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Bounded Nonoscillatory Solutions for Higher-order Nonlinear Dynamic Equations with a Forced Term

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Abstract. In this paper, The author studies the existence of bounded non-oscillatory solutions of a class of forced higher order nonlinear neutral dynamic equations on a time scale $\mathbf{T}$. By using fixed point theorem and some new techniques, the author obtains sufficient conditions for the existence of non-oscillatory solutions for general $p_i(t)$, $f_i(x)$ and $q(t)$ which means that they are allowed oscillate. The results not only generalize and improve the known results stated for differential and difference equations, but also improve some of the results for dynamic equations on time scales. An example is included to illustrate the results.

Keywords: Dynamic equation; higher order; non-oscillation; time scales; neutral.

1 Introduction

The theory of time scales, which has recently received a lot of attention, was introduced by Stefan Hilger in his Ph. D. thesis in 1988 in order to unify continuous and discrete analysis (see Hilger [1]). Several authors have expounded on various aspects of this new theory; see the survey paper by Agarwal et al. [2] and references cited therein. A book on the subject of time scales, by Bohner and Peterson [3],

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summarizes and organizes much of the time scale calculus; we refer also to the last book by Bohner and Peterson [4] for advances in dynamic equations on time scales.

In recent years, there has been much research activity concerning the oscillation and non-oscillation of solutions of various equations on time scales(see[5-15,25-30]). However, there are relatively few papers to discuss the existence of non-oscillatory solutions for higher order nonlinear dynamic equations with forced terms on time scales. Motivated by these works, in this paper, we investigate the existence of non-oscillatory solutions of the following forced higher order neutral dynamic equation

$$\left[x(t)+p(t)x(\tau(t))\right]^{\Delta^m}+\sum_{i=1}^{k}p_i(t)f_i(x(\tau_i(t)))=q(t) \qquad t\in[t_0,\infty)_{\mathbf{T}} \tag{1.1}$$

where $p_i\in C_{rd}([t_0,\infty)_{\mathbf{T}},R)$, $p\in C([t_0,\infty)_{\mathbf{T}},R)$, $\tau,\tau_i\in C([t_0,\infty)_{\mathbf{T}},R)$ with $\lim_{t\to\infty}\tau(t)=\lim_{t\to\infty}\tau_i(t)=+\infty$ and $f_i\in C(R,R)$, $i=1,2,\cdots,k$.

We obtain some sufficient conditions for the existence of a non-oscillatory solution of (1.1) without using nondecreasing conditions on the function $f_i(x)$ with $xf_i(x)>0$ for $x\neq 0$ and any sign conditions on the functions $p_i(t)$, $q(t)$ via Kranoselskii's fixed point theorem and some new techniques. In particular, our results generalize and improve essentially the known results by removing the restrictive conditions on the functions $p(t)$ and $f_i(x)$ ($i=1,2,\cdots,k$) (see[9,11,16-25]).

2 Preliminaries

We state the following conditions, which are needed in the sequel:

(H_1) there exists a constant $p\in(\frac{1}{2},1)$ such that $|p(t)|\le 1-p$ for all $t\ge t_0$;

(H_2) there exists a constant $p\in(0,1)$ such that $0\le p(t)\le p$ for all $t\ge t_0$;

(H_3) there exists a constant $p\in(-1,0]$ such that $p\le p(t)\le 0$ for all $t\ge t_0$.

Let k be a nonnegative integer and $s,t\in\mathbf{T}$, we define two sequences of functions $h_k(t,s)$ and $g_k(t,s)$ as follows:

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$$h_k(t,s)=\begin{cases}1, & k=0,\\ \int_s^t h_{k-1}(\tau,s)\Delta\tau, & k\geq 1.\end{cases}\quad g_k(t,s)=\begin{cases}1, & k=0,\\ \int_s^t g_{k-1}(\sigma(\tau),s)\Delta\tau, & k\geq 1.\end{cases}$$

By Theorems 1.112, Theorems 1.60 of [3] and Lemma 2.2 of [11], then

$$h_k(t,s)=(-1)^k g_k(t,s),\quad h_k^{\Delta_t}(t,s)=\begin{cases}0, & k=0,\\ h_{k-1}(t,s), & k\geq 1.\end{cases}$$

$$g_k^{\Delta_t}(t,s)=\begin{cases}0, & k=0,\\ g_{k-1}(\sigma(t),s), & k\geq 1.\end{cases}$$

where $h_k^{\Delta_t}(t,s)$ and $g_k^{\Delta_t}(t,s)$ denote for each fixed s the derivative of $h_k(t,s)$ and $g_k(t,s)$ with respect to t.

