On Somewhat $\pi gb$-Continuous and Somewhat $\pi gb$-Open Functions

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Abstract: In this paper, we have introduced new continuous and open functions called somewhat $\pi gb$-continuous and somewhat $\pi gb$-open functions by using $\pi gb$-open sets. Further somewhat almost $\pi gb$-open sets, and somewhat $M$-$\pi gb$-open functions are also discussed. Its various characterizations and properties are established.

Keywords: somewhat $\pi gb$-continuous, somewhat $\pi gb$-irresolute, somewhat $\pi gb$-open, somewhat $\pi gb$-dense, somewhat $\pi gb$-seperable, somewhat almost $\pi gb$-open sets and somewhat $M$-$\pi gb$-open functions.

1. Introduction:

Andrijevic [1] introduced a new class of generalized open sets called $b$-open sets in a topological space. This type of sets was discussed by Ekici and Caldas [6] under the name of $\gamma$-open sets. Levine [10] introduced the concept of generalized closed sets in topological space and a class of topological spaces called $T_{1\frac{1}{2}}$ spaces. The concepts of feebly continuous functions and feebly open functions were first introduced and studied by Zdenek Frolik [15]. Gentry and Hoyle [7] introduced and studied the concept of somewhat open functions which are Frolik functions with some conditions being dropped. These ideas are closely related to weakly equivalent topologies which was first introduced by [14]. S.S Benchalli and Priyanka M.Bansali [2] introduced somewhat $b$-continuous and somewhat $b$-open functions in topological spaces.

In this paper, by using $\pi gb$-open sets new functions called somewhat $\pi gb$-continuous, somewhat $\pi gb$-irresolute and somewhat $\pi gb$-open functions are introduced and discussed. Also somewhat almost $\pi gb$-open sets and somewhat $M$-$\pi gb$-open functions are discussed. These findings results in procuring several characterizations and properties of this class.

2. Preliminaries

Throughout this paper $(X, \tau)$ and $(Y, \tau)$ represent non-empty topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset $A$ of a space $(X, \tau)$ $cl(A)$ and $int(A)$ denote the closure of $A$ and the interior of $A$ respectively. $(X, \tau)$ will be replaced by $X$ if there is no chance of confusion.

Definition 2.1: A subset $A$ of a space $X$ is said to be:
1) semi open [9] if $A \subseteq Cl(Int(A))$.
2) a regular open set if $A= int(cl(A))$ and a regular closed set if $A= Cl(int(A))$;
3) $b$-open [1] or sp-open [4], $\gamma$ –open [6] if $A \subseteq Cl(int(A)) \cup int(cl(A))$.

Definition 2.2: A subset $A$ of a space $(X, \tau)$ is called
1) a generalized $b$-closed (briefly $gb$-closed)[8] if $bcl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open.
2) $\pi g$-closed [5] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\pi$-open.
3) $\pi gb$-closed [12] if $bcl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\pi$-open in $(X, \tau)$. 

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By $\pi\text{GB}(\tau)$ we mean the family of all $\pi\text{gb}$-closed subsets of the space $(X, \tau)$.

**Definition 2.3:** A function $f: (X, \tau) \to (Y, \sigma)$ is called

1) $\pi\text{gb}$-continuous [12] if every $f^{-1}(V)$ is $\pi\text{gb}$-closed in $(X, \tau)$ for every closed set $V$ of $(Y, \sigma)$.

2) $\pi\text{gb}$-irresolute [12] if $f^{-1}(V)$ is $\pi\text{gb}$-closed in $(X, \tau)$ for every $\pi\text{gb}$-closed set $V$ in $(Y, \sigma)$.

**Definition 2.4[13]:** A map $f: X \to Y$ is said to be $\pi\text{gb}$-open if for every open set $F$ of $X$, $f(F)$ is $\pi\text{gb}$-open in $Y$.

**Definition 2.5[13]:** A map $f: (X, \tau) \to (Y, \sigma)$ is said to be a $\pi\text{gb}$-open map if the image $f(A)$ is $\pi\text{gb}$-open in $Y$ for every $\pi\text{gb}$-open set $A$ in $X$.

**Definition 2.6:** A subset $D$ of a topological space $X$ is said to be dense (or everywhere dense) in $X$ if the closure of $D$ is equal to $X$. Equivalently, $D$ is dense if and only if $D$ intersects every non-empty open set.

**Definition 2.7** [7]: A function $f: X \to Y$ is said to be somewhat continuous if for $U \in \sigma$ and $f^{-1}(U) \neq \emptyset$ there exists an open set $V$ in $X$ such that $V \neq \emptyset$ and $V \subseteq f^{-1}(U)$.

**Definition 2.8** [11]: A function $f: X \to Y$ is said to be somewhat semi continuous if for $U \in \sigma$ and $f^{-1}(U) \neq \emptyset$ there exists a semi open set $V$ in $X$ such that $V \neq \emptyset$ and $V \subseteq f^{-1}(U)$.

**Remark 2.9** [11]: Every somewhat continuous function is somewhat semi continuous function.

**Definition 2.10** [7]: A function $f: X \to Y$ is said to be somewhat open function provided that for $U \in \tau$ and $U \neq \emptyset$, there exists an open set $V$ in $Y$ such that $V \neq \emptyset$ and $V \subseteq f(U)$.

**Definition 2.11** [11]: A function $f: X \to Y$ is said to be somewhat semi open function provided that for $U \in \tau$ and $U \neq \emptyset$, there exists a semi open set $V$ in $Y$ such that $V \neq \emptyset$ and $V \subseteq f(U)$.

**Remark 2.12** [11]: Every somewhat open function is somewhat semi open function but the converse need not be true in general.

