A STUDY OF FINITE LENGTH THERMOELASTIC PROBLEM OF HOLLOW CYLINDER WITH RADIATION

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Abstract

In this paper, in hollow cylinders it is to be noticed that all possible problems on boundary conditions can be solved by particularizing the method described here. A new finite integral transformation an extension of those given by Sneddon [11] whose kernel is given by cylindrical functions, is used to solve the problem of finding the temperature at any point of a hollow cylinder of any height, with boundary conditions of radiation type on the outside and inside surfaces with independent radiation constants.

Keyword: Hollow Cylinder, Marchi-Zgrablich Transform

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1 Introduction

The main objective of is to solve the problem of finding the temperature at any point of hollow cylinder of any height, when there is heat radiation on its outside and inside surfaces.

In all aforementioned investigation, an axisymmetrically heated plate has been considered. Nasser [2,3] proposed the concept of heat sources in generalized thermoelastic and applied to a thick plate problem. They have not however considered any thermoelastic problem with boundary conditions of radiation type, in which sources are generated according to the linear function of the temperature, which satisfies the time-dependent heat conduction equation.

Nowacki [10] has determined steady-state thermal stresses in a thick circular plate subjected to an axisymmetric temperature distribution on the upper face with zero temperature on the lower face and circular edge.

Kulkarni et al. [1] also studied on quasi-static transient thermal stresses in a thick annular disc. Wankhede et al. [4] studied on modified Marchi-Zgrabrich transformation.

Marchi et al. [5] and Watson [6] discussed on heat conduction problem in hollow cylinders with radiation and defined a theory of Bessel function. Recently, Gaikwad et al.[7] and Ghane et al. [9] also proposed the thermoelastic problem of a thick annular disc due to radiation and transient thermoelastic problem of a semi infinite cylinder with heat sources.

Hiranwar et al. [8] also proposed the thermoelastic problem of thin annular disc due to radiation.

we assume that all functions involved satisfy Dirichlet's conditions in the intervals considered.

2 Statement of the problem

Consider the hollow cylinder whose axis is coincident with the Z-axis, defined by $0 \leq z \leq h$ and $a \leq r \leq b$, where a and b are the external and internal radii, respectively and (r, ϕ, z) are cylindrical coordinates. Let us consider the heat conduction problem with the symmetry with the time, will be the solution of the conduction equations,

The thermoelastic displacement function as [Nowacki] is governed by the Poisson's equation:

$$\nabla^2 \phi = \frac{(1+\nu)}{(1-\nu)} \, a_t \, T \tag{2.1}$$

with $U = 0$ at $r = a$ and $r = b$.

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

ν and a_t are the Poisson's ratio and the linear coefficient of thermal expansion of the material of the plate and T is the temperature of the plate satisfying the differential equation which is given below

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\hbar}\frac{\partial T}{\partial t} \tag{2.2}$$

where $\hbar = \frac{\kappa}{\rho \ C}$, $\hbar$ being the thermal conductivity of the material, ρ is the density and C is the calorific capacity, assumed to be constant. with boundary and initial conditions,

$$T(a, z, t) + k_1 \frac{\partial T(a, z, t)}{\partial r} = f_a(z, t) \ \text{for all} \ 0 < z < h \ \text{and} \ t > 0 \tag{2.3}$$

$$T(b, z, t) + k_2 \frac{\partial T(b, z, t)}{\partial r} = g_b(z, t) \ \text{for all} \ 0 < z < h \ \text{and} \ t > 0 \tag{2.4}$$

where k_1 and k_2 are the radiation constants on the two cylindrical surfaces,

$$T(r, h, t) = 0 \ \text{for all} \ 0 < z < h \ \text{and} \ t > 0 \tag{2.5}$$

$$T(r, 0, t) = 0 \ \text{for all} \ 0 < z < h \ \text{and} \ t > 0 \tag{2.6}$$

$$T(r, z, 0) = T_0(r, z) \ \text{for all} \ 0 < z < h \ \text{and} \ a \leq r \leq b \tag{2.7}$$