Lemma 1 ([12]). Assume that $s,t\in\mathbf{T}$ and $g\in C_{rd}(\mathbf{T}\times\mathbf{T},R)$, then

$$\int_s^t\left[\int_\eta^t g(\eta,\xi)\Delta\xi\right]\Delta\eta=\int_s^t\left[\int_s^{\sigma(\xi)} g(\eta,\xi)\Delta\eta\right]\Delta\xi.$$

Lemma 2 ([12]). Let n be a nonnegative integer, $h\in C_{rd}(\mathbf{T},[0,\infty))$ and $s\in\mathbf{T}$. Then $\int_s^\infty g_n(\sigma(\tau),s)h(\tau)\Delta\tau<\infty$ implies that each of the following is true:

(i) $\int_t^\infty g_j(\sigma(\tau),t)h(\tau)\Delta\tau$ is decreasing for all $t\in\mathbf{T}$ and all $0\leq j\leq n$.

(ii) $\int_t^\infty g_j(\sigma(\tau),t)h(\tau)\Delta\tau<\infty$ for all $t\in\mathbf{T}$ and all $0\leq j\leq n-1$.

Let $BC_{rd}([t_0,\infty)_{\mathbf{T}},R)$ be the Banach space of all bounded rd-continuous functions on $[t_0,\infty)_{\mathbf{T}}$ with sup norm $\|x\|=\sup_{t\geq t_0}|x(t)|$.

Lemma 3 (Arzelá-Ascoli theorem [9]). Suppose that $\Omega\subset BC_{rd}([t_0,\infty)_{\mathbf{T}},R)$ is bounded, uniformly Cauchy and equi-continuous, then Ω is relatively compact.

Lemma 4 (Krasnoselskii's fixed point theorem [9]). Suppose that X is a Banach space and Ω is a bounded, convex and closed subset of X. Suppose further that there exist two operators T_1 and T_2: $\Omega\to X$ such that

(i) $T_1x+T_2y\in\Omega$ for all $x,y\in\Omega$; (ii) T_1 is a contraction mapping; (iii) T_2 is completely continuous. Then T_1+T_2 has a fixed point in Ω.

3 Main results and Examples

We start this section with the following results, which investigate sufficient conditions for the existence of bounded non-oscillatory solutions of (1.1) with $p(t)$ in one of the ranges $(H_1)-(H_3)$.

Theorem 1 Assume that (H_1) holds, and that

$$\int_{t_0}^{\infty} g_{m-1}(\sigma(s),t_0)\,|\,p_i(s)\,|\,\Delta s<\infty,\quad i=1,2,\cdots,k \tag{3.1}$$

and

$$\int_{t_0}^{\infty} g_{m-1}(\sigma(s),t_0)\,|\,q(s)\,|\,\Delta s<\infty. \tag{3.2}$$

Then (1.1) has a bounded nonoscillatory solution $x(t)$ with $\liminf_{t\to\infty}|x(t)|>0$.

Proof. For some $d\neq 0$, we choose d_1,c_1 such that $0<d_1<(2p-1)\,|\,d\,|$ and $d_1+(1-p)\,|\,d\,|<c_1<p\,|\,d\,|$. Let $c=\min\{c_1-d_1-(1-p)\,|\,d\,|,\ p\,|\,d\,|-c_1\}$. By (3.1) and (3.2), there exists $t_1\geq t_0$ sufficiently large, such that

$$\int_{t_1}^{\infty} g_{m-1}(\sigma(s),t_1)\Big(\sum_{i=1}^{k}M_1\,|\,p_i(s)\,|+|\,q(s)\,|\Big)\Delta s\leq c$$

and $\tau(t),\tau_i(t)\geq t_0$ for $t\geq t_1$, where $M_1=\max\limits_{d_1\leq x\leq|d|}\{|\,f_i(x)\,|:1\leq i\leq k\}$. Let $\Omega=\{x\in BC_{rd}([t_0,\infty)_{\mathbf{T}},R):\ \ d_1\leq x(t)\leq|\,d\,|,\quad t\geq t_0\}$. It is easy to verify that Ω is a bounded, convex and closed subset of $BC_{rd}([t_0,\infty)_{\mathbf{T}},R)$.

