gpr-closed and gpr-open mappings

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Abstract: In this paper we discuss new type of closed and open mappings called gpr-closed and gpr-open mappings; its properties and interrelation with other such functions are studied.

Keywords: closed mapping; semi-closed mapping; pre-closed mapping; \(\beta\)-closed mapping; \(\gamma\)-closed mapping, \(\nu\)-closed mapping, open mapping; semi-open mapping; pre-open mapping; \(\beta\)-open mapping; \(\gamma\)-open mapping and \(\nu\)-open mapping

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1 INTRODUCTION

Mappings plays an important role in the study of modern mathematics, especially in Topology and Functional Analysis. Closed and open mappings are one such mappings which are studied for different types of closed sets by various mathematicians for the past many years. Norman Levine introduced the notion of generalized closed sets. After him different mathematicians worked and studied on different versions of generalized closed sets and related topological properties. In this paper we are going to further study weak form of closed and open mappings namely gpr-closed and gpr-open mappings using gpr-closed and gpr-open sets. Basic properties are verified. Throughout the paper \(X, Y\) means a topological spaces \((X, \tau)\) and \((Y, \sigma)\) unless otherwise mentioned without any separation axioms.

2 PRELIMINARIES

Definition 2.01: \(A \subseteq X\) is called
(i) closed[semi-closed; regular-closed] if its complement is open[semi-open; regular-open].
(ii) g-closed[rg-closed] if cl \(A \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open in \(X\).
(iii) pg-closed[gp-closed; gpr-closed] if pcl(A) \(\subseteq U\) whenever \(A \subseteq U\) and \(U\) is pre-open[open; regular-open] in \(X\).
(iv) \(\alpha\)-g-closed if \(\alpha cl(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open in \(X\).
**Definition 2.02:** A function \( f: X \to Y \) is said to be
(i) continuous \( \text{[resp: nearly-continuous; pre-continuous; g-continuous; rg-continuous]} \)
if inverse image of each open set is open \( \text{[resp: regular-open; preopen; g-open; rg-open]} \).
(ii) nearly-irresolute; \( \text{[resp: g-rresolute; rg-irresolute]} \) if inverse image of each regular-open set \( \text{[resp: g-open; rg-open]} \) is regular-open; \( \text{[resp: g-open; rg-open]} \).
(iii) closed if image of each closed set is closed \( \text{[resp: regular-closed; g-closed; rg-closed]} \).
(iv) open if image of each open set is open \( \text{[resp: regular-open; g-open; rg-open]} \).

**Definition 2.03:** \( X \) is said to be \( T_{1/2}[rT_{1/2}] \) if every g-\( [rg-] \)closed set is regular-closed.

**Note 1:** From Definition 2.1 we have the interrelations among closed and open sets.

\[
\begin{align*}
\text{Closed} & \implies \text{g-closed} \implies \alpha g\text{-closed} \implies \text{gp-closed} \implies \text{gpr-closed} \\
r\text{-irresolute} & \implies \text{g-closed} \implies \alpha g\text{-closed} \implies \text{gp-closed} \implies \text{gpr-closed}
\end{align*}
\]

\[
\begin{align*}
\text{Open} & \implies \text{g-open} \implies \alpha g\text{-open} \implies \text{gp-open} \implies \text{gpr-open} \\
r\text{-irresolute} & \implies \text{g-open} \implies \alpha g\text{-open} \implies \text{gp-open} \implies \text{gpr-open}
\end{align*}
\]

### 3 GPR-CLOSED MAPPINGS

**Definition 3.01:** A function \( f: X \to Y \) is said to be gpr-closed if image of every closed set in \( X \) is gpr-closed in \( Y \).