where $f_a(z, t)$ and $g_b(z, t)$ and $T_0(r, z)$ are known. The radial and axial displacement U and W satisfying the uncoupled thermoealstic equations are

$$\nabla^2 U - \frac{U}{r^2} + (1 - 2\nu^{-1})\frac{\partial e}{\partial r} = 2\frac{(1+\nu)}{(1-\nu)} \ a_t \ \frac{\partial T}{\partial r} \tag{2.8}$$

$$\nabla^2 W + (1 + 2\nu^{-1})\frac{\partial e}{\partial z} = 2\frac{(1+\nu)}{(1-\nu)} \ a_t \ \frac{\partial T}{\partial z} \tag{2.9}$$

where

$$e = \frac{\partial U}{\partial r} + \frac{U}{r} + \frac{\partial W}{\partial z}$$

is the volume dilation and

$$U = \frac{\partial \phi}{\partial r} \tag{2.10}$$

$$W = \frac{\partial \phi}{\partial z} \tag{2.11}$$

The stress functions are given by,

$$\sigma_{rr} = (\lambda + 2G)\frac{\partial U}{\partial r} + \lambda\left(\frac{U}{r} + \frac{\partial W}{\partial z}\right) \tag{2.12}$$

$$\sigma_{zz} = (\lambda + 2G)\frac{\partial W}{\partial z} + \lambda\left(\frac{\partial U}{\partial r} + \frac{U}{r}\right) \tag{2.13}$$

$$\sigma_{\theta\theta} = (\lambda + 2G)\frac{U}{r} + \lambda\left(\frac{\partial U}{\partial r} + \frac{\partial W}{\partial z}\right) \tag{2.14}$$

$$\tau_{\theta\theta} = G\left(\frac{\partial W}{\partial r} + \frac{\partial U}{\partial z}\right) \tag{2.15}$$

where $\lambda = \frac{2G\nu}{1-2\nu}$, is the Lame's constant, G is the shear modulus and U and W are the displacement components. The equation (2.1) to (2.15) constitute the mathematical formulation of the problem under consideration.

3 Useful results

Let us define the finite integral transform

$$\overline{f}(n) = \int_a^b x f(x) S_p(k_1, k_2, \mu_n x) dx. \tag{3.1}$$

the inversion theorem is given by

$$f(x) = \sum_n a_n S_p(k_1, k_2, \mu_n x) dx. \tag{3.2}$$

where the sum must be taken over the positive roots of equation,

$$J_p(k_1, \mu_a) G_p(k_2, \mu_b) - J_p(k_2, \mu_b) G_p(k_1, \mu_a) = 0 \tag{3.3}$$

From the orthogonality of the functions defined by equation,

$$\begin{aligned} S_p(k_1, k_2, \mu_n x) =& J_p(\mu_n x)[G_p(k_1, \mu_a) + G_p(k_2, \mu_b)] \\ & - G_p(\mu_n x)[J_p(k_1, \mu_a) + J_p(k_2, \mu_b)] \end{aligned} \tag{3.4}$$

The a_n are given by,

$$a_n = \overline{f}_p(n)/C_n \tag{3.5}$$

where

$$C_n = \int_a^b x[S_p(k_1, k_2, \mu_n x)]^2 dx$$

By using the relation

$$\begin{aligned} \int^z z\ell_u(kz)\overline{\ell}_u(kz)dz =& \frac{1}{4}z^2\left\{2\ell_u(kz)\overline{\ell}(kz)\ell_{u-1}(kz) - \overline{\ell}_{u+1}(kz)\right. \\ & \left. -\ell_{u+1}(kz)\overline{\ell}_{u-1}(kz)\right\} \end{aligned}$$

where $\ell_u(kz)$ and $\overline{\ell}_u(kz)$ are cylindrical functions of order p, then results;

$$C_n = \frac{1}{2}b^2\left\{\widetilde{S}_p^2(k_1,k_2,\mu_n b) - \widetilde{S}_{p-1}(k_1,k_2,\mu_n b)\widetilde{S}_{p+1}(k_1,k_2,\mu_n b)\right\}$$
$$-\frac{1}{2}a^2\left\{\widetilde{S}_p^2(k_1,k_2,\mu_n a) - \widetilde{S}_{p-1}(k_1,k_2,\mu_n a)\widetilde{S}_{p+1}(k_1,k_2,\mu_n a)\right\} \tag{3.6}$$

