A hybrid iterative scheme for a general system of variational inequalities based on mixed nonlinear mappings

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Abstract. The purpose of this paper is to study the strong convergence of a hybrid iterative scheme for finding a common element of the set of a general system of variational inequalities for \( \alpha \)-inverse-strongly monotone mapping and relaxed \((c,d)\)-cocoercive mapping, the set of solutions of a mixed equilibrium problem and the set of common fixed points of a finite family of nonexpansive mappings in a real Hilbert space. Using the demi-closedness principle for nonexpansive mapping, we prove that the iterative sequence converges strongly to a common element of these three sets under some control conditions. Our results extend recent results announced by many others.

Keywords: Nonexpansive mapping; A general system of variational inequality problem; Mixed equilibrium problem; Demi-closedness principle.

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1 Introduction

Let \( H \) be a real Hilbert space with inner product \( \langle \cdot, \cdot \rangle \) and \( C \) be a nonempty closed convex subset of \( H \). Recall that \( T : C \to C \) is nonexpansive if

\[ \|Tx - Ty\| \leq \|x - y\|, \]
for all \( x, y \in C \). The fixed point set of \( T \) is denoted by \( F(T) := \{ x \in C : Tx = x \} \).

Let \( A : C \to H \) be a nonlinear mapping. Then \( A \) is called

(i) **monotone**, if
\[
\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C;
\]

(ii) **\( \alpha \)-strongly monotone**, if there exists a positive real number \( \alpha > 0 \) such that
\[
\langle Ax - Ay, x - y \rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in C;
\]

(iii) **\( L \)-Lipschitz continuous** (or **Lipschitzian**), if there exists a constant \( L \geq 0 \) such that
\[
\|Ax - Ay\| \leq L\|x - y\|, \quad \forall x, y \in C;
\]

(iv) **\( \alpha \)-inverse-strongly monotone**, if there exists a positive real number \( \alpha > 0 \) such that
\[
\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.
\]

It is obvious that every \( \alpha \)-inverse-strongly monotone mapping \( A \) is monotone and Lipschitz continuous. It is known that if \( T \) is a nonexpansive mapping of \( C \) into itself, then \( A = I - T \) is 1/2-inverse strongly monotone, where \( I \) is the identity mapping of \( H \).

A mapping \( A \) is called **relaxed \( c \)-cocoercive**, if there exists a constant \( c > 0 \) such that
\[
\langle Ax - Ay, x - y \rangle \geq (-c)\|Ax - Ay\|^2, \quad \forall x, y \in C.
\]

A mapping \( A \) is called **relaxed \( (c,d) \)-cocoercive**, if there exist two constants \( c, d > 0 \) such that
\[
\langle Ax - Ay, x - y \rangle \geq (-c)\|Ax - Ay\|^2 + d\|x - y\|^2, \quad \forall x, y \in C.
\]

For \( c = 0 \), \( A \) is \( d \)-strongly monotone. This class of mappings is more general than the class of strongly monotone mappings. As a result, we have the following implication:
\[
d \text{-strong monotonicity} \Rightarrow \text{relaxed \( (c,d) \)-cocoercivity}.
\]

For a given nonlinear operator \( A : C \to H \), we consider the following variational inequality problem of finding \( x^* \in C \) such that
\[
\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{1.1}
\]

The set of solutions of the variational inequality (1.1) is denoted by \( VI(C, A) \). Variational inequality theory has emerged as an important tool in studying a wide class
of obstacle, unilateral, free, moving, equilibrium problems arising in several branches of pure and applied sciences in a unified and general framework. The variational inequality problem has been extensively studied and continued in the literature, see, Piri [12], Qin et al. [13], Shehu [14], Wangkeeree and Preechasip [19], Yao et al. [21], Yao et al. [23] and relevant references cited therein.

Next, we focus on a general system of variational inequality problems [in short, GSVI] which is considered by Ceng et al. [2]: find \((x^*, y^*) \in C \times C\) such that

\[
\begin{align*}
\langle \lambda Ay^* + x^* - y^*, x - x^* \rangle &\geq 0, \quad \forall x \in C, \\
\langle \mu Bx^* + y^* - x^*, x - y^* \rangle &\geq 0, \quad \forall x \in C,
\end{align*}
\]  \tag{1.2}

where \(A, B : C \to H\) are two nonlinear mappings, \(\lambda > 0\) and \(\mu > 0\) are two constants. In particular, if \(A = B\), then GSVI (1.2) reduces to find \((x^*, y^*) \in C \times C\) such that

\[
\begin{align*}
\langle \lambda Ay^* + x^* - y^*, x - x^* \rangle &\geq 0, \quad \forall x \in C, \\
\langle \mu Ax^* + y^* - x^*, x - y^* \rangle &\geq 0, \quad \forall x \in C,
\end{align*}
\]  \tag{1.3}

which is defined by Verma [17], and is called the new system of variational inequalities. Further, if we add the requirement that \(x^* = y^*\), then problem (1.3) reduces to the classical variational inequality \(VI(C, A)\). Ceng et al. [2] introduced and studied a relaxed extragradient method for finding a common element of the set of solutions of GSVI (1.2) for the \(\alpha\) and \(\beta\)-inverse-strongly monotone mappings and the set of fixed points of a nonexpansive mapping in a real Hilbert space. Some related works, we refer to see [3, 5, 8, 9, 18, 22].

Recently, in 2012, Ceng et al. [3] considered an iterative method for the system of GSVI (1.2) and obtained a strong convergence theorem for the two different systems of GSVI (1.2) and the set of fixed points of a strict pseudocontraction mapping in a real Hilbert space.