We define two operators T_1 and T_2: $\Omega \to BC_{rd}([t_0,\infty)_{\mathbf{T}}, R)$ as follows:

$$(T_1x)(t) = \begin{cases} -p(t)x(\tau(t)), & t \ge t_1, \\ (T_1x)(t_1), & t_0 \le t \le t_1, \end{cases}$$

$$(T_2x)(t) = \begin{cases} c_1 + (-1)^{m-1}\int_t^{\infty} g_{m-1}(\sigma(s),t)[\sum_{i=1}^{k} p_i(s) f_i(x(\tau_i(s))) - q(s)]\Delta s, t \ge t_1, \\ (T_2x)(t_1), & t_0 \le t \le t_1. \end{cases}$$

Now we show that T_1 and T_2 satisfy the conditions in Lemma 4.

(1) We will show that $T_1x + T_2y \in \Omega$ for any $x, y \in \Omega$. In fact, we have

$$(T_1x)(t) + (T_2y)(t) \ge c_1 - (1-p)|d| - [c_1 - d_1 - (1-p)|d|] = d_1,$$

$$(T_1x)(t) + (T_2y)(t) \le c_1 + (1-p)|d| + p|d| - c_1 = |d|,$$

which implies that $T_1x + T_2y \in \Omega$ for any $x, y \in \Omega$.

(2) We will show that T_1 is a contraction mapping. Indeed, we have

$$|(T_1x)(t) - (T_2y)(t)| \le |p(t)||x(\tau(t)) - y(\tau(t))| \le (1-p)\|x - y\|,$$

which implies that T_1 is a contraction mapping.

(3) We will show that T_2 is a completely continuous mapping.

(i) By the proof of (1), we see that $T_2\Omega \subset \Omega$.

(ii) We consider the continuity of T_2. Let $x_n \in \Omega$ and $\|x_n - x\| \to 0$ as $\mathrm{n} \to \infty$, then $x \in \Omega$ and $|x_n(t) - x(t)| \to 0$ for any $t \in [t_0,\infty)_{\mathbf{T}}$. we have

$$|(T_2x_n)(t) - (T_2x)(t)| \le \int_{t_1}^{\infty} g_{m-1}(\sigma(s),t_1)[\sum_{i=1}^{k} |p_i(s)||f_i(x_n(\tau_i(s))) - f_i(x(\tau_i(s)))|]\Delta s$$

Since

$$g_{m-1}(\sigma(s),t_1)[\sum_{i=1}^{k} |p_i(s)||f_i(x_n(\tau_i(s))) - f_i(x(\tau_i(s)))|] \le 2M_1 g_{m-1}(\sigma(s),t_1)\sum_{i=1}^{k} |p_i(s)|$$

and $|f_i(x_n(\tau_i(s))) - f_i(x(\tau_i(s)))| \to 0$ as $\mathrm{n} \to \infty$ for $i = 1, 2, \cdots, k$. In view of (3.1) and applying the Lebesgue dominated convergence theorem, we conclude that $\lim_{n\to\infty} \| T_2 x_n - T_2 x \| = 0$, which implies that T_2 is continuous on Ω.

(iii) We show that $T_2\Omega$ is uniformly Cauchy. For any $\varepsilon > 0$, $t_2 > t_1$ such that

$$\int_{t_2}^{\infty} g_{m-1}(\sigma(s),t_2)(\sum_{i=1}^{k} M_1 \,|\, p_i(s)\,| + |\, q(s)\,|)\Delta s \le \varepsilon .$$

Then for any $x \in \Omega$ and $t, r \in [t_2, \infty)_{\mathbf{T}}$, we have

$$\left|(T_2 x_n)(t) - (T_2 x)(r)\right| \le \int_t^{\infty} g_{m-1}(\sigma(s),t)[\sum_{i=1}^{k} |\, p_i(s)\,||\, f_i(x(\tau_i(s)))\,| + |\, q(s)\,|]\Delta s$$

$$\le 2\int_{t_2}^{\infty} g_{m-1}(\sigma(s),t_2)[\sum_{i=1}^{k} M_1 \,|\, p_i(s)\,| + |\, q(s)\,|]\Delta s \le 2\varepsilon.$$

This means that $T_2\Omega$ is uniformly Cauchy.

