)x(t)g_j^T(t)] dt \]
\[ + x(t)(G_j^*(t)x(t) + g_j^*(t))^T dB_j(t). \]

Now equation (39) follows by taking the expectation on both sides of the integral form of the above equality. Since \( P(t) \) is the covariance matrix of \( x(t) \), it is of course nonnegative-definite and symmetric. The proof is complete. \( \square \)

Theorem 3.2 tells us that we can have the explicit solution to the linear equation (30) provided we know the corresponding fundamental matrix \( \Phi(t) \). The following theorem shows that in the especial case, the solution of the corresponding homogeneous equation of equation (30) as follows:

**Theorem 3.4** Let \( F^*, G_j^* \) be real-valued Borel measurable bounded functions on \([t_n, T]\). Then

\[ x(t) = x_n \exp \left( \int_{t_n}^{t} [F^*(s) - \frac{1}{2} \sum_{j=1}^{m} G_j^{*2}(s)] ds + \sum_{j=1}^{m} \int_{t_n}^{t} G_j^*(s) dB_j(s) \right) \]

is the unique solution to the scalar linear stochastic differential equation

\[ dx(t) = F^*(t)x(t)dt + \sum_{j=1}^{m} G_j^*(t)x(t)dB_j(t) \]

on \([t_n, T]\) with initial condition \( x(t_n) = x_n \).

**Proof:** For proof see [3]. \( \square \)

### 3.2.2 Singular points and analytical solutions
If \( f_i(x, t) \) and/or \( g_j(x, t) \) are/is singular at \( t_n \), the above derivation is not valid, but \( f_i(x, t) \) and \( g_j(x, t) \) may be approximated by the first three terms of it’s Taylor series expansion around \((x_n, t_{n+1})\). In the similar manner in the section 3.1.1, linearization form of equation (24) as follows:

\[
dx(t) = (f^*(t) + G^*(t)x(t))dt + \sum_{j=1}^{m} (G_j^*(t)x(t))dB_j(t) \tag{43}\]

where \( F^*(\cdot), G^*(\cdot) \) are \( d \times d \) -matrix-valued functions, \( f^*(\cdot), g^*_j(\cdot) \) are \( R^d \) -valued functions and as before, \( B(t) = (B_1(t), \ldots, B_m(t))^T \) is an \( m \)-dimensional Brownian motion and

\[
F^*(t) = \begin{bmatrix}
\frac{\partial f_1(x_n, t_{n+1})}{\partial x_1} & \cdots & \frac{\partial f_1(x_n, t_{n+1})}{\partial x_d} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_d(x_n, t_{n+1})}{\partial x_1} & \cdots & \frac{\partial f_d(x_n, t_{n+1})}{\partial x_d}
\end{bmatrix}_{d \times d}, \quad f^*(t) = \begin{bmatrix} f_{1,1}(t) \\
\vdots \\
\vdots \\
f_{1,d}(t) \end{bmatrix}_{d \times 1}, \tag{44}
\]

\[
G^*_j(t) = \begin{bmatrix}
\frac{\partial g_{1j}(x_n, t_{n+1})}{\partial x_1} & \cdots & \frac{\partial g_{1j}(x_n, t_{n+1})}{\partial x_d} \\
\vdots & \ddots & \vdots \\
\frac{\partial g_{dj}(x_n, t_{n+1})}{\partial x_1} & \cdots & \frac{\partial g_{dj}(x_n, t_{n+1})}{\partial x_d}
\end{bmatrix}_{d \times d}, \quad g^*_j = \begin{bmatrix} g_{1,1}(t) \\
\vdots \\
\vdots \\
g_{1,d}(t) \end{bmatrix}_{d \times 1}, \tag{45}
\]

\[
\begin{align*}
f_{1,j}(t) &= f_j(x_n, t_{n+1}) + \frac{\partial f_j(x_n, t)}{\partial t}|_{(x_n, t_{n+1})} (t - t_{n+1}) - \nabla f_j(x, t)|_{(x_n, t_{n+1})} x_n, \\
f_{2,j}(t) &= \nabla f_j(x, t)|_{(x_n, t_{n+1})}, \\
g_{1,j}(t) &= g_j(x_n, t_{n+1}) + \frac{\partial g_j(x_n, t)}{\partial t}|_{(x_n, t_{n+1})} (t - t_{n+1}) - \nabla g_j(x, t)|_{(x_n, t_{n+1})} x_n, \\
g_{2,j}(t) &= \nabla g_j(x, t)|_{(x_n, t_{n+1})}. \tag{46}
\end{align*}
\]
It is clear that, equations (43) is a linear stochastic differential equation, whose analytical solution may be written as