### 3. Somewhat $\pi\text{gb}$-Continuous Functions

**Definition 3.1** Let $(X, \tau)$ and $(Y, \sigma)$ be any two topological spaces. A function $f: X \to Y$ is said to be somewhat $\pi\text{gb}$-continuous function if for every $U \in \sigma$ and $f^{-1}(U) \neq \emptyset$ there exists a $\pi\text{gb}$-open set $V$ in $X$ such that $V \neq \emptyset$ and $V \subseteq f^{-1}(U)$.

**Example 3.2** Let $X = \{a, b, c\}$, $\tau = \{X, \{a\}, \{a, b\}, \phi\}$, $\sigma = \{X, \phi, \{a\}\}$. Now define a function $f: (X, \tau) \to (X, \sigma)$ as follows: $f(a) = b$, $f(b) = a$, $f(c) = c$. Then clearly $f$ is a somewhat $\pi\text{gb}$-continuous function.

**Theorem 3.3** Every somewhat semi continuous function is somewhat $\pi\text{gb}$-continuous function.

**Proof.** Let $f: X \to Y$ be somewhat semi continuous function. Let $U$ be any open set in $Y$ such that $f^{-1}(U) \neq \emptyset$. Since $f$ is somewhat semi continuous, there exists a semi open set $V$ in $X$ such that $V \neq \emptyset$ and $V \subseteq f^{-1}(U)$. Since every semi open set is $\pi\text{gb}$-open, there exists a $\pi\text{gb}$-open set $V$ such that $V \neq \emptyset$ and $V \subseteq f^{-1}(U)$, which implies that $f$ is somewhat $\pi\text{gb}$-continuous function.

**Remark 3.4** Converse of the above theorem need not be true in general which follows from the following example.

**Example 3.5** Let $X = \{a, b, c\}$, $\tau = \{X, \{a, b\}, \phi\}$, $\sigma = \{X, \phi, \{a\}\}$. Define a function $f: (X, \tau) \to (X, \sigma)$ by $f(a) = a$, $f(b) = b$, $f(c) = c$. Then $f$ is a somewhat $\pi\text{gb}$-continuous function but not somewhat semi continuous function.

**Theorem 3.6** Every somewhat continuous function is somewhat $\pi\text{gb}$-continuous function.

**Proof.** Follows from Theorem 3.3 and Remark 2.4.
Remark 3.7 Converse of the above theorem need not be true in general which follows from the following example.

Example 3.8 Let \( X = \{a, b, c, d\} \), \( \tau = \{X, \{c\}, \phi\} \), \( \sigma = \{X, \phi, \{a\}, \{b\}, \{a, b\}\} \). Define a function \( f: (X, \tau) \rightarrow (X, \sigma) \) by \( f(a) = a, f(b) = c, f(c) = d \) and \( f(d) = b \). Then clearly \( f \) is somewhat \( \pi gb \)-continuous function but not a somewhat continuous function.

Theorem 3.9 Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) and \( g: (Y, \sigma) \rightarrow (Z, \eta) \) be any two functions. If \( f \) is somewhat \( \pi gb \)-continuous function and \( g \) is continuous function, then \( g \circ f \) is somewhat \( \pi gb \)-continuous function.

Proof. Let \( U \subseteq \eta \). Suppose that \( g^{-1}(U) \neq \phi \). Since \( U \subseteq \eta \) and \( g \) is continuous function \( g^{-1}(U) \in \sigma \). Suppose that \( f^{-1}g^{-1}(U) \neq \phi \). Since by hypothesis \( f \) is somewhat \( \pi gb \)-continuous function, there exists a \( \pi gb \)-open set \( V \) in \( X \) such that \( V \neq \phi \) and \( V \subseteq f^{-1}(g^{-1}(U)) \). But \( f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U) \), which implies that \( V \subseteq (g \circ f)^{-1}(U) \). Therefore \( g \circ f \) is somewhat \( \pi gb \)-continuous function.

Remark 3.10 If \( f \) is continuous function and \( g \) is somewhat \( \pi gb \)-continuous function, then it is not necessarily true that \( g \circ f \) is somewhat \( \pi gb \)-continuous function.

Example 3.11 Let \( X = \{a, b, c\} \), \( \tau = \sigma = \{X, \phi, \{a\}, \{b\}, \{a,b\}\} \) and \( \eta = \{X, \phi, \{b,c\}, \{c\}\} \). Define \( f: (X, \tau) \rightarrow (X, \sigma) \) by \( f(a) = a, f(b) = b \) and \( f(c) = c \) and define \( g: (X, \sigma) \rightarrow (X, \eta) \) by \( g(a) = b, g(b) = a \) and \( g(c) = c \). Then clearly \( f \) is continuous function and \( g \) is somewhat \( \pi gb \)-continuous function but \( g \circ f \) is not a somewhat \( \pi gb \)-continuous function.

Definition 3.12 Let \( A \) be a subset of a topological space \( (X, \tau) \). Then \( A \) is said to be \( \pi gb \)-dense in \( X \) if there is no proper \( \pi gb \)-closed set \( C \) in \( X \) such that \( A \subseteq C \subset X \).

Theorem 3.13 Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) be a function. Then the following are equivalent:

(i) \( f \) is somewhat \( \pi gb \)-continuous function.

(ii) If \( C \) is a closed subset of \( Y \) such that \( f^{-1}(C) \neq X \), then there is a proper \( \pi gb \)-closed subset \( D \) of \( X \) such that \( D \supset f^{-1}(C) \).

(iii) If \( M \) is a \( \pi gb \)-dense subset of \( X \) then \( f(M) \) is a dense subset of \( Y \).

Proof. (i) \( \Rightarrow \) (ii) : Let \( C \) be a closed subset of \( Y \) such that \( f^{-1}(C) \neq X \). Then \( Y - C \) is an open set in \( Y \) such that \( f^{-1}(Y - C) = X - f^{-1}(C) \neq \phi \). By hypothesis (i) there exists a \( \pi gb \)-open set \( V \) in \( X \) such that \( V \neq \phi \) and \( V \subseteq f^{-1}(Y - C) = X - f^{-1}(C) \). This means that \( X - V \supset f^{-1}(C) \) and \( X - V = D \) is a \( \pi gb \)-closed set in \( X \). This proves (ii).

