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Second order parallel tensors on generalized Sasakian spaceforms

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Abstract: The object of present paper is to study the symmetric and skew symmetric properties of a second order parallel tensor in a generalized Sasakian space-form.

Keywords: Generalized Sasakian spaceform, Second order parallel tensor, Einstein manifold.

1. INTRODUCTION

In 1926 H.Levy[1] proved that a second order symmetric parallel non-singular tensor on a space of constant curvature is a constant multiple of the metric tensor. In recent papers R.Sharma([2],[3],[4]) generalized Levy's result and also studied a second order parallel tensor on Kaehler space of constant holomorphic sectional curvature as well as contact manifolds.In 1996 U.C.De[5] studied second order parallel tensor on a P-Sasakian manifold.Recently L.Das[6] studied a second order parallel tensor on a α-Sasakian manifold.

In this shown that in a regular generalized Sasakian space-form admits the second order symmetric parallel covariant tensorα is a constant multiple of the associated metric tensor if $\beta \neq 0$,where β is a function satisfying $\xi\beta = 0$. Further, it is shown that on a regular generalized Sasakian space-form there is no nonzero second order skew-symmetric parallel tensor when $\beta \neq 0$.

2. Generalized Sasakian space-forms

The nature of a Riemannian manifold mostly depends on the curvature tensor R of the manifold. It is well known that the sectional curvatures of a manifold determine curvature tensor completely. A Riemannian manifold with constant sectional curvature c is known as real-space-form and its curvature tensor is given by

$$R(X,Y)Z = c[g(Y,Z)X - g(X,Z)Y]. \tag{2.1}$$

A Sasakian manifold with constant ϕ-sectional curvature is a Sasakian-space-form and it has a specific form of its curvature tensor. Similar notion also holds for Kenmotsu and cosymplectic space-forms. In order to generalize such space-forms in a common frame P. Alegre,D.E.Blair and A.Carriazo introduced the notion of generalized Sasakian-space-forms in 2004[7].In this connection it should be mentioned that in 1989 Z.Olszak[8] studied generalized complex-space-forms and proved its existence. A generalized Sasakian-space-form is defined in[7]: Given an almost contact metric manifold $M(\phi,\xi,\eta,g)$, we say that M is generalized Sasakian-space-form if there exist three functions f_1,f_2,f_3 on M such that the curvature tensor R is given by

$$\begin{aligned} R(X,Y)Z &= f_1[g(Y,Z)X - g(X,Z)Y] + f_2[g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z] \\ &+ f_3[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi], \end{aligned} \tag{2.2}$$

for any vector fields X,Y,Z on M. In such a case we denote the manifold as $M(f_1,f_2,f_3)$. In[7] the authors cited several examples of such manifolds. If $f_1 = \frac{c+3}{4}$,$f_2 = \frac{c-1}{4}$ and $f_3 =$

$\frac{c-1}{4}$then a generalized Sasakian space-form with Sasakian structure becomes Sasakian-space-form.

Generalized Sasakian-space-forms and Sasakian-space-forms have been studied by several authors,viz., [7],[9],[10],[11].Symmetry of a manifold is the most important property among its all geometrical properties. Symmetry property of manifolds have been studied by many authors, viz.,[13],[14]. As a weaker notion of locally symmetric manifolds T.Takahashi[17] introduced and studied locally ϕ-symmetric Sasakian manifolds. Symmetry of a manifold primarily depends on curvature tensor and Ricci tensor of the manifold.

3. Preliminaries

A (2n+1)-dimensional Riemannian manifold (M, g) is called an almost contact manifold if the following results hold[9]:

$$\phi^2(X) = -X + \eta(X)\xi, \tag{3.1}$$

$$\phi\xi = 0, \eta(\xi) = 1, \tag{3.2}$$

$$g(X, \xi) = \eta(X), \eta(\phi X) = 0, \tag{3.3}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{3.4}$$

$$g(\phi X, Y) = -g(X, \phi Y), \tag{3.5}$$

$$g(\phi X, X) = 0, \tag{3.6}$$

$$(\nabla_X \eta)(Y) = g(\nabla_X \xi, Y), \tag{3.7}$$

An almost contact metric manifold is called contact metric manifold if

$$d\eta(X, Y) = \phi(X, Y) = g(X, \phi Y). \tag{3.8}$$

ϕ is called the fundamental two form of the manifold. If in addition ξ is a Killing vector the manifold is called a K-contact manifold. It is well known that a contact metric manifold is K-contact if and only if $\nabla_X \xi = -\phi X$, for any vector field X on M. On the other hand a normal contact metric manifold is known as Sasakian manifold. An almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \tag{3.9}$$

