THERMAL STRESSES IN A THIN ANNULAR DISC

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Abstract. This paper is concerned with the direct steady-state thermoelastic problem to determine the temperature, displacement and stress functions of a thin annular disc of thickness $2h$, occupying the space $D: a \leq r \leq b, -h \leq z \leq h$ with the stated boundary conditions. The finite Hankel transform techniques have been used.

Keywords: Annular disc, steady-state problem, direct thermoelastic problem Hankel Transform.

1 Introduction

Grysa and Cialiowski [1], Grysa and Koalowski [2] have studied one-dimensional transient thermoelastic Problems derived the heating temperature and heat flux on the surface of an isotropic infinite slab. Khobragade and Wankhede [3] have studied the inverse steady-state thermoelastic problem to determine the temperature, displacement and stress function on the outer curved surface of a thin
annular disc occupying the space \( D : a \leq r \leq b, 0 \leq z \leq h \). The direct problems of thin circular plate have been studied by Roychaudhari [5] and Wankhede [7].

In the present problem an attempt is made to study the direct steady-state thermoelastic problem to determine the temperature, displacement and stress functions of a thin annular disc of thickness \( 2h \), occupying the space \( D : a \leq r \leq b, -h \leq z \leq h \) with the known boundary conditions. The finite Hankel transform techniques have been used to find the solution of the problem.

2 Result Required

Consider if \( f(x) \) satisfies the Dirichlet’s conditions in the interval \((0, a)\) then its finite Hankel transform in that range is defined to be,

\[
\tilde{f}_\mu(\xi_i) = \int_0^a f(x) J_\mu(x\xi_i) \, dx
\]

(2.1)

where \( \xi_i \) is the root of the transcendental equation

\[
J_\mu(a\xi_i) = 0
\]

(2.2)

then, at any point of \((0, a)\) at which the function \( f(x) \) is continuous,

\[
f(x) = \frac{2}{a^2} \sum \tilde{f}_\mu(\xi_i) \frac{J_\mu(x\xi_i)}{[J'_\mu(a\xi_i)]^2}
\]

(2.3)

where the sum is taken over all the positive roots of the equation (2.2) properties of Hankel transform in (2.1)

If \( f(x) \) satisfies the Dirichlet’s conditions in the interval \((0, a)\) then,

1. Finite Hankel transform of \( \frac{\partial f}{\partial x} \) i. e,
If \( f(x) \) satisfies the Dirichlet’s conditions in the range \( b \leq x \leq a \) and if its finite Hankel transform in that range is defined to be

\[
H[f(x)] = \int_0^a f(x) J_\mu(x \xi) \, dx
\]

where \( \xi_i \) is the root of the transcendental equation

\[
\left[ J_\mu(\xi b) G_\mu(\xi a) - J_\mu(\xi a) G_\mu(\xi b) \right] = 0
\]

then at which the function is continuous

\[
f(x) = \sum_i 2 \xi_i^2 J_\mu(b \xi_i) \overline{J_\mu(b \xi_i)} \left[ J_\mu(x \xi_i) G_\mu(a \xi_i) - J_\mu(a \xi_i) G_\mu(x \xi_i) \right]
\]

Now property of hankel transform defined in (2.4)

\[
\int_a^b \left[ \frac{\partial^2 f}{\partial x^2} + \frac{1}{x} \frac{\partial f}{\partial x} \right] \left[ J_\mu(x \xi_i) G_\mu(a \xi_i) - J_\mu(a \xi_i) G_\mu(x \xi_i) \right] \, dx
\]

\[
= -\xi_i^2 \overline{J_\mu(\xi_i)} + a \left[ J_\mu(x \xi_i) G_\mu(a \xi_i) - J_\mu(a \xi_i) G_\mu(x \xi_i) \right]_{x=a}
\]

\[
+ b \left[ J_\mu(x \xi_i) G_\mu(a \xi_i) - J_\mu(a \xi_i) G_\mu(x \xi_i) \right]_{x=b}
\]

\[
= -\xi_i^2 \overline{J_\mu(\xi_i)}
\]
3 Statement of the Problem

Consider a thin circular plate of thickness \(2h\) occupying the space \(D: a \leq r \leq b, -h \leq z \leq h\). The differential equation governing the displacement function \(U(r, z)\) as in Nowaki [4] is,

\[
\frac{\partial^2 U}{\partial r^2} + \frac{1}{r} \frac{\partial U}{\partial r} = (1 + \nu)a_iT
\]  
\quad (3.1)

with \(U_r = 0\) at \(r = a\) & \(r = b\)  
\quad (3.2)

\( \nu \) and \( a_i \) are the poisson’s ratio and the linear coefficient of thermal expansion of the material of plate and \( T \) is the temperature of the plate satisfying the differential equation

\[
\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0
\]  
\quad (3.3)

subject to the boundary conditions

\[
T(r, z) + \frac{\partial T(r, z)}{\partial z} \bigg|_{z=h} = f(r)
\]  
\quad (3.4)

\[
T(r, z) + \frac{\partial T(r, z)}{\partial z} \bigg|_{z=-h} = 0
\]  
\quad (3.5)

\( T(a, z) = 0 \)  
\quad (3.6)

\( T(b, z) = 0 \)  
\quad (3.7)

The stress function \(\sigma_{rr}\) and \(\sigma_{\theta\theta}\) are given by

\[
\sigma_{rr} = -2\mu \frac{1}{r} \frac{\partial U}{\partial r}
\]  
\quad (3.8)
\[ \sigma_{\theta \theta} = -2\mu \frac{\partial^2 U}{\partial r^2} \]  

(3.9)

where \( \mu \) is the lamé constant while each of the stress functions \( \sigma_{rz} \), \( \sigma_{zz} \), \( \sigma_{th} \) are zero within the plate in the plane state of stress.

