Group Connectivity of Graph with Odd Cycle

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Abstract. Let $G$ be an undirected graph, $A$ be an (additive) abelian group and $A' = A - \{0\}$. A graph $G$ is $A$-connected if $G$ has an orientation $D(G)$ such that for every function $b : V(G) \otimes A$ satisfying $v \in V(G), b(v) = 0$, there is a function $f : E(G) \otimes A'$ such that at each vertex $v \in V(G)$, $\sum_{e \in E_d(v)} f(e) - \sum_{e \in E_o(v)} f(e) = b(v)$.

In this study, we proved that if $G$ has an odd cycle $C$ and for every vertex $v \in V(G), d_c(v) = 3$, then $G$ has no $Z_3$-NZF. Furthermore, we proposed a few applications of this result.

Keywords: integer flow; group connectivity; odd cycle

1 Introduction

The graphs in this paper are finite and may have multiple edges and loops. The terms and notations not defined here are from [1].

Let $D = D(G)$ be an orientation of a graph $G$. If an edge $e \in E(G)$ is directed from a vertex $u$ to a vertex $v$, then let $\text{tail}(e) = u$ and $\text{head}(e) = v$. We call $e$ an out-edge of $u$ and out-edge of $u$ and an in-edge of $v$. For a vertex $v \in V(G)$, let

$E_d(v) = \{e \in E(D) : v = \text{tail}(e)\}$, and $E_o(v) = \{e \in E(D) : v = \text{head}(e)\}$.

We write $D$ for $D(G)$ when its meaning can be understood from the context.

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Let \( A \) denote an (additive) abelian group where the identity of \( A \) is denoted by 0. Let \( A^* \) denote the set of nonzero elements of \( A \). We define

\[
F(G, A) = \{ f : E(G) \mapsto A \}
\]
and

\[
F^*(G, A) = \{ f : V(G) \mapsto A^* \}
\]

Given a function \( f \in F^*(G, A) \), define \( \| f \| : V(G) \mapsto A \) by

\[
\| f \| = \sum_{d \in F(G)} f(e) - \sum_{d \in F(G)} f(e),
\]

Where “+” refers to the addition in \( A \).

Group connectivity was introduced by Jaeger et al. [3] as a generalization of nowhere-zero flows. For a graph \( G \), a function \( b : V(G) \mapsto A \) is called an \( A \)-valued zero sum function on \( G \) if \( \sum_{d \in F(G)} b(v) = 0 \). The set of all \( A \)-valued zero sum functions on \( G \) is denoted by \( Z(G, A) \). Given \( b \in Z(G, A) \), a function \( f \in F^*(G, A) \) is called an \((A, b)\)-nowhere-zero flow (abbreviated as \((A, b)\)-NZF) if \( G \) has an orientation \( D(G) \) such that \( \| f \| = b \). A graph \( G \) is \( A \)-connected if for any \( b \in Z(G, A) \), \( G \) has an \((A, b)\)-nowhere-zero flow. In particular, \( G \) admits an \( A \)-nowhere-zero flow (abbreviated as an \( A \)-NZF) if \( G \) has an \((A, 0)\)-nowhere-zero flow. \( G \) admits a nowhere-zero \( k \)-flow if \( G \) admits a nowhere-zero \( Z_k \)-flow (abbreviated as a \( k \)-NZF), where \( Z_k \) is a cyclic group of order \( k \). Tutte [8] proved that \( G \) admits an \( A \)-NZF with \(|A| = k \) if and only if \( G \) admits a \( k \)-NZF. One notes that if a graph \( G \) is \( A \)-connected and \(|A| \geq k \), then \( G \) admits a \( k \)-NZF. Generally speaking, when \( G \) admits a \( k \)-NZF, \( G \) may not be \( A \)-connected with \(|A| \geq k \). For example, a \( n \)-cycle is \( A \)-connected if and only if \(|A| \geq n + 1 \) given in [6, Lemma 3.3] while for any \( n \), a \( n \)-cycle admits a \( 2 \)-NZF. Thus, group connectivity generalizes nowhere-zero flows.
For an abelian group $A$, let $\mathcal{A}$ be the family of graphs that are $A$-connected. It is observed in [3] that the property $G \hat{A}$ is independent of the orientation of $G$, and that every graph in $\langle A \rangle$ is 2-edge-connected.