(ii) \( \Rightarrow \) (i) : Let \( U \subseteq \sigma \) and \( f^{-1}(U) \neq \phi \). Then \( Y - U \) is closed and \( f^{-1}(Y - U) = X - f^{-1}(U) \neq \phi \). By hypothesis of (ii) there exists a proper \( \pi gb \)-closed set \( D \) such that \( f^{-1}(Y - U) \supset D \). This implies that \( X - D \subset f^{-1}(U) \) and \( X - D \) is \( \pi gb \)-open and \( X - D \neq \phi \).

(ii) \( \Rightarrow \) (iii) : Let \( M \) be a \( \pi gb \)-dense set in \( X \). We have to show that \( f(M) \) is dense in \( Y \). Suppose not, then there exists a proper \( \pi gb \)-closed set \( C \) in \( Y \) such that \( f(M) \subseteq C \subseteq Y \). Clearly \( f^{-1}(C) \neq X \). Hence by (ii) there exists a proper \( \pi gb \)-closed set \( D \) such that \( M \subseteq f^{-1}(C) \subseteq D \subseteq X \). This contradicts the fact that \( M \) is \( \pi gb \)-dense in \( X \).

(iii) \( \Rightarrow \) (ii) : Suppose that (ii) is not true. This means there exists a closed set \( C \) in \( Y \) such that \( f^{-1}(C) \neq X \). But there is no proper \( \pi gb \)-closed set \( D \) in \( X \) such that \( f^{-1}(C) \subseteq D \). This means that \( f^{-1}(C) \) is \( \pi gb \)-dense in \( X \). But by (iii) \( f(f^{-1}(C)) = C \) must be dense in \( Y \), which is contradiction to the choice of \( C \).
Theorem 4.4 Let $(X, \tau)$ and $(Y, \sigma)$ be any two topological spaces, $A$ be an open set in $X$ and $f : (A, \tau|A) \to (Y, \sigma)$ be somewhat $\pi gb$-continuous function such that $f(A)$ is dense in $Y$. Then any extension $F$ of $f$ is somewhat $\pi gb$ – continuous function.

**Proof.** Let $U$ be any open set in $(Y, \sigma)$ such that $F^{-1}(U) \neq \phi$. Since $f(A) \subseteq Y$ is dense in $Y$ and $U \cap f(A) \neq \phi$ it follows that $F^{-1}(U) \cap A \neq \phi$. That is $f^{-1}(U) \cap A \neq \phi$. Hence by hypothesis on $f$, there exists a $\pi gb$-open set $V$ in $A$ such that $V \neq \phi$ and $V \subseteq f^{-1}(U) \subseteq f^{-1}(U)$ which implies $F$ is somewhat $\pi gb$-continuous function.

Theorem 3.15 Let $(X, \tau)$ and $(Y, \sigma)$ be any two topological spaces, $X = A \cup B$ where $A$ and $B$ are regular open and $\pi gb$-closed subsets of $X$ and $f : (X, \tau) \to (Y, \sigma)$ be a function such that $f/A$ and $f/B$ are somewhat $\pi gb$-continuous functions. Then $f$ is somewhat $\pi gb$-continuous function.

**Proof.** Let $U$ be any open set in $(Y, \sigma)$ such that $f^{-1}(U) \neq \phi$. Then $(f/A)^{-1}(U) \neq \phi$ or $(f/B)^{-1}(U) \neq \phi$ or both $(f/A)^{-1}(U) \neq \phi$ and $(f/B)^{-1}(U) \neq \phi$.

**Case 1.** Suppose $(f/A)^{-1}(U) \neq \phi$. Since $f/A$ is somewhat $\pi gb$-continuous, there exists a $\pi gb$-open set $V$ in $A$ such that $V \neq \phi$ and $V \subseteq (f/A)^{-1}(U) \subseteq f^{-1}(U)$. Since $X - V$ is $\pi gb$-closed in $A$ and $A$ is regular open and $\pi gb$-closed in $X$, $X - V$ is $\pi gb$-closed in $X$. Thus $f$ is somewhat $\pi gb$-continuous function.

**Case 2.** Suppose $(f/B)^{-1}(U) \neq \phi$. Since $f/B$ is somewhat $\pi gb$-continuous function, there exists a $\pi gb$-open set $V$ in $B$ such that $V \neq \phi$ and $V \subseteq (f/B)^{-1}(U) \subseteq f^{-1}(U)$. Since $X - V$ is $\pi gb$-closed in $B$ and $B$ is regular open and $\pi gb$-closed in $X$, $X - V$ is $\pi gb$-closed in $X$. Thus $f$ is somewhat $\pi gb$-continuous function.

**Case 3.** Suppose $(f/A)^{-1}(U) \neq \phi$ and $(f/B)^{-1}(U) \neq \phi$. This follows from both the cases 1 and 2. Thus $f$ is somewhat $\pi gb$-continuous function.

Definition 3.16 A topological space $X$ is said to be $\pi gb$-separable if there exists a countable subset $B$ of $X$ which is $\pi gb$-dense in $X$.

Theorem 3.17 If $f$ is somewhat $\pi gb$-continuous function from $X$ onto $Y$ and if $X$ is $\pi gb$-separable, then $Y$ is separable.

**Proof.** Let $f : X \to Y$ be somewhat $\pi gb$-continuous function such that $X$ is $\pi gb$-separable. Then by definition there exists a countable subset $B$ of $X$ which is $\pi gb$-dense in $X$. Then by Theorem 3.13, $f(B)$ is dense in $Y$. Since $B$ is countable $f(B)$ is also countable which is dense in $Y$, which indicates that $Y$ is separable.