**Note 2:** By note 1 and Definitions 2.02(iii) and 3.01 we have the following diagram.

\[
\begin{align*}
\text{Closed map} & \implies \text{g-closed map} \implies \alpha g\text{-closed map} \implies \text{gp-closed map} \implies \text{gpr-closed map} \\
r\text{-irresolute map} & \implies \text{g-closed map} \implies \alpha g\text{-closed map} \implies \text{gp-closed map} \implies \text{gpr-closed map}
\end{align*}
\]

However, we have the following converse part:

**Note 3:** If GPRC(Y) = RC(Y) we have the following implication diagram.

\[
\begin{align*}
\text{Closed map} & \iff \text{g-closed map} \iff \alpha g\text{-closed map} \iff \text{gp-closed map} \iff \text{gpr-closed map} \\
r\text{-irresolute map} & \iff \alpha g\text{-closed map} \iff \text{gp-closed map} \iff \text{gpr-closed map}
\end{align*}
\]

**Example 1:** Let \( X = Y = \{a, b, c\}; \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}. \) and \( \sigma = \{\emptyset, \{a\}, Y\} \)
and let \( f: X \to Y \) be defined as \( f(a) = b; f(b) = c; f(c) = a \), then \( f \) is \textit{gpr-closed}; \( \text{rg-closed but not g-closed}; \) \( \text{gp-closed; r-closed; closed; g\alpha\text{-closed}.} \)

**Theorem 3.01:** If \((Y, \sigma)\) is a discrete space, then \( f \) is closed of all types:

**Example 2:** Let \( X = Y = \{a, b, c\} \) and \( \tau = \{\phi, \{b\}, \{a, b\}, \{b, c\}, X\}; \sigma = \varnothing(Y) \) and let \( f: X \to Y \) be defined as \( f(a) = b; f(b) = a; f(c) = c \), then \( f \) is \textit{gpr-closed}; \( \text{rg-closed; g-closed; gp-closed; r-closed; closed.} \)

**Example 3:** Let \( X = Y = \{a, b, c\} \) and \( \tau = \varnothing(X); \sigma = \{\phi, \{b\}, \{a, b\}, \{b, c\}, Y\} \) and let \( f: X \to Y \) be defined as \( f(a) = c; f(b) = a; f(c) = b \), then \( f \) is \textit{gpr-closed}; \( \text{rg-closed; but not g-closed; r-closed; closed.} \)

**Theorem 3.02:** (i) If \( f \) is closed and \( g \) is \textit{gpr-closed} [\( \text{rg-closed; r-irresolute} \)] then \( g \circ f \) is \textit{gpr-closed}.

(ii) If \( f \) and \( g \) are \( \text{r-irresolute} \), then \( g \circ f \) is \textit{gpr-closed}.

(iii) If \( f \) is \( \text{r-irresolute and g is \textit{gpr-closed} [r-irresolute]} \), then \( g \circ f \) is \textit{gpr-closed}.

**Theorem 3.03:** If \( f: X \to Y \) is \textit{gpr-closed}, then \( \textit{gpr}(\text{cl}\{f(A)\}) \subset f(\text{cl}\{A\}) \)

**Proof:** Let \( A \subset X \) and \( f: X \to Y \) be \textit{gpr-closed} and \( \text{gpr}(\text{cl}\{f(A)\}) \subset f(\text{cl}\{A\}) \)

Combining (1) and (2) we have \( \textit{gpr}(\text{cl}\{f(A)\}) \subset (f(\text{cl}\{A\})) \) for every subset \( A \) of \( X \).

**Remark 1:** converse is not true in general, as shown by

**Example 4:** Let \( X = Y = \{a, b, c\}; \tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}; \sigma = \{\phi, \{a\}, \{c\}, \{a, c\}, Y\}; f: X, \tau \to Y, \sigma \) be the identity map then \( \textit{gpr} (\text{cl}\{f(A)\}) \subset f(\text{cl}\{A\}) \) for every subset \( A \) of \( X \) but \( f \) is not \textit{gpr-closed}. Since \( f(\{c\}) = \{c\} \) is not \textit{gpr-closed}.

**Theorem 3.04:** If \( f \) is \textit{gpr-closed}[\( \text{rg-closed} \) and \( A \subset \text{RC}(X) \), then \( f(A) \) is \( \tau_{\text{gpr-closed}} \). in \( Y \).