The nowhere-zero flow problems were introduced by Tutte in [6, 7, 8] and surveyed by Jaeger in [3] and Zhang in [10]. The following conjecture is due to Tutte.

Conjecture 1.1 (4-flow conjecture, [7]) Every bridgeless graph containing no subdivision of the Petersen graph admits a nowhere-zero 4-flow.

For a 2-edge-connected graph $G$, define the flow number of $G$ as

$$L(G) = \min\{k : G \text{ has a } k \text{-NZF} \}$$

and the group connectivity number of $G$ as

$$L_g(G) = \min\{k : A \text{ is an abelian group with } |A|^3 k, \text{ then } G \hat{A} \}$$

If $G$ is 2-edge-connected, then $L(G)$ and $L_g(G)$ exist as finite numbers, and $L(G) \leq L_g(G)$.

Some of the known results will be presented below which will be utilized in our proofs.

Let $G$ be a graph and let $X \subseteq E(G)$ be an edge subset. The contraction [2] $G/X$ is the graph obtained from $G$ by identifying the two ends of each edge $e$ in $X$ and deleting $e$. If $X = \{e\}$, then we write $G/e$ for $G/\{e\}$. If $H$ is a subgraph of $G$, then we write $G/H$ for $G/E(H)$. Note that even $G$ is a simple graph, the contraction $G/X$ may have multiple edges and (or) loops.

Lemma 1.2 ([4]) Let $A$ be an abelian group, then each of the following statements holds:

(1) $K_1 \hat{\langle A \rangle}$;
(2) If $G \hat{\Gamma} \langle A \rangle$ and $e \hat{\Gamma} E(G)$, then $G/e \hat{\Gamma} \langle A \rangle$.

(3) If $H$ is a sub-graph of $G$, and if $H \hat{\Gamma} \langle A \rangle$ and $G/H \hat{\Gamma} \langle A \rangle$, then $G \hat{\Gamma} \langle A \rangle$.

Lemma 1.3 ([3], [4]) $C_n \hat{\Gamma} \langle A \rangle$ if and only if $|A|^3 = n + 1$, where $C_n$ is a $n$-cycle.

Lemma 1.4 ([3]) Let $G$ be a connected graph with $n$ vertices and $m$ edges, then $\log_2(G) = 2$. If and only if $n = 1$ ($G$ has $m$ loops).

Lemma 1.5 [5] Let $T$ be a connected spanning subgraph of $G$. If for each edge $e \hat{\Gamma} E(T)$, $G$ has a subgraph $H_e$ such that $e \hat{\Gamma} E(H_e)$ and $H_e \hat{\Gamma} \langle A \rangle$, then $G \hat{\Gamma} \langle A \rangle$.

Figure 1: Graph $x_n$

Let $C_n$ and $C'_n$ are two copies of $n$-cycles $(n \geq 3)$. The graph obtained by connecting each vertex in $C_n$ to a vertex in $C'_n$ with a new edge in a certain order is called a Column, denoted as $x_n$ (Shown as Figure 1). Obviously, $x_n$ is a 3-regular graph (Shown as Figure 1).

Lemma 1.6 [11] $\log(x_n) = 4(n - 3)$. 
Let $G$ be a graph and $v \vert V(G)$ be a vertex of degree $m \geq 4$. Suppose $N(v) = v_1, v_2, \ldots$ and $X = \{vv_1, vv_2\}$. The graph $G_{[v,X]}$ is obtained from $G - X$ by adding a new edge that joins $v_1$ and $v_2$.

**Lemma 1.7** [4] Let $A$ be an Abelian group. Let $G$ be a graph and let $v$ be a vertex of $\vert V(G)\vert$ degree $m \geq 4$. If for some $X$ of two edges incident with $v$ in $G$, $G_{[v,X]} \langle A \rangle$, then $G \langle A \rangle$.

### 2 Main Results

**Theorem 2.1** Let $G$ has a odd cycle $C$ and for every vertex $v \vert V(G)$, $d_C(v) = 3$, then $G$ has no $Z_3 - NZF$.

**Proof** By contradiction. If $G$ has a $Z_3 - NZF$, there is a function $f : F(G, Z_3)$, such that $\|f = 0$. Suppose that $C = v_1v_2v_3\ldots v_{2k+1}(= v_1)$ and denote the edge that is correlative with $v_i$ and does not emerge in $C$ as $e'_i$.