Let \(\varphi : C \to \mathbb{R} \cup \{+\infty\}\) be a proper extended real-valued function and \(F\) be a bifunction from \(C \times C\) to \(\mathbb{R}\), where \(\mathbb{R}\) is the set of real numbers. Ceng and Yao [4] considered the following mixed equilibrium problem (in short, MEP):

Find \(x \in C\) such that \(F(x, y) + \varphi(y) \geq \varphi(x), \forall y \in C\).  \tag{1.4}

The set of solution of MEP (1.4) is denoted by \(MEP(F, \varphi)\). It is easy to see that \(x\) is a solution of MEP (1.4) implies that \(x \in \text{dom}\varphi = \{x \in C \mid \varphi(x) < +\infty\}\).

If \(\varphi = 0\), then the MEP (1.4) becomes the following equilibrium problem:

Find \(x \in C\) such that \(F(x, y) \geq 0, \forall y \in C\).  \tag{1.5}
The set of solution of (1.5) is denoted by $EP(F)$.

If $F = 0$, then the MEP (1.4) reduces to the convex minimization problem:

Find $x \in C$ such that $\varphi(y) \geq \varphi(x), \forall y \in C$.

If $\varphi = 0$ and $F(x, y) = \langle Ax, y - x \rangle$ for all $x, y \in C$, where $A$ is a mapping from $C$ into $H$, then MEP (1.4) reduces to the classical variational inequality and $EP(F) = VI(C, A)$. For solving problem MEP (1.4), Ceng and Yao [4] introduced a hybrid iterative scheme for finding a common element of the set $MEP(F, \varphi)$ and the set of common fixed points of finite many nonexpansive mappings in a Hilbert space. Some related works, we refer to see [8, 14, 18, 21].

Recently, in 2012, Kumam and Katchang [9] introduced an iterative algorithm for finding a common element of the set of solutions of a system of mixed equilibrium problems, the set of solutions of a general system of variational inequalities for Lipschitz continuous and relaxed cocoercive mappings, the set of common fixed points for nonexpansive semigroups and the set of common fixed points for an infinite family of strictly pseudocontractive mappings in Hilbert spaces.

Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of $C$ into itself. In 1999, Atsushiba and Takahashi [1] defined the mapping $W_n$ as follows:

$$U_{n,1} = \lambda_{n,1}T_1 + (1 - \lambda_{n,1})I,$$
$$U_{n,2} = \lambda_{n,2}T_2U_{n,1} + (1 - \lambda_{n,2})I,$$
$$U_{n,3} = \lambda_{n,3}T_3U_{n,2} + (1 - \lambda_{n,3})I,$$
$$\vdots$$
$$U_{n,N-1} = \lambda_{n,N-1}T_{N-1}U_{n,N-2} + (1 - \lambda_{n,N-1})I,$$
$$W_n = U_{n,N} = \lambda_{n,N}T_NU_{n,N-1} + (1 - \lambda_{n,N})I,$$

where $\{\lambda_{n,i}\}_{i=1}^N \subseteq [0, 1]$. This mapping is called the $W$-mapping generated by $T_1, T_2, \ldots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$. In 2000, Takahashi and Shimoji [16] proved that if $X$ is a strictly convex Banach space, then $F(W_n) = \bigcap_{i=1}^N F(T_i)$, where $0 < \lambda_{n,i} < 1, \ i = 1, 2, \ldots, N$.

In 2009, Kangtunyakarn and Suantai [7] introduced a new mapping called the $S$-mapping. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of $C$ into itself. For each $n \in \mathbb{N}$, and $j = 1, 2, \ldots, N$, let $\alpha^{(n)}_j = (\alpha^{n,j}_1, \alpha^{n,j}_2, \alpha^{n,j}_3)$ be such that $\alpha^{n,j}_1, \alpha^{n,j}_2, \alpha^{n,j}_3 \in [0, 1]$ with $\alpha^{n,j}_1 + \alpha^{n,j}_2 + \alpha^{n,j}_3 = 1$. They defined the new mapping
$S_n : C \rightarrow C$ as follows:

\[
U_{n,0} = I,
U_{n,1} = \alpha_1^n T_1 U_{n,0} + \alpha_2^n U_{n,0} + \alpha_3^n I,
U_{n,2} = \alpha_1^n T_2 U_{n,1} + \alpha_2^n U_{n,1} + \alpha_3^n I,
U_{n,3} = \alpha_1^n T_3 U_{n,2} + \alpha_2^n U_{n,2} + \alpha_3^n I,
\]

\[\vdots\]

\[
U_{n,N-1} = \alpha_1^n T_{N-1} U_{n,N-2} + \alpha_2^n U_{n,N-2} + \alpha_3^n I,
S_n = U_{n,N} = \alpha_1^n T_N U_{n,N-1} + \alpha_2^n U_{n,N-1} + \alpha_3^n I.
\]

The mapping $S_n$ is called the $S$-mapping generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1^n, \alpha_2^n, \ldots, \alpha_N^n$. Nonexpansivity of each $T_i$ ensures the nonexpansivity of $S_n$.

Motivated and inspired by Ceng et al. [2], Ceng et al. [3], Ceng and Yao [4] and Kangtunyakarn and Suantai [7], we introduce a hybrid iterative scheme for finding a common element of the set of solutions of GSVI (1.2) for $\alpha$-inverse-strongly monotone mapping and relaxed $(c,d)$-cocoercive mapping, the set of solutions of MEP (1.4) and the set of common fixed points of a finite family of nonexpansive mappings in a real Hilbert space. Starting with an arbitrary $v \in C$ and let $x_1 \in C$, we define the sequences $\{x_n\}, \{u_n\}$ and $\{y_n\}$ by

\[
\begin{cases}
F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n} (y - u_n, u_n - x_n) \geq 0, \quad \forall y \in C, \\
y_n = P_C(u_n - \mu B u_n), \\
x_{n+1} = a_n v + b_n x_n + (1 - a_n - b_n) S_n P_C(y_n - \lambda A y_n), \quad n \geq 1,
\end{cases}
\]

where $\lambda > 0$ and $\mu > 0$ are two constants, $\{r_n\} \subset (0, \infty)$ and $\{a_n\}, \{b_n\} \subset [0, 1]$. Using the demi-closedness principle for nonexpansive mappings, we show that the sequence $\{x_n\}$ converges strongly to a common element of those three sets under some control conditions. Our results extend recent results announced by many others.