**Proof:** Let \( A \subset X \) and \( f: X \to Y \) be \textit{gpr-closed} and \( \textit{gpr}(\text{cl}\{f(A)\}) \subset f(\text{cl}\{A\}) \) which in turn implies \( \textit{gpr}(\text{cl}\{f(A)\}) \subset f(A) \) since \( f(A) = f(\text{cl}\{A\}) \). But \( f(A) \subset \textit{gpr}(\text{cl}\{f(A)\}) \). Combining we get \( f(A) = \textit{gpr}(\text{cl}\{f(A)\}) \). Hence \( f(A) \) is \( \tau_{\text{gpr-closed}} \) in \( Y \).

**Corollary 3.01:** (i) If \( f: X \to Y \) is \textit{rg-closed}, then \( \textit{gpr}(\text{cl}\{f(A)\}) \subset f(\text{cl}\{A\}) \)

(ii) If \( f: X \to Y \) is \( \text{r-closed} \), then \( \textit{gpr}(\text{cl}\{f(A)\}) \subset f(\text{cl}\{A\}) \)

(iii) If \( f: X \to Y \) is \( \text{r-closed} \), then \( f(A) \) is \( \tau_{\text{gpr-closed}} \) in \( Y \) if \( A \) is \( \text{closed [r-closed]} \) in \( X \).

**Theorem 3.05:** If \( \textit{rgcl}(A) = \textit{rgcl}(A) \) for every \( A \subset Y \), the following are equivalent:

(i) \( f: X \to Y \) is \textit{gpr-closed} map

(ii) \( \textit{gpr}(\text{cl}\{f(A)\}) \subset f(\text{cl}\{A\}) \)

**Proof:** (i) \( \Rightarrow \) (ii) follows from Theorem 3.3
Theorem 3.06: \( f: X \to Y \) is \( gpr \)-closed iff for each subset \( S \) of \( Y \) and each open set \( U \) containing \( f^{-1}(S) \), there is a \( gpr \)-open set \( V \) of \( Y \) such that \( S \subseteq V \) and \( f^{-1}(V) \subseteq U \).

**Proof:** If \( f \) is \( gpr \)-closed, \( S \subseteq Y \) and \( U \in \tau \), \( f^{-1}(S) \subseteq \text{GPRO}(Y) \) and \( V = Y \setminus f(X \setminus U) \subseteq \text{GPRO}(Y) \). Then \( f^{-1}(V) \subseteq X \setminus f(X \setminus U) = U \).

Conversely, let \( F \) be closed in \( X \), then \( f^{-1}(F^c) \subseteq F^c \). By hypothesis, there exists \( V \in \text{GPRO}(Y) \) such that \( f(F) \subseteq V \) and \( f^{-1}(V) \subseteq F \) and so \( F \subseteq (f^{-1}(V))^c \). Hence \( V^c \supseteq f(F^c) \subseteq (f^{-1}(V))^c \subseteq V^c \Rightarrow f(F) \subseteq V^c \Rightarrow f(F) = V^c \). Thus \( f(F) \) is \( gpr \)-closed in \( Y \) and therefore \( f \) is \( gpr \)-closed.

**Remark 2:** composition of two \( gpr \)-closed maps is not \( gpr \)-closed.

Theorem 3.07: Let \( X, Y, Z \) be topological spaces and every \( gpr \)-closed set is \( r \)-closed in \( Y \), then the composition of two \( gpr \)-closed maps is \( gpr \)-closed.

**Proof:** Let \( A \) be closed in \( X \) \( \Rightarrow f(A) \) is \( gpr \)-closed in \( Y \) \( \Rightarrow f(A) \) is regular-closed in \( Y \) [by assumption] \( \Rightarrow g(A) = gprf(A) \) is \( gpr \)-closed in \( Z \) \( \Rightarrow gf(A) \) is \( gpr \)-closed.

Corollary 3.02: If \( f \) is \( rg \)-closed; \( g \) is \( gpr \)-closed \([ rg \)-closed \)] and \( \text{rg} \)-closed in \( Y \) and \( Y \) is \( rT_{1/2} \), then \( g \cdot f \) is \( gpr \)-closed.