The equations (3.1) to (3.9) constitute the mathematical formulation of the problem under consideration.

4 Solution of the Problem

Applying finite Hankel transform stated in [6] to (3.3) to (3.5) and using (3.6) and (3.7) one obtains,

\[ \frac{d^2 \overline{T}}{dz^2} - \lambda_n^2 \overline{T} = 0 \]  

(4.1)

\[ \left[ \overline{T}(\lambda_n, z) + \frac{d\overline{T}(\lambda_n, z)}{dz} \right]_{z=h} = \overline{f}(\lambda_n) \]  

(4.2)

\[ \left[ \overline{T}(\lambda_n, z) + \frac{d\overline{T}(\lambda_n, z)}{dz} \right]_{z=-h} = 0 \]  

(4.3)

where \( \overline{T} \) denotes the finite Hankel transform of \( T \) and \( \lambda_n \) is the Hankel transform parameter. The equation (4.1) is a second order differential equation whose solution is given by

\[ \overline{T}(\lambda_n, z) = Ae^{\lambda_n z} + Be^{-\lambda_n z} \]  

(4.4)
where $A$ and $B$ are constants.

Using (4.2) and (4.3) in (4.4) we obtain the values of $A$ and $B$ substituting this values in (4.4) and then inversion of finite Hankel transform lead to

$$
T(r, z) = 2 \sum_{n=1}^{\infty} \mathcal{H}^2 \left[ \frac{J^2_0(\lambda_n a)}{J^2_0(\lambda_n b) - J^2_0(\lambda_n a)} \right] \sinh(\lambda_n(z + h)) \cosh(\lambda_n(z + h)) \bigg[ (1 + \lambda_n^2) \sinh(2\lambda_n h) + 2\lambda_n \cosh(2\lambda_n h) \bigg] \bigg[ J_0(r\lambda_n)G_0(b\lambda_n) - J_0(b\lambda_n)G_0(r\lambda_n) \bigg]$$

(4.5)

where $\mathcal{H}(\lambda_n) = \int_0^a f(r) \left[ J_0(r\lambda_n)G_0(b\lambda_n) - J_0(b\lambda_n)G_0(r\lambda_n) \right] dr$ and $\lambda_n$ are the root of the transcendental equation

$$
\left[ J_0(a\lambda_n)G_0(b\lambda_n) - J_0(b\lambda_n)G_0(a\lambda_n) \right] = 0
$$

Equation (4.5) is the desired solution of the given problem.

5 Determination of Thermoelastic Displacement

Substituting this values of $T(r, z)$ from (4.5) in (3.1) one obtains the thermoelastic displacement function $U(r, z)$ as

$$
U(r, z) = -2(1 + \nu)a \sum_{n=1}^{\infty} \mathcal{H}^2 \left[ \frac{J^2_0(\lambda_n a)}{J^2_0(\lambda_n b) - J^2_0(\lambda_n a)} \right] \sinh(\lambda_n(z + h)) \cosh(\lambda_n(z + h)) \bigg[ (1 + \lambda_n^2) \sinh(2\lambda_n h) + 2\lambda_n \cosh(2\lambda_n h) \bigg] \bigg[ J_0(r\lambda_n)G_0(b\lambda_n) - J_0(b\lambda_n)G_0(r\lambda_n) \bigg]
$$

(5.1)
6 Determination of Stress Function

For Using (5.1) in (3.8) and (3.9) the stress functions are obtained as

\[
\sigma_{rr} = \frac{4\mu(1 + \nu) a_i}{r} \sum_{n=1}^{\infty} \lambda_n \bar{f}(\lambda_n) \left[\frac{J_0^2(\lambda_n a)}{J_0^2(\lambda_n b) - J_0^2(\lambda_n a)} \right] \\
\times \left[ \frac{\sinh(\lambda_n(z+h)) + \lambda_n \cosh(\lambda_n(z+h))}{(1 + \lambda_n^2) \sinh(2\lambda_n h) + 2\lambda_n \cosh(2\lambda_n h)} \right] \\
\times \left[ J_1(r \lambda_n) G_0(b \lambda_n) - J_0(b \lambda_n) G_1(r \lambda_n) \right] \\
\text{(6.1)}
\]

\[
\sigma_{\theta\theta} = 4\mu(1 + \nu) a_i \sum_{n=1}^{\infty} \lambda_n \bar{f}(\lambda_n) \left[\frac{J_0^2(\lambda_n a)}{J_0^2(\lambda_n b) - J_0^2(\lambda_n a)} \right] \\
\times \left[ \frac{\sinh(\lambda_n(z+h)) + \lambda_n \cosh(\lambda_n(z+h))}{(1 + \lambda_n^2) \sinh(2\lambda_n h) + 2\lambda_n \cosh(2\lambda_n h)} \right] \\
\times \left[ J_1(r \lambda_n) G_0(b \lambda_n) - J_0(b \lambda_n) G_1(r \lambda_n) \right] \\
\text{(6.2)}
\]

7 Conclusion

In this paper, we discussed completely the direct steady state problem of thermoelastic deformation of a thin annular disc of thickness $2h$, with homogeneous boundary condition of the third kind is maintained on the lower plane surface while on upper plane surface, it is maintained at $f(r)$, which is known function of $r$ and the temperature is maintained at zero on curved surfaces of the annular disc.

The finite Hankel transform technique is used to obtain the numerical results. The temperature, displacement and thermal stresses that are obtained can be applied to the design of useful structures or machines in engineering applications.
References