4. Somewhat $\pi gb$-irresolute function

**Definition 4.1:** A function $f$ is said to be somewhat $\pi gb$-irresolute if for $U \subseteq \pi GBO(\sigma)$ and $f^{-1}(U) \neq \phi$, there exists a non-empty $\pi gb$-open set $V$ in $X$ such that $V \subseteq f^{-1}(U)$.

**Example 4.2:** Let $X = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{b, c\}, X\}$ and $\sigma = \{\phi, \{a\}, \{b\}, X\}$. The function $f : (X, \tau) \to (X, \sigma)$ defined by $f(a) = c$, $f(b) = a$ and $f(c) = b$ is somewhat $\pi gb$-irresolute but not somewhat-irresolute.

**Example 4.3:** Let $X = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{b, a\}, X\}$ and $\sigma = \{\phi, \{a\}, \{b,c\}, X\}$. The function $f : (X, \tau) \to (X, \sigma)$ defined by $f(a) = c$, $f(b) = a$ and $f(c) = b$ is not somewhat $\pi gb$-irresolute and somewhat-irresolute.

**Theorem 4.4:** If $f$ is somewhat $\pi gb$-irresolute and $g$ is $\pi gb$-irresolute, then $g \circ f$ is somewhat $\pi gb$-irresolute.
Proof. Let $U \in \pi\text{GBO}(\eta)$. Suppose that $g^{-1}(U) \neq \phi$. Since $U \in \pi\text{GBO}(\eta)$ and $g$ is $\pi\text{gb}$-irresolute function $g^{-1}(U) \in \pi\text{GBO}(\sigma)$. Suppose that $f^{-1}(g^{-1}(U)) \neq \phi$. Since by hypothesis $f$ is somewhat $\pi\text{gb}$-irresolute function, there exists a $\pi\text{gb}$-open set $V$ in $X$ such that $V \neq \phi$ and $V \subseteq f^{-1}(g^{-1}(U))$. But $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$, which implies that $V \subseteq (g \circ f)^{-1}(U)$. Therefore $g \circ f$ is somewhat $\pi\text{gb}$-continuous function.

**Theorem 4.5** Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function. Then the following are equivalent:

(i) $f$ is somewhat $\pi\text{gb}$-irresolute function.

(ii) If $C$ is a $\pi\text{gb}$-closed subset of $Y$ such that $f^{-1}(C) \neq X$, then there is a proper $\pi\text{gb}$-closed subset $D$ of $X$ such that $D \supseteq f^{-1}(C)$.

(iii) If $M$ is a $\pi\text{gb}$-dense subset of $X$ then $f(M)$ is a $\pi\text{gb}$-dense subset of $Y$.

**Proof.** (i) $\Rightarrow$ (ii) : Let $C$ be a $\pi\text{gb}$-closed subset of $Y$ such that $f^{-1}(C) \neq X$. Then $Y - C$ is an $\pi\text{gb}$-open set in $Y$ such that $f^{-1}(Y - C) = X - f^{-1}(C) \neq \phi$. By hypothesis (i) there exists a $\pi\text{gb}$-open set $V$ in $X$ such that $V \neq \phi$ and $V \subseteq f^{-1}(Y - C) = X - f^{-1}(C)$. This means that $X - V \supseteq f^{-1}(C)$ and $X - V = D$ is a proper $\pi\text{gb}$-closed set in $X$. This proves (ii).

(ii) $\Rightarrow$ (i) : Let $U \in \pi\text{GBO}(\sigma)$ and $f^{-1}(U) \neq \phi$. Then $Y - U$ is $\pi\text{gb}$-closed and $f^{-1}(Y - U) = X - f^{-1}(U) \neq \phi$. By hypothesis of (ii) there exists a proper $\pi\text{gb}$-closed set $D$ such that $f^{-1}(Y - U) \subseteq D$. This implies that $X - D \subset f^{-1}(U)$ and $X - D$ is $\pi\text{gb}$-dense in $X$.

(iii) $\Rightarrow$ (ii) : Suppose that (ii) is not true. This means there exists a $\pi\text{gb}$-closed set $C$ in $Y$ such that $f^{-1}(C) \neq X$. But there is no proper $\pi\text{gb}$-closed set $D$ in $X$ such that $f^{-1}(C) \subseteq D$. This means that $f^{-1}(C)$ is $\pi\text{gb}$-dense in $X$. By (iii) $f(f^{-1}(C)) = C$ must be $\pi\text{gb}$-dense in $Y$, which is contradiction to the choice of $C$.

**Theorem 4.6:** Let $(X, \tau)$ and $(Y, \sigma)$ be any two topological spaces, $X = A \cup B$ where $A$ and $B$ are regular open and $\pi\text{gb}$-closed sets of $X$ and $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function such that $f/A$ and $f/B$ are somewhat $\pi\text{gb}$-irresolute functions. Then $f$ is somewhat $\pi\text{gb}$-irresolute function.

**Proof.** Let $U$ be any $\pi\text{gb}$-open set in $(Y, \sigma)$ such that $f^{-1}(U) \neq \phi$. Then $(f/A)^{-1}(U) \neq \phi$ or $(f/B)^{-1}(U) \neq \phi$ or both $(f/A)^{-1}(U) \neq \phi$ and $(f/B)^{-1}(U) \neq \phi$.

**Case 1.** Suppose $(f/A)^{-1}(U) \neq \phi$. Since $f/A$ is somewhat $\pi\text{gb}$-irresolute, there exists a $\pi\text{gb}$-open set $V$ in $A$ such that $V \neq \phi$ and $V \subseteq (f/A)^{-1}(U) \subseteq f^{-1}(U)$. Since $X - V$ is $\pi\text{gb}$-closed in $A$ and $A$ is regular open and $\pi\text{gb}$-closed sets in $X$, $X - V$ is $\pi\text{gb}$-closed in $X$. Thus $f$ is somewhat $\pi\text{gb}$-irresolute function.