Theorem 3.09: If \( f, g \) be such that \( g \cdot f \) is \( gpr \)-closed \([ rg \)-closed \)]. Then the following are true

(i) If \( f \) is continuous \([ r \)-irresolute \] and surjective, then \( g \) is \( gpr \)-closed

(ii) If \( f \) is \( rg \)-continuous, surjective and \( X \) is \( rT_{1/2} \), then \( g \) is \( gpr \)-closed

Corollary 3.03: For \( f, g \) if \( g \cdot f \) is \( r \)-irresolute. Then the following are true

(i) If \( f \) is continuous \([ r \)-irresolute \] and surjective, then \( g \) is \( gpr \)-closed

(ii) If \( f \) is \( rg \)-continuous, surjective and \( X \) is \( rT_{1/2} \), then \( g \) is \( gpr \)-closed

Theorem 3.10: If \( X \) is \( gpr \)-regular, \( f: X \to Y \) is \( r \)-open, \( rg \)-continuous, \( gpr \)-closed surjection and \( cl(A) = A \) for every \( gpr \)-closed set in \( Y \), then \( Y \) is \( gpr \)-regular.

**Proof:** Let \( p \in U \subseteq \text{GPRO}(Y) \), \( x \in X \) \( \ni \mathcal{f}(x) = p \). Since \( X \) is \( gpr \)-regular and \( f \) is \( rg \)-continuous \( \exists V \subseteq \text{RO}(X) \ni x \in V \subseteq cl(V) \subseteq U \). \( \Rightarrow p \in f(V) \subseteq cl(V) \subseteq U \). \( f \) is \( gpr \)-closed, \( f(cl(V)) \subseteq U \) is \( gpr \)-closed and \( cl(f(cl(V))) = f(cl(V)) \) and \( cl(f(cl(V))) = cl(f(V)) \) \( \Rightarrow \). \( f(V) \subseteq cl(V) \subseteq U \). From (1) and (2) \( p \in f(V) \subseteq cl(V) \subseteq U \) and \( f(V) \) is \( rg \)-open. Hence \( Y \) is \( gpr \)-regular.

Corollary 3.04: If \( X \) is \( gpr \)-regular, \( f: X \to Y \) is \( r \)-open, \( rg \)-continuous, \( gpr \)-closed surjection and \( cl(A) = A \) for every \( rg \)-closed set in \( Y \), then \( Y \) is \( gpr \)-regular.
Theorem 3.11: (i) If \( f \) is \( \text{gpr}-\text{closed} \) \([\text{rg}-\text{closed}] \) and \( A \in \text{RC}(X) \), then \( f_A \) is \( \text{gpr}-\text{closed} \).
(ii) If \( f \) is \( \text{gpr}-\text{closed} \) \([\text{rg}-\text{closed}] \), \( X \) is \( T_{1/2} \) and \( A \in \text{RGC}(X) \), then \( f_A \) is \( \text{gpr}-\text{closed} \).

Corollary 3.05: (i) If \( f \) is r-closed and \( A \) is r-closed set of \( X \), then \( f_A \) is \( \text{gpr}-\text{closed} \).
(ii) If \( f \) is r-closed, \( X \) is \( T_{1/2} \) and \( A \) is rg-closed set of \( X \), then \( f_A \) is \( \text{gpr}-\text{closed} \).

Theorem 3.12: If \( f_i; X_i \rightarrow Y_i \) be \( \text{gpr}-\text{closed} \) \([\text{rg}-\text{closed}] \) for \( i = 1, 2 \). If \( f: X_1 \times X_2 \rightarrow Y_1 \times Y_2 \) is defined as \( f(x_1, x_2) = (f_1(x_1), f_2(x_2)) \). Then \( f: X_1 \times X_2 \rightarrow Y_1 \times Y_2 \) is \( \text{gpr}-\text{closed} \).