2 Preliminaries

In this section, we recall the well known results and give some useful lemmas that will be used in the next section.

Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. For every point $x \in H$, there exists a unique nearest point in $C$, denoted by $P_C x$, such that

\[
\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.
\]
$P_C$ is called the metric projection of $H$ onto $C$. It is well known that $P_C$ is a nonexpansive mapping of $H$ onto $C$ and satisfies

$$\langle x - y, P_C x - P_C y \rangle \geq \| P_C x - P_C y \|^2, \quad \forall x, y \in H. \quad (2.1)$$

Obviously, this immediately implies that

$$\| (x - y) - (P_C x - P_C y) \|^2 \leq \| x - y \|^2 - \| P_C x - P_C y \|^2, \quad \forall x, y \in H. \quad (2.2)$$

Recall that, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle x - y, P_C x - P_C y \rangle \leq 0,$$

$$\| x - y \|^2 \geq \| x - P_C x \|^2 + \| P_C x - y \|^2, \quad (2.3)$$

for all $x \in H$ and $y \in C$; see Goebel and Kirk [6] for more details.

For solving the mixed equilibrium problem, let us give the following assumptions for the bifunction $F, \varphi$ and the set $C$:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) $F$ is monotone, i.e. $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(A3) For each $y \in C$, $x \mapsto F(x, y)$ is weakly upper semicontinuous;

(A4) For each $x \in C$, $y \mapsto F(x, y)$ is convex;

(A5) For each $x \in C$, $y \mapsto F(x, y)$ is lower semicontinuous;

(B1) For each $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$F(z, y_x) + \varphi(y_x) + \frac{1}{r} \langle y_x - z, z - x \rangle < \varphi(z).$$

(B2) $C$ is a bounded set.

In the sequel, we shall need to use the following lemmas.

**Lemma 2.1.** ([11]) Let $C$ be a nonempty closed convex subset of $H$. Let $F$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A5) and let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:

$$T_r(x) = \left\{ z \in C : F(z, y) + \varphi(y) + \frac{1}{r} \langle y - z, z - x \rangle \geq \varphi(z), \quad \forall y \in C \right\}$$

for all $x \in H$. Then the following conclusions hold:

(1) For each $x \in H$, $T_r(x) \neq \emptyset$;

(2) $T_r$ is single-valued;

(3) $T_r$ is firmly nonexpansive, i.e. for any $x, y \in H$,

$$\| T_r(x) - T_r(y) \|^2 \leq \langle T_r(x) - T_r(y), x - y \rangle;$$

(4) $T_r$ is nonexpansive, i.e. for any $x, y \in H$,

$$\| T_r(x) - T_r(y) \|^2 \leq \langle T_r(x) - T_r(y), x - y \rangle;$$

(5) $T_r$ is nonexpansive, i.e. for any $x, y \in H$,
Lemma 2.2. ([20]) Assume \( \{a_n\} \) is a sequence of nonnegative real numbers such that
\[
a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n,
\]
where \( \{\gamma_n\} \) is a sequence in \((0, 1)\) and \( \{\delta_n\} \) is a sequence such that
(i) \( \sum_{n=1}^{\infty} \gamma_n = \infty; \)
(ii) \( \limsup_{n \to \infty} \delta_n/\gamma_n \leq 0 \) or \( \sum_{n=1}^{\infty} |\delta_n| < \infty. \)
Then \( \lim_{n \to \infty} a_n = 0. \)

Lemma 2.3. ([10]) Let \((H, \langle \cdot, \cdot \rangle)\) be an inner product space. Then, for all \( x, y, z \in H \) and \( \alpha, \beta, \gamma \in [0, 1] \) with \( \alpha + \beta + \gamma = 1 \), we have
\[
\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2
\]
\[
- \alpha\gamma\|x - z\|^2 - \beta\gamma\|y - z\|^2.
\]

Lemma 2.4. ([15]) Let \( \{x_n\} \) and \( \{y_n\} \) be bounded sequences in a Banach space \( X \) and let \( \{b_n\} \) be a sequence in \([0, 1]\) with \( 0 < \liminf_{n \to \infty} b_n \leq \limsup_{n \to \infty} b_n < 1 \). Suppose \( x_{n+1} = (1 - b_n)y_n + b_n x_n \) for all integers \( n \geq 1 \) and \( \limsup_{n \to \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0 \). Then, \( \lim_{n \to \infty} \|y_n - x_n\| = 0. \)

Lemma 2.5. ([6]) Demi-closedness principle. Assume that \( T \) is a nonexpansive self-mapping of a nonempty closed convex subset \( C \) of a real Hilbert space \( H \). If \( T \) has a fixed point, then \( I - T \) is demi-closed: that is, whenever \( \{x_n\} \) is a sequence in \( C \) converging weakly to some \( x \in C \) (for short, \( x_n \rightharpoonup x \in C \)), and the sequence \( \{(I - T)x_n\} \) converges strongly to some \( y \) (for short, \( (I - T)x_n \to y \)), it follows that \( (I - T)x = y \). Here \( I \) is the identity operator of \( H \).

