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Group Connectivity of Graph with Odd Cycle

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Abstract. Let G be an undirected graph, A be an (additive) abelian group and $A^* = A - \{0\}$. A graph G is A-connected if G has an orientation $D(G)$ such that for every function $b: V(G) \to A$ satisfyin $v \in V(G), b(v) = 0$, there is a function $f: E(G) \to A^*$ such that at each vertex $v \in V(G)$, $\sum_{e \in E_D^+(v)} f(e) - \sum_{e \in E_D^-(v)} f(e) = b(v)$. In this study, we proved that if G has an odd cycle C and for every vertex $v \in V(G), d_C(v) = 3$, then G has no $Z_3 - NZF$. Furthermore, we proposed a few applications of this result.

Keywords: integer flow; group connectivity; odd cycle

1 Introduction

The graphs in this paper are finite and may have multiple edges and loops. The terms and notations not defined here are from [1].

Let $D = D(G)$ be an orientation of a graph G. If an edge $e \in E(G)$ is directed from a vertex u to a vertex v, then let $tail(e) = u$ and $head(e) = v$. We call e an out-edge of u and out-edge of u and an in-edge of v. For a vertex $v \in V(G)$, let

$$E_D^-(v) = \{e \in E(D) : v = tail(e)\}, \text{ and } E_D^+(v) = \{e \in E(D) : v = head(e)\}.$$

We write D for $D(G)$ when its meaning can be understood from the context.

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Let A denote an (additive) abelian group where the identity of A is denoted by 0. Let A^* denote the set of nonzero elements of A. We define

$$F(G,A)=\{f:E(G)\mapsto \quad \text{and} \quad F^*(G,A)=\{f:E(G)\mapsto$$

Given a function $f\in F(G,A)$, define $\partial f:V(G)\mapsto$ by

$$\partial f=\sum_{e\in E_D^+(v)} f(e)-\sum_{e\in E_D^-(v)} f(e),$$

Where "Σ" refers to the addition in A.

Group connectivity was introduced by Jaeger *et al.* [3] as a generalization of nowhere-zero flows. For a graph G, a function $b:V(G)\mapsto$ is called an A-valued zero sum function on G if $\sum_{v\in V(G)} b(v)=0$. The set of all A-valued zero sum functions on G is denoted by $Z(G,A)$. Given $b\in Z(G,A)$, a function $f\in F^*(G,A)$ is called an (A,b)-nowhere-zero flow(abbreviated as $(A,b)-NZF$) if G has an orientation $D(G)$ such that $\partial f=b$. A graph G is A-connected if for any $b\in Z(G,A)$, G has an (A,b)-nowhere-zero flow. In particular, G admits an A-nowhere-zeros flow(abbreviated as an $A-NZF$) if G has an $(A,0)$-nowhere-zero flow. G admits a nowhere-zero k-flow if G admits a nowhere-zero Z_k-flow(abbreviated as an $k-NZF$), where Z_k is a cyclic group of order k. Tutte [8] proved that G admits a $A-NZF$ with $|A|=k$ if and only if G admits a $k-NZF$. One notes that if a graph G is A-connected and $|A|\geq k$, then G admits a $k-NZF$. Generally speaking, when G admits a $k-NZF$, G may not be A-connected with $|A|\geq k$. For example, a n-cycle is A-connected if and only if $|A|\geq n+1$ given in [6, Lemma 3.3] while for any n, a n-cycle admits a $2-NZF$. Thus, group connectivity generalizes nowhere-zero flows.

For an abelian group A,let A be the family of graphs that are A-connected .It is observed in [3] that the property $G \in A$ is independent of the orientation of G, and that every graph in $\langle A \rangle$ is 2-edge-connected.

The nowhere-zero flow problems were introduced by Tutte in [6, 7, 8] and surveyed by Jaeger in [3] and Zhang in [10]. The following conjecture is due to Tutte. Partial results of this conjecture can be found in [3] and others. However, it is still open.

Conjecture 1.1 (4-flow conjecture, [7]) Every bridgeless graph containing no subdivision of the Petersen graph admits a nowhere-zero 4-flow.

For a 2-edge-connected graph G, define the flow number of G as

$$\Lambda(G)=\min\{k : \text{if } G \text{ has a } k\text{-}NZF\}$$

and the group connectivity number of G as

$$\Lambda g(G)=\min\{k : \text{if } A \text{ is an abelian group with } |A| \geq k\text{, then } G \in A\}$$

If G is 2-edge-connected, then $\Lambda(G)$ and $\Lambda g(G)$ exist as finite numbers, and $\Lambda(G) \leq \Lambda g(G)$.