Proof: Let \( U_1 \times U_2 \subseteq X_1 \times X_2 \) where \( U_i \in \text{RC}(X_i) \) for \( i = 1, 2 \). Then \( f(U_1 \times U_2) = f_1(U_1) \times f_2(U_2) \in \text{GPRC}(Y_1 \times Y_2) \). Hence \( f \) is \( \text{gpr}-\text{closed} \).

Theorem 3.13: Let \( h: X \rightarrow Y \) be \( \text{gpr}-\text{closed} \)[rg-closed]. Let \( f_i; X \rightarrow X_i \) be defined as: \( h(x) = (x_1, x_2) \) and \( f_i(x) = x_i \). Then \( f_i; X \rightarrow X_i \) is \( \text{gpr}-\text{closed} \) for \( i = 1, 2 \).

Proof: Let \( U_i \times X_2 \in \text{RC}(X_1 \times X_2) \), then \( f_i(U_i) = h(U_i \times X_2) \in \text{GPRC}(X_i) \), therefore \( f_i \) is \( \text{gpr}-\text{closed} \). Similarly we have \( f_2 \) is \( \text{gpr}-\text{closed} \) and thus \( f \) is \( \text{gpr}-\text{closed} \) for \( i = 1, 2 \).

Corollary 3.06: (i) If \( f_i; X_i \rightarrow Y_i \) be r-closed for \( i = 1, 2 \). Let \( f_i; X_i \times X_2 \rightarrow Y_1 \times Y_2 \) be defined as follows: \( f(x_1, x_2) = (f_1(x_1), f_2(x_2)) \). Then \( f_i; X_i \times X_2 \rightarrow Y_1 \times Y_2 \) is \( \text{gpr}-\text{closed} \).
(ii) Let \( h; X \rightarrow X_1 \times X_2 \) be r-closed. Let \( f_i; X \rightarrow X_i \) be defined as: \( h(x) = (x_1, x_2) \) and \( f_i(x) = x_i \). Then \( f_i; X \rightarrow X_i \) is \( \text{gpr}-\text{closed} \) for \( i = 1, 2 \).

4 \text{ GPR-OPEN MAPPINGS}

Definition 4.01: A function \( f; X \rightarrow Y \) is said to be \( \text{gpr}-\text{open} \) if image of every open set in \( X \) is \( \text{gpr}-\text{open} \) in \( Y \).

Note 4: By note 1 and Definitions 2.02(iv) and 4.01 we have the following diagram.

\[
\text{Open map} \rightarrow \text{g-open map} \rightarrow \text{ag-open map} \rightarrow \text{gp-open map} \rightarrow \text{gpr-open map}
\]

However, we have the following converse part:

Note 5: Converse is true if GPRO\( (Y) = \text{RO}(Y) \)

\[
\text{Open map} \leftrightarrow \text{g-open map} \leftrightarrow \text{ag-open map} \leftrightarrow \text{gp-open map} \leftrightarrow \text{gpr-open map}
\]

\[
\text{r-irresolute map} \leftrightarrow \leftrightarrow \leftrightarrow \leftrightarrow \leftrightarrow \leftrightarrow \leftrightarrow \leftrightarrow \leftrightarrow \leftrightarrow \leftrightarrow \leftrightarrow \leftrightarrow \text{rg-open map}
\]

Example 5: Let \( X = Y = \{a, b, c\}; \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\} \) and \( \sigma = \{\emptyset, \{a\}, Y\} \) and let \( f; X \rightarrow Y \) be defined as \( f(a) = b; f(b) = c; f(c) = a \), then \( f \) is \( \text{gpr-open} \); rg-open but not g-open; gp-open; r-open; open; ag-open.
**Theorem 4.01:** If \((Y, \sigma)\) is a discrete space, then \(f\) is open of all types:

**Example 6:** Let \(X = Y = \{a, b, c\}\) and \(\tau = \{\phi, \{b\}, \{a, b\}, \{b, c\}, X\}; \sigma = \phi(Y)\) and let \(f : X \rightarrow Y\) be defined as \(f(a) = b; f(b) = a; f(c) = c\), then \(f\) is gpr-open; rg-open; g-open; gp-open; \(\alpha\)g-open; r-open; open.