The following lemma is an immediate consequence of an inner product.

Lemma 2.6. In a real Hilbert space \( H \), there holds the inequality
\[
\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \quad \forall x, y \in H.
\]

Lemma 2.7. ([7]) Let \( C \) be a nonempty closed convex subset of a strictly convex Banach space \( X \). Let \( \{T_i\}_{i=1}^{N} \) be a finite family of nonexpansive mappings of \( C \) into itself with \( \bigcap_{i=1}^{N} F(T_i) \neq \emptyset \) and let \( \alpha_j = (\alpha_{1j}, \alpha_{2j}, \alpha_{3j}), \; j = 1, 2, \ldots, N \), where
\[
\alpha_{1j}, \alpha_{2j}, \alpha_{3j} \in [0, 1], \; \alpha_{1j} + \alpha_{2j} + \alpha_{3j} = 1, \; \alpha_{1j} \in (0, 1) \text{ for all } j = 1, 2, \ldots, N-1, \; \alpha_{1N} \in (0, 1]
\]
and \( \alpha_{2j}, \alpha_{3j} \in [0, 1] \text{ for all } j = 1, 2, \ldots, N. \) Let \( S \) be the \( S \)-mapping generated by \( T_1, T_2, \ldots, T_N \) and \( \alpha_1, \alpha_2, \ldots, \alpha_N \). Then \( F(S) = \bigcap_{i=1}^{N} F(T_i). \)
Lemma 2.8. ([7]) Let $C$ be a nonempty closed convex subset of a Banach space $X$. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of $C$ into itself and for all $n \in \mathbb{N}$ and all $j \in \{1, 2, \ldots, N\}$, let $\alpha^{(n)}_j = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j})$, $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j)$ where $\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j} \in [0, 1]$, $\alpha_1^j, \alpha_2^j, \alpha_3^j \in [0, 1]$, $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$.

Suppose $\alpha^{n,j}_i \to \alpha^j_i$ as $n \to \infty$ for all $i \in \{1, 3\}$ and all $j = 1, 2, 3, \ldots, N$. Let $S$ and $S_n$ be the $S$-mappings generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$ and $T_1, T_2, \ldots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_N^{(n)}$, respectively. Then $\lim_{n \to \infty} \|S_n x - S x\| = 0$ for every $x \in C$.

Lemma 2.9. ([2]) For given $x^*, y^* \in C$, $(x^*, y^*)$ is a solution of GSVI (1.2) if and only if $x^*$ is a fixed of the mapping $G : C \to C$ defined by

$$G(x) = P_C[P_C(x - \mu Bx) - \lambda A P_C(x - \mu Bx)], \ \forall x \in C,$$

where $y^* = P_C(x^* - \mu Bx^*)$.

Throughout this paper, the set of fixed points of the mapping $G$ is denoted by GSVI$(C, A, B)$.

3 Main Results

In this section, we prove a strong convergence theorem of the hybrid iterative scheme (1.6) to a common element of the set of solutions of GSVI (1.2) for $\alpha$-inverse -strongly monotone mapping and relaxed $(c, d)$-cocoercive mapping, the set of solutions of MEP (1.4) and the set of common fixed points of a finite family of nonexpansive mappings in a real Hilbert space.

Next, we prove some lemmas which are very useful for our consideration.

Lemma 3.1. Let $A : C \to H$ be $\alpha$-inverse-strongly monotone and let $B : C \to H$ be $L$-Lipschitzian and relaxed $(c', d')$-cocoercive. Let the mapping $G : C \to C$ be defined by

$$G(x) = P_C[P_C(x - \mu Bx) - \lambda A P_C(x - \mu Bx)], \ \forall x \in C.$$

If $\lambda \in (0, 2\alpha]$ and $0 < \mu \leq \frac{2(d' - c'L^2)}{L^2}$. Then $G$ is nonexpansive.
Proof. For any $x, y \in C$, we have
\[
\|G(x) - G(y)\| = \|P_C[P_C(x - \mu Bx) - \lambda A P_C(x - \mu Bx)]
\]
\[
- P_C[P_C(y - \mu By) - \lambda A P_C(y - \mu By)]\|^2
\]
\[
\leq \|P_C(x - \mu Bx) - \lambda A P_C(x - \mu Bx)
\]
\[
- (P_C(y - \mu By) - \lambda A P_C(y - \mu By))\|
\]
\[
= \|(I - \lambda A) P_C(I - \mu B)x - (I - \lambda A) P_C(I - \mu B)y\|.
\]
It is well known that if $A : C \to H$ be $\alpha$-inverse-strongly monotone and $B : C \to H$ be $L$-Lipschitzian and relaxed $(c', d')$-cocoercive, $I - \lambda A$ and $I - \mu B$ are nonexpansive, where $\lambda \in (0, 2\alpha]$ and $0 < \mu \leq \frac{2(c' - d')^2}{L^2}$. It follows that $(I - \lambda A) P_C(I - \mu B)$ is nonexpansive, which implies that $G$ is nonexpansive. \hfill \Box