4 Solution of the problem

From equation (10.2.2) of the transformation defined in (3.1), we get,

$$\overline{T}(n,z,t) = \int_a^b rT(r,z,t)S_0(k_1,k_2,\mu_n r)dr$$

where μ_n are positive roots of equation (3.3) with $p = 0$, and taking into account (2.3), we obtain

$$\hbar\frac{\partial^2\overline{T}(n,z,t)}{\partial z^2} - \frac{\partial\overline{T}(n,z,t)}{\partial t} - \mu_n^2\hbar\overline{T}(n,z,t) = \hbar\chi(z,t) \tag{4.1}$$

where

$$\chi(z,t) = \frac{a}{k_1}S_0(k_1,k_2,\mu_n a)f_a(z,t) - \frac{b}{k_2}S_0(k_1,k_2,\mu_n b)g_b(z,t) \tag{4.2}$$

is obviously a known function: To solve differential equation (4.1), let us introduce the Fourier transform, in the variable z.

$$\overline{\overline{T}}(n,m,t) = \int_0^h \overline{T}(n,z,t)\sin(m\pi z/h)dz$$

From this and the property:

$$\int_0^h \frac{\partial^2}{\partial z^2}\overline{T}(n,z,t)\sin\left(\frac{m\pi z}{h}\right)dz = \frac{m\pi}{h}[(-1)^{m+1}\overline{T}(n,h,t) + \overline{T}(n,0,t)]$$
$$-\frac{m^2\pi^2}{h^2}\overline{\overline{T}}(n,m,t)$$

and remembering (2.4), (4.1) is transformed into,

$$\frac{d\overline{\overline{T}}(n,m,t)}{dt} + \hbar(\mu_n^2 + m^2\pi^2/h^2)\overline{\overline{T}}(n,m,t) = -\hbar\chi_s(m,t) \tag{4.3}$$

where

$$\chi_s(m,t) = \int_0^h \chi(z,t)\sin(m\pi z/h)dz$$

By introducing the Laplace transform $L[\psi(t)] = \int_0^\infty e^{-pt}\psi(t)dt$ and taking into arround the initial condition Eq.(2.5), Eq.(4.3) is transformed into

$$L[\overline{\overline{T_0}}(n,m,t)] = \overline{\overline{T}}(n,m)/p + \hbar(\mu_n^2 + m^2\pi^2/h^2)$$
$$-\hbar L[\chi_s(m,t)]/p + \chi(\mu_n^2 + m^2\pi^2/h^2) \tag{4.4}$$

Now, if we apply operator L^{-1} to Eq. (4.4) and remember the convolution theorem for Laplace transforms, we obtain.

$$\overline{\overline{T_0}}(n,m,t) = \overline{\overline{T}}(n,m)\exp[-\hbar(\mu_n^2 + m^2\pi^2/h^2)t]$$
$$-\hbar\int_0^t \chi_s(m,u)\exp[-\hbar(\mu_n^2 + m^2\pi^2/h^2)(t-u)]du \tag{4.5}$$

Using now, the inversion theorem for our Fourier transform, there results

$$\overline{T}(n,z,t) = \frac{2}{h}\sum_{m=1}^{\infty}\overline{\overline{T}}_0(n,m,t)\sin(\frac{m\pi z}{h}) \tag{4.6}$$