The proof is similar for other two cases.

**Definition 4.7:** If $X$ is a set and $\tau$ and $\sigma$ are topologies for $X$, then $\tau$ is said to be equivalent to $\sigma$ [3] provided if $U \in \tau$ and $U \neq \phi$, then there is an open set $V$ in $(X, \sigma)$ such that $V \neq \phi$ and $V \subseteq U$ and if $U \in \sigma$ and $U \neq \phi$, then there is an open set $V$ in $(X, \tau)$ such that $V \neq \phi$ and $V \subseteq U$. 

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**Definition 5.1:** If $X$ is a set and $\tau$ and $\sigma$ are topologies for $X$, then $\tau$ is said to be $\pi gb$-equivalent to $\sigma$ provided if $U \in \tau$ and $U \neq \emptyset$, then there is a $\pi gb$-open set $V$ in $(X, \sigma)$ such that $V \neq \emptyset$ and $V \subseteq U$ and if $U \in \sigma$ and $U \neq \emptyset$ then there is a $\pi gb$-open set $V$ in $(X, \tau)$ such that $V \neq \emptyset$ and $V \subseteq U$.

**Theorem 4.9:** Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be somewhat continuous function and let $\tau'$ be a topology for $X$, which is $\pi gb$-equivalent to $\tau$ then the function $f : (X, \tau') \rightarrow (Y, \sigma)$ is somewhat $\pi gb$-continuous function.

**Proof.** Let $U$ be any open set in $(Y, \sigma)$ such that $f^{-1}(U) \neq \emptyset$. Since by hypothesis $f : (X, \tau) \rightarrow (Y, \sigma)$ is somewhat continuous by definition there exists an open set $O$ in $(X, \tau)$ such that $O \neq \emptyset$ and $O \subseteq f^{-1}(U)$. Since $O$ is an open set in $(X, \tau)$ such that $O \neq \emptyset$. Hence $O \subseteq f^{-1}(U)$ Thus for any open set $U$ in $(Y, \sigma)$ such that $f^{-1}(U) \neq \emptyset$ there exist a $\pi gb$-open set $V$ in $(X, \tau')$ such that $V \subseteq f^{-1}(U)$. So $f : (X, \tau') \rightarrow (Y, \sigma)$ is somewhat $\pi gb$-continuous function.

**Theorem 4.10:** Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be somewhat $\pi gb$-continuous function and let $\sigma'$ be a topology for $Y$ which is equivalent to $\sigma$. Then $f : (X, \tau) \rightarrow (Y, \sigma')$ is somewhat $\pi gb$-continuous function.

**Proof.** Let $U$ be an open set in $(Y, \sigma)$ such that $f^{-1}(U) \neq \emptyset$ which implies $O \neq \emptyset$. Since $\sigma$ and $\sigma'$ are equivalent there exists an open set $W$ in $(Y, \sigma')$ such that $W \neq \emptyset$ and $W \subseteq U$. Now, $W$ is an open set such that $W \neq \emptyset$ which implies $f^{-1}(W) \neq \emptyset$. Now by hypothesis $f : (X, \tau) \rightarrow (Y, \sigma)$ is somewhat $\pi gb$-continuous function. Therefore there exists a $\pi gb$-open set $V$ in $X$, such that $V \subseteq f^{-1}(W)$. Now $W \subseteq U$ implies $f^{-1}(W) \subseteq f^{-1}(U)$. Hence $W \subseteq f^{-1}(U)$. Thus for any open set $U$ in $(Y, \sigma)$ such that $f^{-1}(U) \neq \emptyset$ there exist a $\pi gb$-open set $V$ in $(X, \tau')$ such that $V \subseteq f^{-1}(U)$. So $f : (X, \tau') \rightarrow (Y, \sigma)$ is somewhat $\pi gb$-continuous function.

**Theorem 4.11:** Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be somewhat $\pi gb$-irresolute surjection and let $\tau'$ be a topology for $X$, which is $\pi gb$-equivalent to $\tau$ then the function $f : (X, \tau') \rightarrow (Y, \sigma)$ is somewhat $\pi gb$-irresolute function.

**Proof.** Let $U$ be any $\pi gb$-open set in $(Y, \sigma)$ such that $f^{-1}(U) \neq \emptyset$ which implies $O \neq \emptyset$. Since by hypothesis $f : (X, \tau) \rightarrow (Y, \sigma)$ is somewhat $\pi gb$-irresolute, by definition there exists an $\pi gb$-open set $O$ in $(X, \tau)$ such that $O \neq \emptyset$ and $O \subseteq f^{-1}(U)$. Since $O$ is an $\pi gb$-open set in $(X, \tau)$ such that $O \neq \emptyset$ and since by hypothesis $\tau$ is $\pi gb$-equivalent to $\tau'$ by definition there exists a $\pi gb$-open set $V$ in $(X, \tau')$ such that $V \neq \emptyset$ and $V \subseteq O \subseteq f^{-1}(U)$. Hence $O \subseteq f^{-1}(U)$ Thus for any open set $U$ in $(Y, \sigma)$ such that $f^{-1}(U) \neq \emptyset$ there exist a $\pi gb$-open set $V$ in $(X, \tau')$ such that $V \subseteq f^{-1}(U)$. So $f : (X, \tau') \rightarrow (Y, \sigma)$ is somewhat $\pi gb$-irresolute function.

**Theorem 4.12:** Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be somewhat $\pi gb$-irresolute surjection function and let $\sigma'$ be a topology for $Y$ which is equivalent to $\sigma$. Then $f : (X, \tau) \rightarrow (Y, \sigma')$ is somewhat $\pi gb$-irresolute function.