for any vector field X,Y. In 1967, D.E.Blair introduced the notion of quasi-Sasakian manifold to unify Sasakian and cosymplectic manifolds[19]. Again in 1987, Z.Olszak introduced and characterized three-dimensional quasi-Sasakian manifolds[18]. An almost contact metric manifold of dimension three is quasi-Sasakian if and only if

$$\nabla_X \xi = -\beta\phi X, \tag{3.10}$$

for all X $\in$ TM and a function β such that $\xi\beta = 0$. As a consequence of (3.10), we get

$$(\nabla_X \eta)(Y) = g(\nabla_X \xi, Y) = -\beta g(\phi X, Y), \tag{3.11}$$

$$(\nabla_X \eta)(\xi) = -\beta g(\phi X, \xi) = 0. \tag{3.12}$$

Clearly such a quasi-Sasakian manifold is cosymplectic if and only if $\beta = 0$. It is known that[20] for a three-dimensional quasi-Sasakian manifold the Riemannian curvature tensor satisfies

$$R(X, Y)\xi = \beta^2[\eta(Y)X - \eta(X)Y] + d\beta(Y)\phi X - d\beta(X)\phi Y. \tag{3.13}$$

For a (2n+1)-dimensional generalized Sasakian-space-form, we have

$$\begin{aligned} R(X, Y)Z &= f_1[g(Y, Z)X - g(X, Z)Y] + f_2[g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z] \\ &+ f_3[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi], \end{aligned} \tag{3.14}$$

$$R(X, Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y], \tag{3.15}$$

$$R(\xi, X)Y = (f_1 - f_3)[g(X, Y)\xi - \eta(Y)X], \tag{3.16}$$

$$g(R(\xi, X)Y, \xi) = (f_1 - f_3)g(\phi X, \phi Y), \tag{3.17}$$

$$R(\xi, X)\xi = (f_1 - f_3)\phi^2 X, \tag{3.18}$$

$$S(X, Y) = (2nf_1 + 3f_2 - f_3)g(X, Y) - (3f_2 + (2n - 1)f_3)\eta(X)\eta(Y), \tag{3.19}$$

$$Q\xi = 2n(f_1 - f_3)\xi, \tag{3.20}$$

$$S(\phi X, \phi Y) = S(X, Y) + 2n(f_3 - f_1)\eta(X)\eta(Y), \tag{3.21}$$

$$r = 2n(2n-1)f_1 + 6nf_2 - 4nf_3. \tag{3.22}$$

Here S is the Ricci tensor and r is the scalar curvature tensor of the space-forms. A generalized Sasakian space-form of dimension greater than three is said to be conformally flat if and only if Weyl-conformal curvature tensor vanishes. It is known that[11] a $(2n+1)$-dimensional$(n > 1)$ generalized Sasakian space-form is conformally flat if and only if $f_2 = 0$.

4. Second order parallel tensors

Definition 4.1. *A tensor α of second order is said to be a parallel tensor if $\nabla\alpha = 0$, where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g.*

Let α be a (0,2)-symmetric tensor field on a generalized Sasakian space-form M, such that $\nabla\alpha = 0$. Applying the Ricci identity[2], we get

$$\nabla^2\alpha(X, Y; Z, W) - \nabla^2\alpha(X, Y; W, Z) = 0 \tag{4.1}$$

which gives

$$\alpha(R(W, X)Y, Z) + \alpha(Y, R(W, X)Z) = 0, \tag{4.2}$$

for arbitrary vector fields W,X,Y,Z on M. The substitution of $Y = Z = \xi$ in (4.2) gives

$$\alpha(\xi, R(W, X)\xi) = 0 \tag{4.3}$$

since α is symmetric. By using the expression (3.14) for generalized Sasakian spaceform and (3.18)in the above equation, we get

$$(f_1 - f_3)[g(X, \xi)\alpha(W, \xi) - g(W, \xi)\alpha(X, \xi)] = 0. \tag{4.4}$$

Definition 4.2. *$M^{2n+1}(\xi)$ is called regular if $(f_1 - f_3 \neq 0)$.*

In order to obtain a characterization of such manifolds we consider:

Definition 4.3. *([16]) ξ is called semi-torse forming vector field for (M,g) if, for all vector fields X:*

$$R(X,\xi)\xi = 0. \tag{4.5}$$

From (3.16) we get

$$R(X,\xi)\xi = (f_1 - f_3)(X - \eta(X)\xi) \tag{4.6}$$

and therefore, if $X \in ker\eta = \xi^{\perp}$,then $R(X,\xi)\xi = (f_1 - f_3)X$ and we obtain:

Proposition 1. *For $M^{2n+1}(\xi)$ the following are equivalent: i) is regular,*
ii)ξ is not semi-torse forming,
iii)$S(\xi,\xi) \neq 0$ i.e.,ξ is non-degenerate with respect to S,
iv) $Q(\xi) \neq 0$ i.e.,ξ does not belong to the kernel of Q.
In particular,if ξ is parallel $(\nabla\xi = 0)$then M is not regular.