Finally, by the inversion theorem (10.3.2), and by substitution of (4.5) in (4.6) we have,

$$\begin{aligned} T(r,z,t) = \frac{2}{h}\sum_n\sum_m \frac{1}{C_n}&\Big\{\overline{\overline{T}}_0(n,m)\exp[-\hbar(\mu_n^2 + m^2\pi^2/h^2)t] \\ &-\hbar\int_0^t \chi_s(m,u)\exp[\hbar(\mu_n^2+m^2\pi^2/h^2)(t-u)]\Big\} \\ &\times\ \sin(\frac{m\pi z}{h})S_0(k_1,k_2,\mu_n r) \end{aligned} \tag{4.7}$$

where C_n are given by Eq.(3.6) with $p = 0$

Determination of thermoelastic displacement

Substituting the values of $T(r,z,t)$ from Eq.(4.7) in Eq. (2.1), one obtains the thermoelastic displacement function $\phi(r,z,t)$ as,

$$\begin{aligned} \phi(r,z,t) = \left(\frac{1+\nu}{1-\nu}\right)\frac{a_t}{2h}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{C_n}&\Big\{\overline{\overline{T}}_0(n,m)\exp[-\hbar(\mu_n^2+\frac{m^2\pi^2}{h^2})t] \\ -\hbar\int_0^t \chi_s(m,u)\exp[-\hbar(\mu_n^2+\frac{m^2\pi^2}{h^2})(t-u)]du\Big\}& \\ \times\ \sin(\frac{m\pi z}{h})r^2 S_0(k_1,k_2,\mu_n r)& \end{aligned} \tag{4.8}$$

Determination of displacement functions

using Eq. (4.8) in Eq.(2.10) and Eq.(2.11), one obtains the radial and axial displacement U and W as

$$\begin{aligned} U(r,z,t) = \left(\frac{1+\nu}{1-\nu}\right)\frac{a_t}{2h}\sum_{n=1}^{\infty}\frac{1}{C_n}&\Big\{\sum_{m=1}^{\infty}\overline{\overline{T}}_0(n,m)\exp[-\hbar(\mu_n^2+\frac{m^2\pi^2}{h^2})t] \\ -\hbar\int_0^t \chi_s(m,u)\exp[-\hbar(\mu_n^2+\frac{m^2\pi^2}{h^2})(t-u)]du\Big\}&\sin(\frac{m\pi z}{h}) \\ \times\ [r(2S_0(k_1,k_2,\mu_n r)) + \mu_n r S_0^{'}(k_1,k_2,\mu_n r))]& \end{aligned} \tag{4.9}$$

$$\begin{aligned} W(r,z,t) = \left(\frac{1+\nu}{1-\nu}\right)\frac{\pi a_t}{2h}\sum_{n=1}^{\infty}\frac{1}{C_n}&\Big\{\sum_{m=1}^{\infty} m\left[\overline{\overline{T}}_0(n,m)\exp[-\hbar(\mu_n^2+\frac{m^2\pi^2}{h^2})t]\right] \\ -\hbar\int_0^t \chi_s(m,u)\exp[-\hbar(\mu_n^2+\frac{m^2\pi^2}{h^2})(t-u)]du\Big\}& \\ \times\ \cos(\frac{m\pi z}{h})[r^2 S_0(k_1,k_2,\mu_n r))]& \end{aligned} \tag{4.10}$$

Determination of stress function

$$\sigma_{rr} = \left(\frac{1+\nu}{1-\nu}\right)\frac{a_t}{2h}\sum_{n=1}^{\infty}\frac{1}{C_n}\left\{\sum_{m=1}\infty[\overline{\overline{T}}_0(n,m) - \hbar\int_0^t \chi_s(m,u)du\right\}\sin(\frac{m\pi z}{h})$$
$$\times\left\{(\lambda+2G)\left[2S_0(k_1,k_2,\mu_n r) + (1+2r)\mu_n S_0^{'}(k_1,k_2,\mu_n r)\right.\right.$$
$$\left.+\mu_n^2 r^2 S_0^{''}(k_1,k_2,\mu_n r)\right] + \lambda\left[2S_0(k_1,k_2,\mu_n r) + \mu_n r S_0^{'}(k_1,k_2,\mu_n r)\right.$$
$$\left.\left.-\frac{m^2\pi^2 r^2}{h^2}S_0(k_1,k_2,\mu_n r)\right]\right\} \tag{4.11}$$