Theorem 3.2. Let $C$ be a nonempty closed and convex subset of a real Hilbert space $H$. Let $F$ be a function from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A5) and $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let $A : C \to H$ be $\alpha$-inverse-strongly monotone and $B : C \to H$ be a $L$-Lipschitzian and relaxed $(c', d')$-cocoercive mapping. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive self-mappings of $C$ such that $\Omega = \bigcap_{i=1}^N F(T_i) \cap \text{GSVI}(C, A, B) \cap \text{MEP}(F, \varphi) \neq \emptyset$. For all $j \in \{1, 2, \ldots, N\}$, let $\alpha_j^{(n)} = (\alpha_j^{1,j}, \alpha_j^{2,j}, \alpha_j^{3,j})$ be such that $\alpha_j^{1,j} = (\alpha_j^{1,j}, \alpha_j^{2,j}, \alpha_j^{3,j}) \in [0, 1], \alpha_j^{1,j} + \alpha_j^{2,j} + \alpha_j^{3,j} = 1, \{\alpha_j^{1,j}\}_{j=1}^{N-1} \subset [\eta_1, \theta_1]$ with $0 < \eta_1 \leq \theta_1 < 1, \{\alpha_j^{1,j}\}_{j=1}^N \subset [\eta_N, 1]$ with $0 < \eta_N \leq 1$ and $\{\alpha_j^{3,j}\}_{j=1}^{N} \subset [0, \theta_2]$ with $0 \leq \theta_2 < 1$. Let $S_n$ be the $S$-mappings generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots, \alpha_N^{(n)}$. Assume that either (B1) or (B2) holds and that $v$ is an arbitrary point in $C$. Let $x_1 \in C$ and $\{x_n\}, \{u_n\}, \{y_n\}$ be the sequences defined by
\[
\begin{cases}
F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}(y - u_n, u_n - x_n) \geq 0, & \forall y \in C, \\
y_n = P_C(u_n - \mu B u_n), \\
x_{n+1} = a_n v + b_n x_n + (1 - a_n - b_n) S_n P_C(y_n - \lambda A y_n), & n \geq 1,
\end{cases}
\]
where $\lambda \in (0, 2\alpha]$ and $0 < \mu < \frac{2(c' - d')^2}{L^2}$. Suppose that the following conditions hold:

(C1) $\lim_{n \to \infty} a_n = 0$ and $\sum_{n=1}^{\infty} a_n = \infty$;
(C2) $0 < \liminf_{n \to \infty} b_n \leq \limsup_{n \to \infty} b_n < 1$;
(C3) $\liminf_{n \to \infty} r_n > 0$ and $\lim_{n \to \infty} |r_{n+1} - r_n| = 0$;
(C4) $\lim_{n \to \infty} |a_{n+1,i} - a_{n,i}| = 0$ for all $i \in \{1, 2, \ldots, N\}$ and $\lim_{n \to \infty} |a_{n+1,j} - a_{n,j}| = 0$ for all $j \in \{2, 3, \ldots, N\}$. Then $\{x_n\}$ converges strongly to $\bar{x} = P_\Omega(v)$ and $(\bar{x}, \overline{\varphi})$ is a solution of GSVI (1.2), where $\overline{\varphi} = P_C(\bar{x} - \mu B \bar{x})$. 


Proof. Let \( x^* \in \Omega \) and \( \{ T_{r_n} \} \) be a sequence of mappings defined as in Lemma 2.1. It follows from Lemma 2.9 that
\[
x^* = P_C[P_C(x^* - \mu Bx^*) - \lambda A_P(x^* - \mu Bx^*)].
\]
Put \( y^* = P_C(x^* - \mu Bx^* \) and \( t_n = P_C(y_n - \lambda A y_n) \), then \( x^* = P_C(y^* - \lambda A y^*) \) and
\[
x_{n+1} = a_n v + b_n x_n + (1 - a_n - b_n) S_n t_n.
\]
By nonexpansiveness of \( I - \lambda A, I - \mu B, P_C \) and \( T_{r_n} \), we have
\[
\| t_n - x^* \|^2 = \| P_C(I - \lambda A)y_n - P_C(I - \lambda A)y^* \|^2 \\
\leq \| y_n - y^* \|^2 = \| P_C(I - \mu B)u_n - P_C(I - \mu B)x^* \|^2 \\
\leq \| u_n - x^* \|^2 = \| T_{r_n} x_n - T_{r_n} x^* \|^2 \leq \| x_n - x^* \|^2, \quad (3.1)
\]
which, implies that
\[
\| x_{n+1} - x^* \| = \| a_n v + b_n x_n + (1 - a_n - b_n) S_n t_n - x^* \| \\
\leq a_n \| v - x^* \| + b_n \| x_n - x^* \| + (1 - a_n - b_n) \| t_n - x^* \| \\
\leq a_n \| v - x^* \| + b_n \| x_n - x^* \| + (1 - a_n - b_n) \| x_n - x^* \| \\
\leq \max \{ \| v - x^* \|, \| x_1 - x^* \| \}.
\]
Thus, \( \{ x_n \} \) is bounded. Consequently, the sequences \( \{ u_n \}, \{ y_n \}, \{ t_n \}, \{ A y_n \}, \{ B u_n \} \) and \( \{ S_n t_n \} \) are also bounded. Also, observe that
\[
\| t_{n+1} - t_n \| = \| P_C(y_{n+1} - \lambda A y_{n+1}) - P_C(y_n - \lambda A y_n) \| \\
\leq \| y_{n+1} - y_n \| \\
= \| P_C(u_{n+1} - \mu B u_{n+1}) - P_C(u_n - \mu B u_n) \| \\
\leq \| u_{n+1} - u_n \|. \quad (3.2)
\]
On the other hand, from \( u_n = T_{r_n} x_n \in \text{dom} \varphi \) and \( u_{n+1} = T_{r_{n+1}} x_{n+1} \in \text{dom} \varphi \), we have
\[
F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \quad (3.3)
\]
and
\[
F(u_{n+1}, y) + \varphi(y) - \varphi(u_{n+1}) + \frac{1}{r_{n+1}} \langle y - u_{n+1}, u_{n+1} - x_{n+1} \rangle \geq 0, \quad \forall y \in C. \quad (3.4)
\]
Putting \( y = u_{n+1} \) in (3.3) and \( y = u_n \) in (3.4), we have
\[
F(u_n, u_{n+1}) + \varphi(u_{n+1}) - \varphi(u_n) + \frac{1}{r_n} \langle u_{n+1} - u_n, u_n - x_n \rangle \geq 0,
\]
and
\[ F(u_{n+1}, u_n) + \varphi(u_n) - \varphi(u_{n+1}) + \frac{1}{r_n+1} \langle u_n - u_{n+1}, u_n - x_{n+1} \rangle \geq 0. \]