**Theorem 4.05:** If \(X = Y = \{a, b, c\}\) and \(\tau = \{\phi, \{b\}, \{a, b\}, \{b, c\}, Y\}\) and let \(f : X \rightarrow Y\) be defined as \(f(a) = c; f(b) = a; f(c) = b\), then \(f\) is gpr-open; rg-open; but not g-open; r-open; open.

**Theorem 4.02:** (i) If \(f\) is open and \(g\) is gpr-open[rg-open; r-open] then \(gf\) is gpr-open
(ii) If \(f\) and \(g\) are r-irresolute then \(gf\) is gpr-open
(iii) If \(f\) is r-irresolute and \(g\) is gpr-open then \(gf\) is gpr-open

**Theorem 4.03:** If \(f : X \rightarrow Y\) is gpr-open, then \(f(A^o) \subset gpr\left(\{f(A)\}^o\right)\)

**Proof:** Let \(A \subset X\) and \(f : X \rightarrow Y\) is gpr-open gives \(f(A^o)\) is gpr-open in \(Y\) and \(f(A^o) \subset f(A)\) which in turn gives \(gpr(f(A)\}^o) \subset gpr(\{f(A)\}^o)\) \(------------------------------------(1)\)

Since \(f(\{A\}^o)\) is gpr-open in \(Y\), \(gpr(f(\{A\}^o)) = f(\{A\}^o)\) \(------------------------------------(2)\)

Combining (1) and (2) we have \(f(A^o) \subset gpr(\{f(A)\}^o)\) for every subset \(A\) of \(X\).

**Remark 3:** converse is not true in general, as shown by

**Example 8:** Let \(X = Y = \{a, b, c\}; \tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}, \sigma = \{\phi, \{a\}, \{c\}, \{a, c\}, Y\}\) \(f(X, \tau) \rightarrow (Y, \sigma)\) be the identity map then \(f(A^o) \subset gpr(\{f(A)\}^o)\) for every subset \(A\) of \(X\) but \(f\) is not gpr-open. Since \(f(\{a, b\}) = \{a, b\}\) is not gpr-open.

**Theorem 4.04:** If \(f\) is gpr-open[rg-open] and \(A \in RO(X)\), then \(f(A)\) is \(\tau_{gpr}\) open in \(Y\).

**Proof:** Let \(A \subset X\) and \(f : X \rightarrow Y\) is gpr-open \(\Rightarrow gpr(\{f(A)\}^o) \subset f(A)\) which in turn implies gpr(\{f(A)\}^o) \(\subset f(A)\), since \(f(A) = f(\{A\}^o)\). But \(f(A) \subset gpr(\{f(A)\}^o)\). Combining we get \(f(A) = gpr(\{f(A)\}^o)\). Hence \(f(A)\) is \(\tau_{gpr}\) open in \(Y\).

**Corollary 4.01:** (i) If \(f : X \rightarrow Y\) is rg-open, then \(f(\{A\}^o) \subset gpr(\{f(A)\}^o)\)
(ii) If \(f : X \rightarrow Y\) is r-open, then \(f(\{A\}^o) \subset gpr(\{f(A)\}^o)\)
(iii) If \(f : X \rightarrow Y\) is r-open, then \(f(A)\) is \(\tau_{gpr}\) open in \(Y\) if \(A\) is open[rg-open] set in \(X\).

**Theorem 4.05:** If \(gpr(\{A\}^o) = rg(\{A\}^o)\) for every \(A \subset Y\), the following are equivalent:
(i) \(f : X \rightarrow Y\) is gpr-open map
(ii) \(f(\{A\}^o) \subset gpr(\{f(A)\}^o)\)

**Theorem 4.06:** \(f : X \rightarrow Y\) is gpr-open iff for each subset \(S\) of \(Y\) and each open set \(U\) containing \(f^{-1}(S)\), there is a gpr-open set \(V\) of \(Y\) such that \(S \subset V\) and \(f^{-1}(V) \subset U\).