$$\sigma_{zz} = \left(\frac{1+\nu}{1-\nu}\right)\frac{a_t}{2h}\sum_{n=1}^{\infty}\frac{1}{C_n}\left\{\sum_{m=}^{\infty}[\overline{\overline{T}}_0(n,m) - \hbar\int_0^t \chi_s(m,u)du\right\}\sin(\frac{m\pi z}{h})$$
$$\times\left\{\lambda\left[4S_0(k_1,k_2,\mu_n r) + (1+3r)\mu_n S_0^{'}(k_1,k_2,\mu_n r)\right.\right.$$
$$\left.\left.-(\lambda+2G)\frac{m^2\pi^2 r^2}{h^2}S_0(k_1,k_2,\mu_n r)\right]\right\} \tag{4.12}$$

$$\sigma_{\theta\theta} = \left(\frac{1+\nu}{1-\nu}\right)\frac{a_t}{2h}\sum_{n=}^{\infty}\frac{1}{C_n}\left\{\sum_{m=1}^{\infty}[\overline{\overline{T}}_0(n,m) - \hbar\int_0^t \chi_s(m,u)du\right\}\sin(\frac{m\pi z}{h})$$
$$\times\left\{(\lambda+2G)\left[2S_0(k_1,k_2,\mu_n r) + \mu_n r S_0^{''}(k_1,k_2,\mu_n r)\right]\right.$$
$$+\lambda\left[\left(2-\frac{m^2\pi^2 r^2}{h^2}\right)S_0(k_1,k_2,\mu_n r) + (1+2r)\mu_n S_0^{'}(k_1,k_2,\mu_n r)\right.$$
$$\left.\left.+\mu_n^2 r^2 S_0^{''}(k_1,k_2,\mu_n r)\right]\right\} \tag{4.13}$$

$$\tau_{rz} = G\left(\frac{1+\nu}{1-\nu}\right)\frac{\pi a_t}{2h^2}\sum_{n=1}^{\infty}\frac{1}{C_n}\left\{\sum_{m=1}^{\infty}m[\overline{\overline{T}}_0(n,m)e^{-\hbar} - \hbar\int_0^t \chi_s(m,u)e^{-\hbar}du\right\}$$
$$\times\left\{\left[2\cos(\frac{m\pi z}{h}) - \frac{m\pi r}{h}\sin(\frac{m\pi z}{h})\right]rS_0(k_1,k_2,\mu_n r)\right.$$
$$\left.+\mu_n\cos(\frac{m\pi z}{h})r^2 S_0^{'}(k_1,k_2,\mu_n r)\right\} \tag{4.14}$$

5 Special Case

Set,

$$\chi_s(m,u) = T_0\delta(r-a)\delta(r-b)e^t \tag{5.1}$$

After substituting the values of $\chi_s(m,u)$ in Eq.(4.7), we obtain the results,

$$T(r,z,t) = \frac{2}{h}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{C_n}\left\{\overline{\overline{T}}_0(n,m)\exp[-\hbar(\mu_n^2 + m^2\pi^2/h^2)t]\right.$$
$$-\hbar T_0 t\delta(r-a)\delta(r-b)\left(\mu_n^2 + \frac{m^2\pi^2}{h^2}\right)e^{-t^2\left(\mu_n^2+\frac{m^2\pi^2}{h^2}\right)}$$
$$\left.\times\left[1 - e^{-t^2\left(\mu_n^2+\frac{m^2\pi^2}{h^2}\right)}\right]\right\}\sin\left(\frac{m\pi z}{h}\right)S_0(k_1,k_2,\mu_n r) \tag{5.2}$$

Similarly, Substituting the value of equation (5.1) into equation (4.8) to (4.14), one obtains the expressions for the temperature and stresses respectively. This is the required results of the given problem.

6 Numerical results, discussion and remarks

To interpret the numerical computations, we consider material properties of Aluminum metal, which can be commonly used in both, wrought and cast forms. The low density of aluminum results in its extensive use in an aerospace industry and in other transportation fields. Its resistance to corrosion leads to use in food and chemical handling (cookware, pressure vessels, etc.) and to architectural uses tends to expand more than the inner surface leading inner part being under tensile stress. The foregoing analysis are performed by setting the radiation coefficients constants, $k_i = 0.86(i = 1, 3)$ and $k_i = 1(i = 2, 4)$ so as to obtain considerable mathematical simplicities. The derived numerical results for temperature distribution and stress function has been illustrated graphically with available additional sectional heat on its flat surface $z = 1$.