\[
x(t) = \Phi(t)(x_0) + \int_{t_0}^{t} \Phi^{-1}(s) [f^*(s) - \sum_{j=1}^{m} G_j^*(s)g_j^*(s)] ds
\]

\[+ \sum_{j=1}^{m} \int_{t_0}^{t} \Phi^{-1}(s)g_j^*(s)dB_j(s),\]

where \( \Phi(t) \) is the fundamental matrix of corresponding homogeneous equation (43) and \( F^*, f^*, G_j^* \) and \( g_j^* \) are defined as above.

4 Error analysis and convergence

In this section, we perform the estimating error and convergence for the stochastic differential equation (24).

Equations (35) and (47) provides the following nonlinear mapping

\[
x(t_{n+1}) = \Phi(t_{n+1})(x_n) + \int_{t_n}^{t_{n+1}} \Phi^{-1}(s) [f^*(s) - \sum_{j=1}^{m} G_j^*(s)g_j^*(s)] ds +
\]

\[+ \sum_{j=1}^{m} \int_{t_n}^{t_{n+1}} \Phi^{-1}(s)g_j^*(s)dB_j(s),\]

The nonlinear mapping or difference equation corresponding to equation (48) have been derived by considering that \( F^*, f^*, G_j^* \) and \( g_j^* \) are function of two variable \( x \) and \( t \). Moreover, since these equation have been obtained by approximating \( f_j \) and \( g_{ij} \) by the first three terms of its Taylor series expansion, while second- and higher-order terms have been neglected, the local step size \( t_{n+1} - t_n \) may be determined from the condition that the second-order terms be much smaller than the first-order ones, or from the condition that \( x_{n+1} \) does not differ significantly from \( x_n \).

Since the truncated nonlinear mappings or difference equations corresponding to (48) is an approximate solution of (30) or (43), hence, the error function \( e(x) \) for (24) is defined as follows:

\[ e(x) = |x(t) - X(t)|, \]
where $x(t)$ is exact solution of equation (24) and $X(t)$ the approximate solution of equation (24) defined as follow

$$X(t) = \Phi(t)x_0 + \int_{t_0}^{t} \Phi^{-1}(s)[f^*(s) - \sum_{j=1}^{m} G_j^*(s)g_j^*(s)]ds$$

$$+ \sum_{j=1}^{m} \int_{t_0}^{t} \Phi^{-1}(s)g_j^*(s)dB_j(s).$$

(49)

Note that, in the error function $e(x)$ if $t_n$ is a regular point or singular point, then the relations $F^*, f^*, G_j^*$ and $g_j^*$ defined in section 3.1 or 3.2, respectively.

Now, by substituting the solutions $x(t_r)$, $(r = 0, 1, ...$) in the error function $e(x)$ we have

$$e(x) = |x(t_r) - X(t_r)|,$$

then our aim is $e(x_r) \leq 10^{k_r}$ $(k_r$ is any positive integer). If we prescribe $\max(10^{k_r}) = 10^k$, then we decrease the step size $t_{r+1} - t_r$ as long as the following inequality holds at each point $x_r$:

$$e(x_r) \leq 10^{-k}.$$ 

In other words, by decreasing step size $t_{r+1} - t_r$ the error function $e(x_r)$ approaches zero.

Note that, the convergence of this method is such as Taylor's series expansion.

5 Conclusion

This methods provide piecewise closed-form solution which depend of the Jacobian and the derivative of the right-hand side of the stochastic differential equation with respect to the time and the dependent variables. It has been shown that, for regular problems, linearization method is robust, stable and accurate techniques whose accuracy is not a strong function of either the linearization point or the linearization with respect to the time and the dependent variables.

For stochastic differential equation with singular points, it has been found that linearization method provide very accurate results even for relatively large and equal step sizes. The linearization method has an accuracy comparable to the quasilinearization technique for regular problems when fixed steps are employed. The
accuracy of the latter is higher than of the former for hyper singular problems, but its convergence depends on the initial guess, whereas this method provide piecewise solutions in close-form, do not need an initial guess, are not iterative, and may use variable step sizes according to the evolution of the solution.

References