Suppose the direction of $e_i$ in $D(G)$ is from $v_i$ to $v_{i+1}$, and $v_i$ is the tail of edge $e'_i$ in $D(G)$. For every $i(1 \leq i \leq 2k)$, $f(e_i)$, $f(e'_i)$, for otherwise, by

$$\hat{\alpha} \left( \sum_{d N(v)} f(e) - \sum_{d N(v)} f(e_1) + f(e'_i) - f(e) = 0 \right)$$

we know that $f(e'_i) = 0$, which is contradicted to the assumption that $f : F(G, Z_3)$. In addition, because the value of $f(e)$ is only 1 or 2,

$$f(e_1) = f(e_2) = \ldots$$

Thus, by

$$\hat{\alpha} \left( \sum_{d N(v)} f(e) - \sum_{d N(v)} f(e_1) + f(e'_i) - f(e_2k+i) = 0 \right)$$


we know that \( f(e') = 0 \), which is also contradicted to \( f \hat{1} F^*(G, Z) \). So the assumption is wrong.

Let \( C_n^1, C_n^2, \cdots \) are \( m \) copies of \( n \)-cycles. The graph obtained by connecting each vertex in \( C_i \) to a vertex in \( C_{i+1} \) with a new edge in a certain order is called a Cone, denoted as \( V(m, n) \). From the definition we know that \( x_n @ V(2,n) \).

**Corollary 2.2** \( \text{Lg}(V_{2k+1,n}) = 4(k \hat{1} Z) \).

**Proof** By theorem 2.1, we conclude that \( \text{Lg}(V_{2k+1,n}) > 3 \). Since every edge of \( V(2k+1,n) \) lies in \( x_n \), we conclude that \( \text{Lg}(V_{2k+1,n}) \leq 4 \) by lemma 1.5 and 1.6. So \( \text{Lg}(V_{2k+1,n}) = 4 \).

A single fan \( F_n \) is a graph obtained from a \( n \)-path \((n^3,2)\) \( v_1, v_2, \cdots \) by adding a new vertex \( x \) and then joining the new vertex to all vertices in the path. This new vertex \( x \) is called the center of \( F_n \). A double fan \( F_{2n} \) is a graph obtained from a \( n \)-path \((n^3,2)\) \( v_1, v_2, \cdots \) by adding two new vertexes \( x \) and \( y \) and then joining these two vertexes to all the vertices in the path. These new vertexes \( x \) and \( y \) are called the centers of \( F_{2n} \) (Shown as Figure 2).

![Figure 2](image)

**Theorem 2.3** \( \text{Lg}(F_{2n}) = 3 \).
Proof Since every edge of $F_{2n}$ lies on a 3-cycle, by Lemma 1.3 and Lemma 1.5, $\Lambda g(F_{2n}) \leq 4. For d(x) \geq 3$, let be the graph obtained by adding a new edge in $F_{2n} - \{xv_1, xv_2\}$ that connecting $xv_1$ and $xv_2$. Contracting the 2-cycle in $H$, there is still a 2-cycle. Continue this process, we can obtain a graph which has two vertices with several multiple edges. By Lemma 1.2(3), we know that $H \hat{1} \langle Z_3 \rangle$, and by Lemma 1.7, $F_{2n} \hat{1} \langle Z_3 \rangle$. By lemma 1.4, we can conclude that $\Lambda g(F_{2n}) = 3$.

![Graph](image_url)

Figure 3: Graph $H_n$

The graph $H_n = F_n \hat{\omega} C_3$ (Shown as Figure 3) is obtained from $F_n$ and $C_3$ by adding three edges which it is $cx, av_b, bv_a$.

**Corollary 2.4** $\Lambda g(H_n) = 4$.

**Proof** Proof since $H_n$ has an odd cycle such that every vertex in it has degree 3, so by Theorem 2.1, $H_n$ has no $Z3 \cap NZF$. Thus $\Lambda g(H_n) \geq 4$. By contracting cycle $C = abca$, every edges of $H_n/C$ lies in a 3-cycles, so by lemma 1.2 and lemma 1.4, we conclude that $\Lambda g(H_n) \leq 4$. The conclusion is established.
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