**Proof:** Assume \(f\) is gpr-open, \(S \subset Y\) and \(U \in (\tau, f^{-1}(S))\), then \(f(U) \in GPRO(Y)\) and \(V = \)
Y \circ f(X - U) \in \text{GPRO}(Y). f^{-1}(S) \subseteq U \Rightarrow S \subseteq V \text{ and } f^{-1}(V) = X - f^{-1}(f(X - U)) = X - (X - U) = U.

Conversely let \( F \in \tau \), then \( f^{-1}(f(F)) \subseteq F \) and there exists \( V \in \text{GPRO}(Y) \) such that \( f(F) \subseteq V \) and \( f^{-1}(V) \subseteq F \) and so \( F \subseteq f^{-1}(V)^c \). Hence \( V^c \subseteq f \circ f^{-1}(V)^c \subseteq V^c \Rightarrow f(F) \subseteq V^c \Rightarrow f(F) = V^c \). Thus \( f(F) \in \text{GPRO}(Y) \) and therefore \( f \) is \( \text{gpr} \)-open.

**Remark 4:** composition of two \( \text{gpr} \)-open maps is not \( \text{gpr} \)-open.

**Theorem 4.07:** Let \( X, Y, Z \) be topological spaces and every \( \text{gpr} \)-open set is \( r \)-open in \( Y \), then the composition of two \( \text{gpr} \)-open maps is \( \text{gpr} \)-open.

**Proof:** Let \( A \) be \( r \)-open in \( X \Rightarrow f(A) \) is \( \text{gpr} \)-open in \( Y \Rightarrow f(A) \) is \( r \)-open in \( Y \) [by assumption] \( \Rightarrow g(f(A)) \) is \( \text{gpr} \)-open in \( Z \Rightarrow g \circ f(A) \) is \( \text{gpr} \)-open in \( Z \Rightarrow g \circ f \) is \( \text{gpr} \)-open.

**Theorem 4.08:** If \( f \) is \( r \)-regular; \( g \) is \( \text{gpr} \)-open[\( \text{rg} \)-open] and \( Y \) is \( rT_{1/2} \), then \( g \circ f \) is \( \text{gpr} \)-open.

**Proof:**
(i) Let \( A \) be \( r \)-open in \( X \Rightarrow f(A) \) is \( \text{rg} \)-open in \( Y \Rightarrow f(A) \) is \( r \)-open in \( Y \) [since \( Y \) is \( rT_{1/2} \)] \( \Rightarrow g(f(A)) \) is \( \text{gpr} \)-open in \( Z \Rightarrow g \circ f(A) \) is \( \text{gpr} \)-open in \( Z \Rightarrow g \circ f \) is \( \text{gpr} \)-open.

(ii) Since every \( g \)-open set is \( r \)-open, this part follows from the above case.

**Corollary 4.02:** If \( f \) is \( \text{gopen}[\text{rg-open}] \); \( g \) is \( r \)-open and \( Y \) is \( T_{1/2} \{rT_{1/2}\} \), then \( g \circ f \) is \( \text{gopen} \).

**Theorem 4.09:** If \( f \) and \( g \) be two mappings such that \( g \circ f \) is \( \text{gpr-open}[\text{rg-open}] \). Then the following are true
(i) If \( f \) is continuous[\( \text{r-irresolute} \)] and surjective, then \( g \) is \( \text{gpr-open} \)
(ii) If \( f \) is \( r \)-continuous, surjective and \( X \) is \( rT_{1/2} \), then \( g \) is \( \text{gpr-open} \)

**Corollary 4.03:** If \( f; g \) be such that \( g \circ f \) is \( r \)-open. Then the following are true
(i) If \( f \) is \( \text{continuous}[\text{r-continuous}] \) and surjective, then \( g \) is \( \text{gpr-open} \)
(ii) If \( f \) is \( \text{rg-continuous}, \text{surjective and } X \) is \( rT_{1/2} \), then \( g \) is \( \text{gpr-open} \)

**Theorem 4.10:** If \( X \) is \( \text{gpr-regular}, f:X \rightarrow Y \) is \( r \)-open, \( \text{rg-continuous}, \text{gpr-open} \) surjection and \( A' = A \) for every \( \text{gpr-open} \) set in \( Y \), then \( Y \) is \( \text{gpr-regular} \).