Table 1- Material properties and parameters used in this study.

Property values are nominal.

Modulus of Elasticity,E($dynes/cm^2$)	6.9×1011
Shear modulus, G($dynes/cm^2$)	2.7× 1011
Poisson ratio, υ	0.281
Thermal expansion coefficient,α_t(cm/cm-0c)	25.5×10-6
Thermal diffusivity,κ(cm^2/sec)	0.86
Thermal conductivity, λ (cal-cm/0c/sec/cm^2)	0.48
Inner radius, a(cm)	1
outer radius, b(cm)	4
Thickness,h(cm)	2

7 Conclusion

In this study, we treated the two-dimensional thermoelastic problem of a finite length hollow cylinder in which sources are generated according to the linear function of the temperature. We successfully established and obtained the temperature distribution, displacements and stress functions with additional sectional heat of the cylinder. Then, in order to examine the validity of two-dimensional thermoelastic boundary value problem, we analyze, as a particular case with mathematical and numerical calculations were carried out. Moreover, assigning suitable values to the parameters and functions in the equations of temperature, displacements and stresses respectively, expressions of special interest can be derived for any particular case. We may conclude that the system of equations proposed in this study can be adapted to design of useful structures or machines in engineering applications in the determination of thermoelastic behavior with radiation. The expression (5.2) is represented graphically in the radial and axial direction. Any particular case of

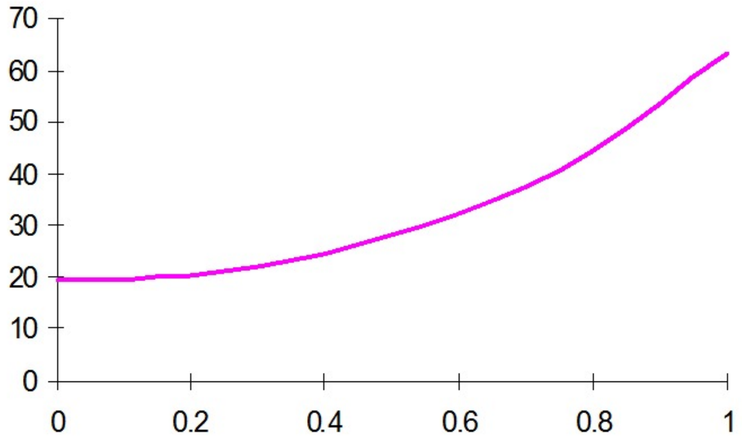

Figure 1: The temperature distribution $T(r,z,t)$ in radial direction

special interest can be derived by assigning suitable values to the parameter and functions int the expression (4.7)

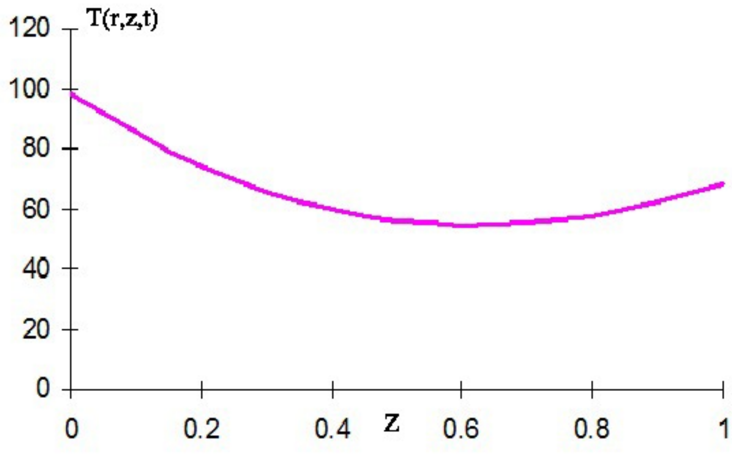

Figure 2: The temperature distribution $T(r, z, t)$ in axial direction

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