**Proof.** Let $U$ be an open set in $(Y, \sigma)$ such that $f^{-1}(U) \neq \emptyset$ which implies $O \neq \emptyset$. Since $\sigma$ and $\sigma'$ are equivalent there exists an open set $W$ in $(Y, \sigma)$ such that $W \neq \emptyset$ and $W \subseteq U$. Now, $W$ is an open set such that $W \neq \emptyset$. Now by hypothesis $f : (X, \tau) \rightarrow (Y, \sigma)$ is somewhat $\pi gb$-irresolute function. Therefore there exists a $\pi gb$-open set $V$ in $X$, such that $V \subseteq f^{-1}(W)$. Now $W \subseteq U$ implies $f^{-1}(W) \subseteq f^{-1}(U)$. Hence $W \subseteq f^{-1}(U)$ Thus for any open set $U$ in $(Y, \sigma)$ such that $f^{-1}(U) \neq \emptyset$ there exist a $\pi gb$-open set $V$ in $(X, \tau')$ such that $V \subseteq f^{-1}(U)$. So $f : (X, \tau') \rightarrow (Y, \sigma)$ is somewhat $\pi gb$-irresolute function.

**5. Somewhat $\pi gb$-open functions**

**Definition 5.1:** A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be somewhat $\pi gb$-open function provided that for $U \in \tau$ and $U \neq \emptyset$ there exists a $\pi gb$-open set $V$ in $Y$ such that $V \neq \emptyset$ and $V \subseteq f(U)$. 


Example 5.2: Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{a, b\}, \{a, c\}\}$ and $\sigma = \{X, \phi, \{a\}\}$. Define a function $f: (X, \tau) \rightarrow (X, \sigma)$ by $f(a) = b$, $f(b) = c$, $f(c) = a$. Then clearly $f$ is somewhat $\pigb$-open function.

Theorem 5.3: Every somewhat semi-open function is somewhat $\pigb$-open function.

Proof. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a somewhat semi-open function. Let $U \in \tau$ and $U \neq \phi$. Since $f$ is somewhat semi-open there exists a semi-open set $V$ in $Y$ such that $V \neq \phi$ and $V \subseteq f(U)$. But every semi-open set is $\pigb$-open. Therefore there exists a $\pigb$-open set $V$ in $Y$ such that $V \neq \phi$ and $V \subseteq f(U)$, which implies that $f$ is somewhat $\pigb$-open function.

Remark 5.4: Converse of the above theorem need not be true in general, which follows from the following example.

Example 5.5 Let $X = \{a, b, c, d\}$. Let $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{X, \phi\}$. Define a function $f: (X, \tau) \rightarrow (X, \sigma)$ by $f(a) = c$, $f(b) = a$, $f(c) = c$. Then clearly $f$ is somewhat $\pigb$-open function which is not a somewhat semi-open function.

Theorem 5.6: Every somewhat open function is somewhat $\pigb$-open function.

Proof. Let $f: (X, \tau) \rightarrow (X, \sigma)$ be somewhat open function. Let $U \in \tau$ and $U \neq \phi$. Since $f$ is somewhat open function, there exists an open set $V$ in $Y$ such that $V \neq \phi$ and $V \subseteq f(U)$. But every open set is $\pigb$-open. So there exists a $\pigb$-open set $V$ in $Y$ such that $V \neq \phi$ and $V \subseteq f(U)$. Thus $f$ is somewhat $\pigb$-open function.

Remark 5.7: Converse of the above theorem need not be true in general, which follows from the following example.

Example 5.8 Let $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{X, \{a\}, \{a, c\}, \phi\}$. Define a function $f: (X, \tau) \rightarrow (X, \sigma)$ as follows $f(a) = b$, $f(b) = c$, $f(d) = a$, $f(c) = a$. Then clearly $f$ is somewhat $\pigb$-open function but not a somewhat open function.

Theorem 5.9: If $f: (X, \tau) \rightarrow (Y, \sigma)$ is an open map and $g: (Y, \sigma) \rightarrow (Z, \eta)$ is somewhat $\pigb$-open map then $g \circ f: (X, \tau) \rightarrow (Z, \eta)$ is somewhat $\pigb$-open map.

Proof. Let $U \in \tau$. Suppose that $U \neq \phi$. Since $f$ is an open map $f(U)$ is open and $f(U) \neq \phi$. Thus $f(U) \in \sigma$ and $f(U) \neq \phi$. Since $g$ is somewhat $\pigb$-open map and $f(U) \in \sigma$ such that $f(U) \neq \phi$ there exists a $\pigb$-open set $V \in \eta$, $V \subseteq g(f(U))$, which implies $g \circ f$ is somewhat $\pigb$-open function.

Theorem 5.10: If $f: (X, \tau) \rightarrow (Y, \sigma)$ is a one-one and onto mapping, then the following conditions are equivalent.

(i) $f$ is somewhat $\pigb$-open map.

(ii) If $C$ is a closed subset of $X$ such that $f(C) \neq Y$, then there is a $\pigb$-closed subset $D$ of $Y$ such that $D \neq Y$ and $D \supseteq f(C)$

Proof. (i) $\Rightarrow$ (ii): Let $C$ be any closed subset of $X$ such that $f(C) \neq Y$. Then $X - C$ is open in $X$ and $X - C \neq \phi$. Since $f$ is somewhat $\pigb$-open, there exists a $\pigb$-open set $V \neq \phi$ in $Y$ such that $V \subseteq f(X - C)$. Put $D = Y - V$. Clearly $D$ is $\pigb$-closed in $Y$ and we claim that $D \neq Y$. For if $D = Y$, then $V = \phi$ which is a contradiction. Since $V \subseteq f(X - C)$, $D = Y - V \supseteq Y - f([f(X - C)]) = f(C)$.