From the monotonicity of \( F \), we obtain that
\[ \langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}} \rangle \geq 0, \]
and hence
\[ \langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}} (u_{n+1} - x_{n+1}) \rangle \geq 0. \]

Then, we have
\[
\|u_{n+1} - u_n\|^2 \leq \left\langle u_{n+1} - u_n, x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right) (u_{n+1} - x_{n+1}) \right\rangle
\leq \|u_{n+1} - u_n\| \left\{ \|x_{n+1} - x_n\| + |1 - \frac{r_n}{r_{n+1}}| \|u_{n+1} - x_{n+1}\| \right\},
\]
and hence
\[
\|u_{n+1} - u_n\| \leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}} |r_{n+1} - r_n| \|u_{n+1} - x_{n+1}\|. \tag{3.5}
\]

It follows from (3.2) and (3.5) that
\[
\|t_{n+1} - t_n\| \leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}} |r_{n+1} - r_n| \|u_{n+1} - x_{n+1}\|. \tag{3.6}
\]

Let \( x_{n+1} = b_n x_n + (1 - b_n) z_n \). Then, we obtain
\[
z_{n+1} - z_n = \frac{x_{n+1} - b_n x_n}{1 - b_n} - \frac{x_n + b_n x_n}{1 - b_n} = a_{n+1} v + (1 - a_{n+1} - b_{n+1}) S_{n+1} t_{n+1} - \frac{a_n v + (1 - a_n - b_n) S_n t_n}{1 - b_n}
= a_{n+1} \frac{(v - S_{n+1} t_{n+1})}{1 - b_{n+1}} + \frac{a_n}{1 - b_n} (S_n t_n - v) + S_n t_{n+1} - S_{n+1} t_n. \tag{3.7}
\]

Next, we estimate \( \|S_{n+1} t_{n+1} - S_n t_n\| \).
For each $k \in \{2, 3, \ldots, N\}$, we have

$$
\|U_{n+1,k} - U_{n,k}\| = \|\alpha_1^{n+1,k} T_k U_{n+1,k-1} - \alpha_2^{n+1,k} U_{n+1,k-1} + \alpha_3^{n+1,k}\|
$$

$$
= \|\alpha_1^{n+1,k} (T_k U_{n+1,k-1} - T_k U_{n,k-1}) + \alpha_2^{n+1,k} (U_{n+1,k-1} - U_{n,k-1}) + (\alpha_3^{n+1,k} - \alpha_3^{n,k}) t_n\|
$$

$$
\leq \|\alpha_1^{n+1,k} T_k U_{n+1,k-1} - \alpha_2^{n+1,k} U_{n+1,k-1} + \alpha_3^{n+1,k} (T_k U_{n,k-1} - U_{n,k-1}) + |\alpha_3^{n+1,k} - \alpha_3^{n,k}| T_k U_{n,k-1} t_n\|
$$

$$
+ |\alpha_3^{n+1,k} - \alpha_3^{n,k}| \|U_{n,k-1}\| + |\alpha_3^{n+1,k} - \alpha_3^{n,k}| \|t_n\|
$$

$$
\leq \|U_{n+1,k-1} - U_{n,k-1}\| + |\alpha_3^{n+1,k} - \alpha_3^{n,k}| \|T_k U_{n,k-1}\|
$$

$$
+ |\alpha_3^{n+1,k} - \alpha_3^{n,k}| \|t_n\| + |\alpha_3^{n,k} - \alpha_3^{n+1,k}| \|U_{n,k-1}\|
$$

$$
\leq \|U_{n+1,k-1} - U_{n,k-1}\| + |\alpha_3^{n+1,k} - \alpha_3^{n,k}| \|T_k U_{n,k-1}\|
$$

$$
+ |\alpha_3^{n+1,k} - \alpha_3^{n,k}| \|t_n\| + |\alpha_3^{n,k} - \alpha_3^{n+1,k}| \|U_{n,k-1}\|
$$

$$
= \|U_{n+1,k-1} - U_{n,k-1}\| + |\alpha_3^{n+1,k} - \alpha_3^{n,k}| \|T_k U_{n,k-1}\| + \|U_{n,k-1}\|
$$

$$
+ |\alpha_3^{n+1,k} - \alpha_3^{n,k}| \|t_n\| + \|U_{n,k-1}\|.
$$

(3.8)

It follow from (3.8) that

$$
\|S_{n+1} - S_n\| = \|U_{n+1,N} - U_{n,N}\|
$$

$$
\leq \|U_{n+1,N} - U_{n,N}\| + \sum_{j=2}^{N} |\alpha_3^{n+1,j} - \alpha_3^{n,j}| \|T_j U_{n,j-1}\| + \|U_{n,j-1}\|
$$

$$
+ \sum_{j=2}^{N} |\alpha_3^{n+1,j} - \alpha_3^{n,j}| \|t_n\| + \|U_{n,j-1}\|
$$

$$
= |\alpha_3^{n+1,1} - \alpha_3^{n,1}| \|t_n\| + \|U_{n,j-1}\|
$$

$$
+ \sum_{j=2}^{N} |\alpha_3^{n+1,j} - \alpha_3^{n,j}| \|T_j U_{n,j-1}\| + \|U_{n,j-1}\|
$$

$$
+ \sum_{j=2}^{N} |\alpha_3^{n+1,j} - \alpha_3^{n,j}| \|t_n\| + \|U_{n,j-1}\|.
$$
This together with the condition \((C4)\), we obtain
\[
\lim_{n \to \infty} \|S_{n+1}t_n - S_nt_n\| = 0. \tag{3.9}
\]