(iv) We show that $T_2\Omega$ is equicontinuous on $[t_0, t_2]_{\mathbf{T}}$ for any $t_2 \in [t_0, \infty)_{\mathbf{T}}$. Without loss of generality, we assume that $t_2 \ge t_1$. For any $\varepsilon > 0$, choose $\delta = \varepsilon / \int_{t_0}^{\infty} g_{m-2}(\sigma(s),t_0)[\sum_{i=1}^{k} M_1 \,|\, p_i(s)\,| + |\, q(s)\,|]\Delta s$. Then for any $x \in \Omega$, when $t, r \in [t_0, t_2]_{\mathbf{T}}$ with $|\, t - r\,| < \delta$, by Lemma 1 and Lemma 2, we have

$$\left|(T_2 x)(t) - (T_2 x)(r)\right| = |\int_t^{\infty} g_{m-1}(\sigma(s),t)[\sum_{i=1}^{k} p_i(s) f_i(x(\tau_i(s))) - q(s)]\Delta s$$

$$-\int_r^{\infty} g_{m-1}(\sigma(s),r)[\sum_{i=1}^{k} p_i(s) f_i(x(\tau_i(s))) - q(s)]\Delta s \,|$$

$$= |\int_t^{r} \int_{\theta}^{\infty} g_{m-2}(\sigma(s),\theta)[\sum_{i=1}^{k} p_i(s) f_i(x(\tau_i(s))) - q(s)]\Delta s \Delta\theta \,|$$

$$\le |\int_t^{r} \int_{t_0}^{\infty} g_{m-2}(\sigma(s),t_0)[\sum_{i=1}^{k} p_i(s) f_i(x(\tau_i(s))) - q(s)]\Delta s \Delta\theta \,|$$

$$\le \delta \int_{t_0}^{\infty} g_{m-2}(\sigma(s),t_0)[\sum_{i=1}^{k} M_1 \,|\, p_i(s)\,| + |\, q(s)\,|]\Delta s < \varepsilon.$$

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This means that $T_2\Omega$ is equicontinuous on $[t_0,t_2]_{\mathbf{T}}$ for any $t_2 \in [t_0,\infty)_{\mathbf{T}}$. Hence, by Lemma 3, $T_2\Omega$ is a completely continuous mapping. It follows from Lemma 4 that there exists $x \in \Omega$ such that $(T_1+T_2)x = x$, which is the desired bounded solution of (1.1) with $\liminf_{t\to\infty} |x(t)| > 0$. The proof is completed.

Theorem 2 Assume that (H_2), (3.1) and (3.2) hold, then (1.1) has a bounded nonoscillatory solution $x(t)$ with $\liminf_{t\to\infty} |x(t)| > 0$.

Proof. By (3.1) and (3.2), we choose a $t_1 \ge t_0$ sufficiently large such that

$$\int_{t_1}^{\infty} g_{m-1}(\sigma(s),t_1)(\sum_{i=1}^{k} M_2 |p_i(s)| + |q(s)|)\Delta s \le 1-p$$

and $\tau(t),\tau_i(t) \ge t_0$ for $t \ge t_1$, where $M_2 = \max_{1-p\le x\le 3}\{|f_i(x)|: 1\le i\le k\}$. Now we define a bounded, convex and closed subset Ω of $BC_{rd}([t_0,\infty)_{\mathbf{T}},R)$:

$\Omega = \{x \in BC_{rd}([t_0,\infty)_{\mathbf{T}},\square): \quad 1-p \le x(t) \le 3, \quad t \ge t_0\}$.

Define two operators T_1 and T_2: $\Omega \to BC_{rd}([t_0,\infty)_{\mathbf{T}},R)$ as follows:

$$(T_1x)(t) = \begin{cases} -p(t)x(\tau(t)), & t \ge t_1, \\ (T_1x)(t_1), & t_0 \le t \le t_1, \end{cases}$$

$$(T_2x)(t) = \begin{cases} 2+p+(-1)^{m-1}\int_t^{\infty} g_{m-1}(\sigma(s),t)[\sum_{i=1}^{k} p_i(s) f_i(x(\tau_i(s))) - q(s)]\Delta s, t \ge t_1, \\ (T_2x)(t_1), \qquad t_0 \le t \le t_1. \end{cases}$$

The rest of the proof is similar to that of Theorem 1 and hence omitted. The proof is completed.