Some of the known results will be presented below which will be utilized in our proofs.

Let G be a graph and let $X \subseteq E(G)$ be an edge subset. The contraction [2] G/X is the graph obtained from G by identifying the two ends of each edge e in X and deleting e. If $X=\{e\}$,then we write G/e for $G/\{e\}$. If H is a subgraph of G, then we write G/H for $G/E(H)$. Note that even G is a simple graph, the contraction G/X may have multiple edges and (or) loops.

Lemma 1.2 ([4]) Let A be an abelian group, then each of the following statements holds:

(1) $K_1 \in \langle A \rangle$;

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(2) If $G \in \langle A \rangle$ and $e \in E(G)$, then $G/e \in \langle A \rangle$;

(3) If H is a sub-graph of G, and if $H \in \langle A \rangle$ and $G/H \in \langle A \rangle$, then $G \in \langle A \rangle$.

Lemma 1.3 ([3], [4]) $C_n \in \langle A \rangle$ if and only if $|A| \geq n+1$, where C_n is a n-cycle.

Lemma 1.4 ([3]) Let G be a connected graph with n vertices and m edges, then $\mathrm{Lg}(G)=2$. If and only if $n=1$ (G has m loops).

Lemma 1.5 [5] Let T be a connected spanning subgraph of G. If for each edge $e \in E(T)$, G has a subgraph H_e such that $e \in E(H_e)$ and $H_e \in \langle A \rangle$, then $G \in \langle A \rangle$.

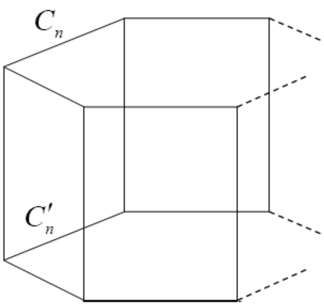


Figure 1: Graph x_n

Let C_n and C'_n are two copies of n-cycles($n \geq 3$).The graph obtained by connecting each vertex in C_n to a vertex in C'_n with a new edge in a certain order is called a Column, denoted as x_n (Shown as Figure 1). Obviously, x_n is a 3-regular graph (Shown as Figure 1).

Lemma 1.6 [11] $\mathrm{Lg}(x_n)=4(n \geq 3)$.

Let G be a graph and $v \in V(G)$ be a vertex of degree $m \geq 4$. Suppose $N(v) = v_1, v_2, \cdots$ [illegible] and $X = \{vv_1, vv_2\}$. The graph $G_{[v,X]}$ is obtained from $G - X$ by adding a new edge that joins v_1 and v_2 .

Lemma 1.7 [4] Let A be an Abelian group. Let G be a graph and let be a vertex of $v \in V(G)$ degree $m \geq 4$. If for some X of two edges incident with v in G, $G_{[v,X]} \in \langle A \rangle$, then $G \in \langle A \rangle$.

2 Main Results

Theorem 2.1 Let G has a odd cycle C and for every vertex $v \in V(G)$, $d_C(v) = 3$, then G has no Z_3 - NZF.

Proof By contradiction. If G has a Z_3 - NZF, there is a function $f \in F^*(G, Z_3)$, such that $\partial f = 0$. Suppose that $C = v_1 e_1 v_2 e_2 \cdots$ [illegible] $v_{2k+2} (= v_1)$ and denote the edge that is correlative with v_i and does not emerge in C as $e_i^{'}$. Suppose the direction of e_i in $D(G)$ is from v_i to v_{i+1} , and v_i is the tail of edge $e_i^{'}$ in $D(G)$. For every $i (1 \leq i \leq 2k)$, $f(e_i) \neq f(e_{i+1})$, for otherwise, by

$$\sum_{e \in N^+(v_{i+1})} f(e) - \sum_{e \in N^-(v_{i+1})} f(e) = f(e_{i+1}) + f(e_{i+1}^{'}) - f(e_i) = 0$$

we know that $f(e_i^{'}) = 0$, which is contradicted to the assumption that $f \in F^*(G, Z_3)$. In addition, because the value of $f(e)$ is only 1 or 2, $f(e_1) = f(e_3) = \cdots$ [illegible]) Thus, by

$$\sum_{e \in N^+(v_1)} f(e) - \sum_{e \in N^-(v_1)} f(e) = f(e_1) + f(e_1^{'}) - f(e_{2k+1}) = 0$$