In the following we restrict to the regular case. Returning to (4.4), with $W = \xi$ then we obtain:

$$\eta(X)\alpha(\xi,\xi) - \alpha(X,\xi) = 0. \tag{4.7}$$

Moreover, by differentiating (4.7) covariantly along Y,we get

$$[g(\nabla_Y X,\xi)+g(X,\nabla_Y\xi)]\alpha(\xi,\xi)+2g(X,\xi)\alpha(\nabla_Y\xi,\xi)-[\alpha(\nabla_Y X,\xi)+\alpha(X,\nabla_Y\xi)] = 0. \tag{4.8}$$

Put $X = \nabla_Y X$ in (4.7)

$$g(\nabla_Y X,\xi)\alpha(\xi,\xi) - \alpha(\nabla_Y X,\xi) = 0. \tag{4.9}$$

From (4.8) and (4.9), we get

$$-\beta g(X,\phi Y)\alpha(\xi,\xi) - 2\beta g(X,\xi)\alpha(\phi Y,\xi) + \beta\alpha(X,\phi Y) = 0. \tag{4.10}$$

Replace X by ϕY in (4.7), we have

$$\alpha(\phi Y,\xi) = 0. \tag{4.11}$$

From (4.10) and (4.11) it follows that

$$-\beta[g(X,\phi Y)\alpha(\xi,\xi) - \alpha(X,\phi Y)] = 0. \tag{4.12}$$

Replacing Y by ϕY in (4.12) and using (3.1),(4.7), we get

$$-\beta[\alpha(X,Y) - g(X,Y)\alpha(\xi,\xi)] = 0. \tag{4.13}$$

By differentiating (4.13) covariantly along any vector field on M, it can be easily seen that $\alpha(\xi,\xi)$ is constant when $\beta \neq 0$. Hence we can state the following theorem:

Theorem 4.1. *A parallel second order symmetric covariant tensor in a regular Generalized Sasakian space forms is a constant multiple of the metric tensor when $\beta \neq 0$.*

Next, Let M be a generalized Sasakian space-form and α be a parallel 2-form. Putting $Y = W = \xi$ in (4.2), we obtain

$$\alpha(R(\xi,X)\xi,Z) + \alpha(\xi,R(\xi,X)Z) = 0. \tag{4.14}$$

Using (3.16) and (3.18) then we get,

$$(f_1 - f_3)[\alpha(\xi,Z)\eta(X) - \alpha(X,Z) + \alpha(\xi,\xi)g(X,Z) - \alpha(\xi,X)\eta(Z)] = 0, \tag{4.15}$$

Here also we restrict to the regular case, ie.,$(f_1 - f_3) \neq 0$. Then the above equation becomes

$$\alpha(X,Z) = \alpha(\xi,Z)\eta(X) + \alpha(\xi,\xi)g(X,Z) - \alpha(\xi,X)\eta(Z), \tag{4.16}$$

since α is 2-form, that is α is (0,2) skew-symmetric tensor. Therefore $\alpha(\xi,\xi) = 0$ and hence (4.16) reduces to

$$\alpha(X,Z) = \alpha(\xi,Z)\eta(X) - \alpha(\xi,X)\eta(Z). \tag{4.17}$$

Now, let A be (1,1) tensor field which is metrically equivalent to α, ie,$\alpha(X,Y) = g(AX,Y)$. Then from (4.17) we have

$$g(AX,Z) = \eta(X)g(A\xi,Z) - \eta(Z)g(A\xi,X) \tag{4.18}$$

and thus

$$AX = \eta(X)A\xi - g(A\xi,X)\xi \tag{4.19}$$

Since α is parallel it follows that A is parallel. Hence using $\nabla_X\xi = -\beta\phi X$, we get

$$\nabla_X(A\xi) = A(\nabla_X\xi) = A(-\beta\phi X). \tag{4.20}$$

Thus

$$\nabla_{\phi X}(A\xi) = A(-\beta\phi^2 X) = \beta[AX - \eta(X)A\xi]. \quad (4.21)$$