(ii) $\Rightarrow$ (i): Let $U$ be any non-empty open set in $X$. Put $C = X - U$. Then $C$ is a closed subset of $X$ and $f(X - U) = f(C) = Y - f(U)$ implies $f(C) \neq \phi$. Therefore, by (ii) there is a $\pigb$-closed subset $D$ of $Y$ such that $D \neq Y$ and $f(C) \subseteq D$. Put $V = Y - D$. Clearly $V$ is a $\pigb$-open set and $V \neq \phi$. Further,
Theorem 5.11 Let \( f : (X, \tau) \to (Y, \sigma) \) be somewhat \( \pi gb \)-open function and \( A \) be any open subset of \( X \). Then \( f/A : (A, \tau/A) \to (Y, \sigma) \) is also somewhat \( \pi gb \)-open function.

**Proof.** Let \( U \in \tau/A \) such that \( U \neq \emptyset \). Since \( U \) is open in \( A \) and \( A \) is open in \((X, \tau)\), \( U \) is open in \((X, \tau)\) and since by hypothesis \( f : (X, \tau) \to (Y, \sigma) \) is somewhat \( \pi gb \)-open function, there exists a \( \pi gb \)-open set \( V \) in \( Y \), such that \( V \subset f(U) \). Thus, for any open set \( U \in (A, \tau/A) \) with \( U \neq \emptyset \), there exists a \( \pi gb \)-open set \( V \) in \( Y \) such that \( V \subset f(U) \) which implies \( f/A \) is somewhat \( \pi gb \)-open function.

**Theorem 5.12** Let \( (X, \tau) \) and \( (Y, \sigma) \) be any two topological spaces and \( X = A \cup B \) where \( A \) and \( B \) are open subsets of \( X \) and \( f : (X, \tau) \to (Y, \sigma) \) be a function such that \( f/A \) and \( f/B \) are somewhat \( \pi gb \)-open, then \( f \) is also somewhat \( \pi gb \)-open function.

**Proof.** Let \( U \) be any open subset of \((X, \tau)\) such that \( U \neq \emptyset \). Since \( X = A \cup B \), either \( A \cap U \neq \emptyset \) or \( B \cap U \neq \emptyset \) or both \( A \cap U \neq \emptyset \) and \( B \cap U \neq \emptyset \). Since \( U \) is open in \((X, \tau)\), \( U \) is open in both \((A, \tau/A)\) and \((B, \tau/B)\).

**Case (i):** Suppose that \( U \cap A \neq \emptyset \) where \( U \cap A \) is open in \( \tau/A \). Since by hypothesis \( f/A \) is somewhat \( \pi gb \)-open function, there exists a \( \pi gb \)-open set \( V \in (Y, \sigma) \) such that \( V \subset f(U \cap A) \subset f(U) \), which implies \( f \) is somewhat \( \pi gb \)-open function.

**Case (ii):** Suppose that \( U \cap B \neq \emptyset \), where \( U \cap B \) is open in \((B, \tau/B)\). Since by hypothesis \( f/B \) is somewhat \( \pi gb \)-open function, there exists a \( \pi gb \)-open set \( V \) in \((Y, \sigma)\) such that \( V \subset f(U \cap B) \subset f(U) \), which implies that \( f \) is also somewhat \( \pi gb \)-open function.

**Case (iii):** Suppose that both \( U \cap B \neq \emptyset \) and \( U \cap A \neq \emptyset \). Then obviously \( f \) is somewhat \( \pi gb \)-open function from the case (i) and case (ii). Thus \( f \) is somewhat \( \pi gb \)-open function.

6. Somewhat almost \( \pi gb \)-open functions

**Definition 6.1:** A function \( f \) is said to be somewhat almost \( \pi gb \)-open provided that if \( U \in RO(\tau) \) and \( U \neq \emptyset \), then there exists a non-empty \( \pi gb \)-open set \( V \) in \( Y \) such that \( V \subset f(U) \).

**Example 6.2:** Let \( X = \{a, b, c\} \), \( \tau = \{\phi, \{a\}, X\} \) and \( \sigma = \{\phi, \{a\}, \{b, c\}, X\} \). The function \( f: (X, \tau) \to (X, \sigma) \) defined by \( f(a) = a, f(b) = c \) and \( f(c) = b \) is somewhat almost \( \pi gb \)-open, somewhat \( \pi gb \)-open and somewhat open.

**Theorem 6.3:** For a bijective function \( f \), the following are equivalent:

- (i) \( f \) is somewhat almost \( \pi gb \)-open.
- (ii) If \( C \) is regular closed in \( X \), such that \( f(C) \neq Y \), then there is a \( \pi gb \)-closed subset \( D \) of \( Y \) such that \( D \neq Y \) and \( D \supset f(C) \).

**Proof:** (i) \( \Rightarrow \) (ii): Let \( C \in RC(X) \) such that \( f(C) \neq Y \). Then \( X-C \neq \emptyset \in RO(X) \). Since \( f \) is somewhat almost \( \pi gb \)-open, there exists \( V \neq \emptyset \in \pi GBO(Y) \) such that \( V \subset f(X-C) \). Put \( D = Y-V \). Clearly \( D \neq \emptyset \in \pi GBC(Y) \). If \( D = Y \), then \( V = \emptyset \), which is a contradiction. Since \( V \subset f(X-C) \), \( D = Y-V \supset (Y- f(X-C)) = f(C) \).
(ii) \(\Rightarrow\) (i): Let \(U \neq \phi \in \text{RO}(X)\). Then \(C = X-U \in \text{RC}(X)\) and \(f(X-U) = f(C) = Y\). Since \(f(U) = Y\) and \(f(C) = Y\), then \(f(U) \neq f(C)\). This implies \(f(U) \neq Y\). Then by (ii), there is \(D \neq \phi \in \pi\text{GBO}(Y)\) and \(f(C) \subset D\). Clearly \(V = Y-D \neq \phi \in \pi\text{GBO}(Y)\). Also, \(V = Y-D \subset Y\). Since \(V = Y-D\), we have \(f(X-U) = f(U)\).