**Proof:** Let \( p \in \text{GPRO}(Y) \), \( \exists \ x \in X \ni f(x) = p \). Since \( X \) is \( \text{gpr-regular} \) and \( f \) is \( \text{rg-continuous} \), \( \exists \ V \in \text{GRO}(X) \ni x \in V \subseteq V \subseteq f^{-1}(U) \) which implies \( p \in f(V) \subseteq f(U) \). (1)
Since \( f \) is \( \text{gpr-open} \), \( f(V) \subseteq U \) is \( \text{gpr-open} \) and \( \{f(V)^c\}^c = f(V^c) \) and \( \{f(V^c)^c\} = f(V)^c \). (2)
From (1) and (2) \( p \in \{f(V)^c\}^c \subseteq f(V^c) \subseteq U \) and \( f(V) \) is \( \text{rg-open} \). Hence \( Y \) is \( \text{gpr-regular} \).

**Corollary 4.04:** If \( X \) is \( \text{gpr-regular}, f:X \rightarrow Y \) is \( r \)-open, \( \text{rg-continuous}, \text{gpr-open} \) surjection and \( A' = A \) for every \( \text{rg-open} \) set in \( Y \), then \( Y \) is \( \text{gpr-regular} \).

**Theorem 4.11:**
(i) If \( f \) is \( \text{gpr-open}[\text{rg-open}] \) and \( A \in \text{RO}(X) \), then \( f_A \) is \( \text{gpr-open} \).
(ii) If \( f \) is \( \text{gpr-open} \), \( X \) is \( rT_{1/2} \) and \( A \in \text{RGO}(X) \), then \( f_A \) is \( \text{gpr-open} \).
Corollary 4.05: (i) If \( f \) is r-open and \( A \) is r-open set of \( X \), then \( f_{A} \) is gpr-open.
(ii) If \( f \) is r-open, \( X \) is r\(T_{1/2}\) and \( A \) is rg-open set of \( X \), then \( f_{A} \) is gpr-open.

Theorem 4.12: If \( f_{i}:X \rightarrow Y_{i} \) be gpr-open[rg-open] for \( i = 1, 2 \). If \( f:X_{1} \times X_{2} \rightarrow Y_{1} \times Y_{2} \) is defined as \( f(x_{1}, x_{2}) = (f_{1}(x_{1}), f_{2}(x_{2})) \). Then \( f:X_{1} \times X_{2} \rightarrow Y_{1} \times Y_{2} \) is gpr-open.

Theorem 4.13: Let \( h: X \rightarrow X_{1} \times X_{2} \) be gpr-open[rg-open]. Let \( f_{i}: X \rightarrow X_{i} \) be defined as: \( h(x) = (x_{1}, x_{2}) \) and \( f(x) = x_{i} \). Then \( f_{i}: X \rightarrow X_{i} \) is gpr-open for \( i = 1, 2 \).

Corollary 4.06: (i) If \( f_{i}: X_{i} \rightarrow Y_{1} \) be r-open for \( i = 1, 2 \). Let \( f_{i}:X_{1} \times X_{2} \rightarrow Y_{1} \times Y_{2} \) be defined as follows: \( f(x_{1}, x_{2}) = (f_{1}(x_{1}), f_{2}(x_{2})) \). Then \( f:X_{1} \times X_{2} \rightarrow Y_{1} \times Y_{2} \) is gpr-open.
(ii) Let \( h: X \rightarrow X_{1} \times X_{2} \) be r-open. Let \( f_{i}: X \rightarrow X_{i} \) be defined as: \( h(x) = (x_{1}, x_{2}) \) and \( f(x) = x_{i} \). Then \( f_{i}: X \rightarrow X_{i} \) is gpr-open for \( i = 1, 2 \).

References


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