Theorem 3 Assume that (H_3), (3.1) and (3.2) hold, then (1.1) has a bounded nonoscillatory solution $x(t)$ with $\liminf_{t\to\infty} |x(t)| > 0$.

Proof. By (3.1) and (3.2), there exists $t_1 \ge t_0$ sufficiently large such that

$$\int_{t_1}^{\infty} g_{m-1}(\sigma(s),t_1)(\sum_{i=1}^{k} M_3 \,|\, p_i(s)\,| + |\, q(s)\,|)\Delta s \le \frac{1+p}{3}$$

and $\tau(t), \tau_i(t) \ge t_0$ for $t \ge t_1$, where $M_3 = \max_{\frac{1+p}{3} \le x \le \frac{4}{3}} \{|\, f_i(x)\,| : 1 \le i \le k\}$. Let

$\Omega = \{x \in BC_{rd}([t_0,\infty)_{\mathbf{T}}, R) : \ \frac{1+p}{3} \le x(t) \le \frac{4}{3}, \quad t \ge t_0\}$. It is easy to verify that

Ω is a bounded, convex and closed subset of $BC_{rd}([t_0,\infty)_{\mathbf{T}}, R)$.

Define two operators T_1 and T_2 : $\Omega \to BC_{rd}([t_0,\infty)_{\mathbf{T}}, R)$ as follows:

$$(T_1x)(t) = \begin{cases} -p(t)x(\tau(t)), & t \ge t_1, \\ (T_1x)(t_1), & t_0 \le t \le t_1, \end{cases}$$

$$(T_2x)(t) = \begin{cases} 1+p+(-1)^{m-1}\int_t^{\infty} g_{m-1}(\sigma(s),t)[\sum_{i=1}^{k} p_i(s)\, f_i(x(\tau_i(s))) - q(s)]\Delta s, t \ge t_1, \\ (T_2x)(t_1), \qquad t_0 \le t \le t_1. \end{cases}$$

The rest of the proof is similar to that of Theorem 1 and hence omitted.

Remark 1 Theorem 1-Theorem 3 not only unify the known results for differential and difference equations corresponding to (1.1), but also extend and improve essentially the existing results of [9,11,16-25] because we do not assume that f_i is Lipschitzian nor nondecreasing with $xf_i(x) > 0$, and allow oscillatory $p(t)$ and $p_i(t)$.

Example Let $\mathbf{T} = \{q^n : n \in N_0, q > 1\}$, consider the dynamic equation

$$(x(t) - \frac{1}{\sqrt{q}}x(\rho(t)))^{\Delta^4} + \frac{(1-\sqrt{q})(q+1)^2(q^2+1)(q^2+q+1)}{q^{10}t^3(t+q^2)^3}x^2(\rho^3(t))$$
$$= \frac{2(1-\sqrt{q})(q+1)^2(q^2+1)(q^2+q+1)}{q^{10}t^5} \tag{3.3}$$

By the definition of $g_k(t,s)$, we have

$$\int_{t_0}^{\infty} g_{4-1}(\sigma(s),t_0)\,|\,p_1(s)\,|\,\Delta s \le \frac{(\sqrt{q}-1)(q+1)^2(q^2+1)(q^2+q+1)}{q^{10}} \int_{t_0}^{\infty} \frac{1}{s^3}\Delta s < \infty,$$

$$\int_{t_0}^{\infty} g_{4-1}(\sigma(s),t_0)\,|\,q(s)\,|\,\Delta s \le \frac{2(\sqrt{q}-1)(q+1)^2(q^2+1)(q^2+q+1)}{q^{10}} \int_{t_0}^{\infty} \frac{1}{s^2}\Delta s < \infty.$$

All the conditions of Theorem 3 hold, therefore, (3.3) admits a bounded non-oscillatory solution $x(t)$ with $\liminf_{t\to\infty} |x(t)| > 0$. In fact, $x(t) = 1 + \frac{1}{t}$ is a solution of (3.3).

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