Therefore, We have from (4.19) and (4.21)

$$\nabla_{\phi X}(A\xi) = -\beta g(A\xi, X)\xi. \quad (4.22)$$

Now, from (4.19) we get

$$g(A\xi, \xi) = 0. \quad (4.23)$$

From (4.22) and (4.23),we have

$$g(\nabla_{\phi X}(A\xi), A\xi) = 0. \quad (4.24)$$

Replacing ϕX by X in (4.24) and using $\nabla_\xi \xi = 0$,we get

$$g(\nabla_X(A\xi), A\xi) = 0, \quad (4.25)$$

for any X, and thus $\| A\xi \|$=constant on M. From (4.25) we deduce that

$$g(A(\nabla_X \xi), A\xi) = -g(\nabla_X \xi, A^2\xi) = 0. \quad (4.26)$$

Replace X by ϕX in the above equation, it follows that

$$g(\nabla_{\phi X}\xi, A^2\xi) = g(-\beta\phi^2 X, A^2\xi) = 0. \quad (4.27)$$

So

$$g(X, A^2\xi) = g(\eta(X)\xi, A^2\xi), if \beta \neq 0 \quad (4.28)$$

and hence

$$A^2\xi = - \| A\xi \|^2 \xi \quad (4.29)$$

Differentiating the above equation covariantly along X, we obtain

$$\nabla_X(A^2\xi) = A^2(\nabla_X \xi) = - \| A\xi \|^2 (\nabla_X \xi), \quad (4.30)$$

which in turn gives

$$A^2(-\beta\phi X) = - \| A\xi \|^2 (-\beta\phi X). \quad (4.31)$$

Replacing ϕX by X, in (4.29) to get

$$A^2 X = - \| A\xi \|^2 X. \quad (4.32)$$

Now, if $\parallel A\xi \parallel\neq 0$, then $\frac{1}{\|A\xi\|}$ A is an almost complex structure on M. In fact,(J, g) is a Kahler structure on M. The fundamental 2-form is $g(JX, Y) = \lambda g(AX, Y) = \lambda\alpha(X, Y)$, with $\lambda = \frac{1}{\|A\xi\|} = constant$. But (4.19) means

$$\alpha(X, Z) = \eta(X)\alpha(\xi, Z) - \eta(Z)\alpha(\xi, X) \tag{4.33}$$

and thus α is degenerate, which is a conradiction. Therefore $\parallel A\xi \parallel= 0$ and hence $\alpha = 0$. Hence we can state the following theorem:

Theorem 4.2. *On a regular generalized Sasakian space-form there is no non-zero second order skew symmetric parallel tensor if $\beta \neq 0$.*

Corollary 4.1. *A locally Ricci symmetric ($\nabla S = 0$) regular generalized Sasakian space-form is an Einstein manifold.*

Remark 4.3. *The following statements for generalized Sasakian spaceform are equivalent. The manifold is (i) Einstein (ii)locally Ricci symmetric (iii) Ricci semi-symmetric that is $R \cdot S = 0$.*

The implication (i)$\Longrightarrow$(ii)$\Longrightarrow$(iii) is trivial. Now we prove the implication (iii)$\Longrightarrow$(i) and $R.S = 0$ means exactly (4.2) with replaced α by S that is

$$(R(X, Y) \cdot S)(U, V) = -S(R(X, Y)U, V) - S(U, R(X, Y)V). \tag{4.34}$$

Considering $R \cdot S = 0$ and putting $X = \xi$ in (4.34), we have

$$S(R(\xi, Y)U, V) + S(U, R(\xi, Y)V) = 0. \tag{4.35}$$

By using (3.16) and (3.19), we obtain

$$\begin{aligned} &2n(f_1 - f_3)^2 g(Y, U)\eta(V) - (f_1 - f_3)\eta(U)S(Y, V) + 2n(f_1 - f_3)^2 g(Y, V)\eta(U) \\ &- (f_1 - f_3)\eta(V)S(U, Y) = 0. \end{aligned} \tag{4.36}$$

Again by putting $U = \xi$ in the above equation and by using (3.2),(3.3) and (3.19), we obtain

$$S(Y, V) = 2n(f_1 - f_3)g(Y, V). \tag{4.37}$$

In conclusion:

Proposition 2. *A Ricci semi-symmetric regular generalized Sasakian spaceform is Einstein.*

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