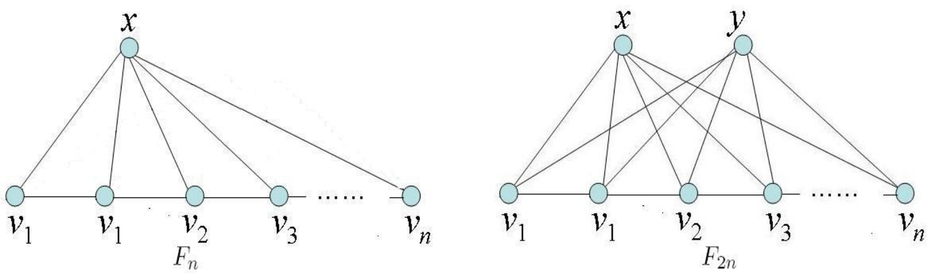
we know that $f(e_1^{'})=0$, which is also contradicted to $f \in F^*(G,Z_3)$. So the assumption is wrong.

Let $C_n^1, C_n^2, \cdots$ are m copies of n-copies of n-cycles. The graph obtained by connecting each vertex in C_i to a vertex in C_{i+1} with a new edge in a certain order is called a Cone, denoted as $V(m,n)$. From the definition we know that $x_n \cong V(2,n)$.

Corollary 2.2 $\mathrm{Lg}(V_{2k+1,n})=4 (k \in Z)$.

Proof By theorem 2.1, we conclude that $\mathrm{Lg}(V_{2k+1,n})>3$. Since every edge of $V(2k+1,n)$ lies in a x_n, we conclude that $\mathrm{Lg}(V_{2k+1,n}) \le 4$ by lemma 1.5 and 1.6. So $\mathrm{Lg}(V_{2k+1,n})=4$.

A single fan F_n is a graph obtained from a n-path ($n \ge 2$) $v_1, v_2, \cdots$ by adding a new vertex x and then joining the new vertex to all vertices in the path. This new vertex x is called the center of F_2 . A double fan F_{2n} is a graph obtained from a n-path ($n \ge 2$) $v_1, v_2, \cdots$ by adding two new vertexes x and y and then joining these two vertexes to all the vertices in the path. These new vertices x and y are called the centers of F_{2n} (Shown as Figure 2).



Theorem 2.3 $\mathrm{Lg}(F_{2n})=3$.

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Proof Since every edge of F_{2n} lies on a 3-cycle, by Lemma 1.3 and Lemma 1.5, $\mathrm{Lg}(F_{2n})$ £ 4 .For $d(x)$ ³ 4 , let be the graph obtained by adding a new edge in F_{2n} - $\{(xv_1, xv_2)\}$ that connecting xv_1 and xv_2. Contracting the 2-cycle in H, there is still a 2-cycle. Continue this process, we can obtain a graph which has two vertices with several multiple edges. By Lemma 1.2(3), we know that H Î $\langle Z_3 \rangle$, and by Lemma 1.7, F_{2n} Î $\langle Z_3 \rangle$. By lemma 1.4, we can conclude that $\mathrm{Lg}(F_{2n}) = 3$.

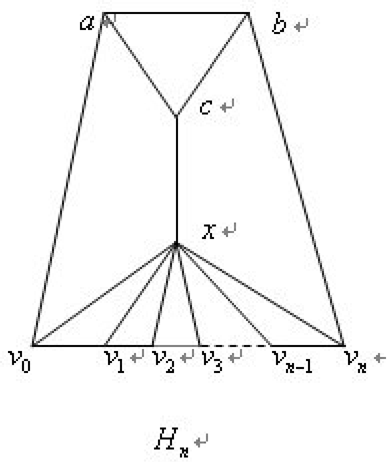

Figure 3: Graph H_n

The graph $H_n = F_n$ Å C_3 (Shown as Figure 3) is obtained from F_n and C_3 by adding three edges which it is cx, av_0, bv_n .

Corollary 2.4 $\mathrm{Lg}(H_n) = 4$.

Proof Proof since Hn has a odd cycle such that every vertex in it has degree 3, so by Theorem 2.1,Hn has no Z3 − N ZF . Thus $\Lambda g\,(H_n) \geq 4$. By contracting cycle C = abca, every edges of H_n/C lies in a 3-cycles,so by lemma 1.2 and lemma 1.4, we conclude that $\Lambda g\,(H_n) \leq 4$. The conclusion is established.

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