It follows from \((3.6)\) that
\[
\|S_{n+1}t_{n+1} - S_nt_n\| \leq \|t_{n+1} - t_n\| + \|S_{n+1}t_n - S_nt_n\|
\]
\[
\leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}}|r_{n+1} - r_n|\|u_{n+1} - x_{n+1}\|
\]
\[
+ \|S_{n+1}t_n - S_nt_n\|. \tag{3.10}
\]

By \((3.7)\) and \((3.10)\), we have
\[
\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \leq \frac{a_{n+1}}{1 - b_{n+1}}\|v - S_{n+1}t_{n+1}\| + \frac{a_n}{1 - b_n}\|S_nt_n - v\|
\]
\[
+ \|S_{n+1}t_{n+1} - S_nt_n\| - \|x_{n+1} - x_n\|
\]
\[
\leq \frac{a_{n+1}}{1 - b_{n+1}}\|v - S_{n+1}t_{n+1}\| + \frac{a_n}{1 - b_n}\|S_nt_n - v\|
\]
\[
+ \frac{1}{r_{n+1}}|r_{n+1} - r_n|\|u_{n+1} - x_{n+1}\|
\]
\[
+ \|S_{n+1}t_n - S_nt_n\|.
\]

This together with \((C1)-(C3)\) and \((3.9)\), we obtain that
\[
\lim_{n \to \infty} \|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \leq 0.
\]

Hence, by Lemma 2.4, we get \(\|x_n - z_n\| \to 0\) as \(n \to \infty\). Consequently,
\[
\lim_{n \to \infty} \|x_{n+1} - x_n\| = \lim_{n \to \infty} (1 - b_n)\|z_n - x_n\| = 0. \tag{3.11}
\]

From \((C3)\), \((3.2)\) and \((3.5)\), we also have \(\|u_{n+1} - u_n\| \to 0\), \(\|t_{n+1} - t_n\| \to 0\) and \(\|y_{n+1} - y_n\| \to 0\), as \(n \to \infty\).

Since
\[
x_{n+1} - x_n = a_n(v - x_n) + (1 - a_n - b_n)(S_nt_n - x_n),
\]

therefore
\[
\|S_nt_n - x_n\| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.12}
\]

Next, we prove that \(\lim_{n \to \infty} \|x_n - u_n\| = 0\). From Lemma 2.1(3), we have
\[
\|u_n - x^*\|^2 = \|T_n x_n - T_n x^*\|^2 \leq \langle T_n x_n - T_n x^*, x_n - x^* \rangle
\]
\[
= \langle u_n - x^*, x_n - x^* \rangle = \frac{1}{2} \left\{ \|u_n - x^*\|^2 + \|x_n - x^*\|^2 - \|x_n - u_n\|^2 \right\}.
\]
Hence
\[ \|u_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|x_n - u_n\|^2. \]  
(3.13)

From Lemma 2.3, (3.1) and (3.13), we have
\[ \|x_{n+1} - x^*\|^2 \leq a_n\|v - x^*\|^2 + b_n\|x_n - x^*\|^2 + (1 - a_n - b_n)\|t_n - x^*\|^2 \]
\[ \leq a_n\|v - x^*\|^2 + b_n\|x_n - x^*\|^2 + (1 - a_n - b_n)\|u_n - x^*\|^2 \]
\[ \leq a_n\|v - x^*\|^2 + b_n\|x_n - x^*\|^2 \]
\[ + (1 - a_n - b_n)[\|x_n - x^*\|^2 - \|x_n - u_n\|^2] \]
\[ \leq a_n\|v - x^*\|^2 + \|x_n - x^*\|^2 - (1 - a_n - b_n)\|x_n - u_n\|^2. \]

It follows that
\[ (1 - a_n - b_n)\|x_n - u_n\|^2 \leq a_n\|v - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \]
\[ \leq a_n\|v - x^*\|^2 + (\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2)\|x_{n+1} - x_n\|. \]

From the conditions (C1), (C2) and (3.11), we obtain
\[ \lim_{n \to \infty} \|x_n - u_n\| = 0. \]  
(3.14)

Since
\[ \|S_n t_n - u_n\| \leq \|S_n t_n - x_n\| + \|x_n - u_n\|, \]

it follows from (3.12) and (3.14) that
\[ \lim_{n \to \infty} \|S_n t_n - u_n\| = 0. \]  
(3.15)

Next, we show that \(\|Ay_n - Ay^*\| \to 0\) and \(\|Bu_n - Bx^*\| \to 0\) as \(n \to \infty\). From (3.1), we have
\[ \|x_{n+1} - x^*\|^2 \leq a_n\|v - x^*\|^2 + b_n\|x_n - x^*\|^2 + (1 - a_n - b_n)\|t_n - x^*\|^2 \]
\[ = a_n\|v - x^*\|^2 + b_n\|x_n - x^*\|^2 \]
\[ + (1 - a_n - b_n)\|PC(y_n - \lambda Ay_n) - PC(y^* - \lambda Ay^*)\|^2 \]
\[ \leq a_n\|v - x^*\|^2 + b_n\|x_n - x^*\|^2 \]
\[ + (1 - a_n - b_n)\|(y_n - \lambda Ay_n) - (y^* - \lambda Ay^*)\|^2 \]
\[ \leq a_n\|v - x^*\|^2 + b_n\|x_n - x^*\|^2 \]
\[ + (1 - a_n - b_n)[\|y_n - y^*\|^2 + \lambda(\lambda - 2\alpha)\|Ay_n - Ay^*\|^2] \]
\[ \leq a_n\|v - x^*\|^2 + \|x_n - x^*\|^2 \]
\[ + (1 - a_n - b_n)\lambda(\lambda - 2\alpha)\|Ay_n - Ay^*\|^2, \]
and