**Theorem 6.4:** The following statements are equivalent:

(i) \(f\) is somewhat almost \(\pi\text{gb-open}\).

(ii) If \(A\) is a \(\pi\text{gb-dense}\) subset of \(Y\), then \(f^{-1}(A)\) is a dense subset of \(X\).

**Proof:** (i) \(\Rightarrow\) (ii): Let \(A\) be a \(\pi\text{gb-dense}\) subset of \(Y\). If \(f^{-1}(A)\) is not dense in \(X\), then there exists \(B \in \text{RC}(X)\) such that \(f^{-1}(A) \subset B \subset X\). Since \(f\) is somewhat almost \(\pi\text{gb-open}\) and \(X-B \in \text{RO}(X)\), there exists \(C \neq \phi \in \pi\text{GBO}(Y)\) such that \(C \subset f(X-B)\). Therefore, \(C \subset f(X-B) \subset f(f^{-1}(Y-A)) \subset Y-A\). That is, \(A \subset Y-C \subset Y\). Now, \(Y-C\) is a \(\pi\text{gb-closed}\) set and \(A \subset Y-C \subset Y\). This implies that \(A\) is not a \(\pi\text{gb-dense}\) set in \(Y\), which is a contradiction. Therefore, \(f^{-1}(A)\) is a dense subset in \(X\).

(ii) \(\Rightarrow\) (i): If \(A \neq \phi \in \text{RO}(X)\). We want to show that \(\text{int}(\pi\text{gb-}(f(A))) \neq \phi\). Suppose \(\text{int}(\pi\text{gb-}(f(A))) = \phi\). Then, \(\pi\text{gb-cl}\{(f(A))\} = Y\). Then by (ii), \(f^{-1}(Y-f(A))\) is dense in \(X\). But \(f^{-1}(Y-f(A)) \subset X-A\). Now, \(X-A \in \text{RC}(X)\). Therefore, \(f^{-1}(Y-f(A)) \subset X-A\) gives \(X = \text{cl}\{(f^{-1}(Y-f(A))\} \subset X-A\). Thus \(A = \phi\), which contradicts \(A \neq \phi\). Therefore, \(\text{int}(\pi\text{gb-}(f(A))) \neq \phi\). Hence \(f\) is somewhat almost \(\pi\text{gb-open}\).

**Theorem 6.5:** Let \((X, \tau)\) and \((Y, \sigma)\) be any two topological spaces, \(X = A \cup B\) where \(A\) and \(B\) are regular open subsets of \(X\) and \(f: (X, \tau) \rightarrow (Y, \sigma)\) be a function such that \(f/A\) and \(f/B\) are somewhat almost \(\pi\text{gb-open}\) functions. Then \(f\) is somewhat almost \(\pi\text{gb-open}\).

**Proof:** Let \(U\) be any regular open set in \(X\). Since \(X = A \cup B\), either \(A \cap U \neq \phi\) or \(B \cap U \neq \phi\) or both \(A \cap U \neq \phi\) and \(B \cap U \neq \phi\). Since \(U\) is regular open in \(X\), \(U\) is regular open in both \(A\) and \(B\).

Case (i): If \(A \cap U \neq \phi \in \text{RO}(A)\). Since \(f/A\) is somewhat almost \(\pi\text{gb-open}\), there exists a \(\pi\text{gb-open}\) set \(V\) of \(Y\) such that \(V \subset f(U \cap A) \subset f(U)\), which implies that \(f\) is a somewhat almost \(\pi\text{gb-open}\). Case (ii): If \(B \cap U \neq \phi \in \text{RO}(B)\). Since \(f/B\) is somewhat almost \(\pi\text{gb-open}\), there exists a \(\pi\text{gb-open}\) set \(V\) of \(Y\) such that \(V \subset f(U \cap B) \subset f(U)\), which implies that \(f\) is somewhat almost \(\pi\text{gb-open}\). Case (iii): Suppose that both \(A \cap U \neq \phi\) and \(B \cap U \neq \phi\). Then by case (i) and (ii) \(f\) is somewhat almost \(\pi\text{gb-open}\).

7. Somewhat \(M\)- \(\pi\text{gb-open}\) function

**Definition 7.1:** A function \(f\) is said to be somewhat \(M\)- \(\pi\text{gb-open}\) provided that if \(U \in \pi\text{GBO}(\tau)\) and \(U \neq \phi\), then there exists a non-empty \(\pi\text{gb-open}\) set \(V\) in \(Y\) such that \(V \subset f(U)\).

**Example 7.2:** Let \(X = \{a, b, c\}, \tau = \{\phi, \{a\}, \{a, b\}\}\) and \(\sigma = \{\phi, \{a\}, \{b, c\}, X\}\). The function \(f: (X, \tau) \rightarrow (X, \sigma)\) defined by \(f(a) = a, f(b) = c\) and \(f(c) = b\) is somewhat quasi \(\pi\text{gb-open}\).

**Theorem 7.3:** For a bijective function \(f\), the following are equivalent:

(i) \(f\) is somewhat \(M\)- \(\pi\text{gb-open}\).
(ii) If $C \in \pi\text{GBC}(X)$, such that $f(C) \neq Y$, then there is a $\pi\text{g}_b$-closed subset $D$ of $Y$ such that $D \neq Y$ and $D \ni f(C)$.

**Proof:** (i) $\Rightarrow$ (ii): Let $C \in \pi\text{GBC}(X)$, such that $f(C) \neq Y$. Then $X-C \neq \emptyset$. If $X-C \neq \emptyset$, then $X-C \neq \emptyset$ and $X-C \neq \emptyset$.

**References:**