\[ \|x_{n+1} - x^*\|^2 \leq a_n \|v - x^*\|^2 + b_n \|x_n - x^*\|^2 + (1 - a_n - b_n)\|t_n - x^*\|^2 \]
\[ \leq a_n \|v - x^*\|^2 + b_n \|x_n - x^*\|^2 + (1 - a_n - b_n)\|y_n - y^*\|^2 \]
\[ \leq a_n \|v - x^*\|^2 + b_n \|x_n - x^*\|^2 \]
\[ + (1 - a_n - b_n)\|u_n - \mu Bu_n\| - (x^* - \mu Bx^*)\|^2 \]
\[ \leq a_n \|v - x^*\|^2 + b_n \|x_n - x^*\|^2 \]
\[ + (1 - a_n - b_n)\|u_n - x^*\|^2 - 2\mu\langle u_n - x^*, Bu_n - Bx^* \rangle \]
\[ + \mu^2\|Bu_n - Bx^*\|^2 \]
\[ \leq a_n \|v - x^*\|^2 + b_n \|x_n - x^*\|^2 \]
\[ + (1 - a_n - b_n)\|u_n - x^*\|^2 + 2\mu c \|Bu_n - Bx^*\|^2 \]
\[ - 2\mu d \|u_n - x^*\|^2 + \mu^2\|Bu_n - Bx^*\|^2 \]
\[ \leq a_n \|v - x^*\|^2 + b_n \|x_n - x^*\|^2 \]
\[ + (1 - a_n - b_n)\|x_n - x^*\|^2 \]
\[ + (2\mu c + \mu^2 - \frac{2\mu d}{L^2})\|Bu_n - Bx^*\|^2 \]
\[ \leq a_n \|v - x^*\|^2 + \|x_n - x^*\|^2 \]
\[ + (1 - a_n - b_n)(2\mu c + \mu^2 - \frac{2\mu d}{L^2})\|Bu_n - Bx^*\|^2. \]

Therefore, we have

\[ -(1 - a_n - b_n)\lambda(\lambda - 2\alpha)\|Ay_n - Ay^*\|^2 \]
\[ \leq a_n \|v - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_{n+1} - x_n\|, \]

and

\[ -(1 - a_n - b_n)(2\mu c + \mu^2 - \frac{2\mu d}{L^2})\|Bu_n - Bx^*\|^2 \]
\[ \leq a_n \|v - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_{n+1} - x_n\|. \]

This together with (3.11), (C1) and (C2), we obtain

\[ \|Ay_n - Ay^*\| \to 0 \text{ and } \|Bu_n - Bx^*\| \to 0 \text{ as } n \to \infty. \] (3.16)

Next, we prove that \(\|S_n t_n - t_n\| \to 0\) as \(n \to \infty\). From (2.1) and nonexpansiveness
of $I - \mu B$, we get

$$
\|y_n - y^*\|^2 = \|P_C(u_n - \mu Bu_n) - P_C(x^* - \mu Bx^*)\|^2
\leq \langle (u_n - \mu Bu_n) - (x^* - \mu Bx^*), y_n - y^* \rangle
\leq \frac{1}{2} \left[ \| (u_n - \mu Bu_n) - (x^* - \mu Bx^*) \|^2 + \| y_n - y^* \|^2 \ight]
- \| (u_n - \mu Bu_n) - (x^* - \mu Bx^*) - (y_n - y^*) \|^2
\leq \frac{1}{2} \left[ \| u_n - x^* \|^2 + \| y_n - y^* \|^2 - \| (u_n - x^*) - (y_n - y^*) \|^2 \ight]
+ 2\mu \langle (u_n - x^*) - (y_n - y^*), Bu_n - Bx^* \rangle - \mu^2 \| Bu_n - Bx^* \|^2.
$$

By (3.1), we obtain

$$
\|y_n - y^*\|^2 \leq \|u_n - x^*\|^2 - \| (u_n - x^*) - (y_n - y^*) \|^2
+ 2\mu \langle (u_n - x^*) - (y_n - y^*), Bu_n - Bx^* \rangle - \mu^2 \| Bu_n - Bx^* \|^2
\leq \|x_n - x^*\|^2 - \| (u_n - x^*) - (y_n - y^*) \|^2
+ 2\mu \langle (u_n - x^*) - (y_n - y^*), Bu_n - Bx^* \rangle - \mu^2 \| Bu_n - Bx^* \|^2.
$$

Hence

$$
\|x_{n+1} - x^*\|^2 \leq a_n \| v - x^* \|^2 + b_n \| x_n - x^* \|^2 + (1 - a_n - b_n) \| y_n - y^* \|^2
\leq a_n \| v - x^* \|^2 + b_n \| x_n - x^* \|^2 + (1 - a_n - b_n) \left[ \| x_n - x^* \|^2 - \| (u_n - x^*) - (y_n - y^*) \|^2 \ight]
+ 2\mu \langle (u_n - x^*) - (y_n - y^*), Bu_n - Bx^* \rangle - \mu^2 \| Bu_n - Bx^* \|^2
\leq a_n \| v - x^* \|^2 + \| x_n - x^* \|^2
- (1 - a_n - b_n) \| (u_n - x^*) - (y_n - y^*) \|^2
+ (1 - a_n - b_n) 2\mu \| (u_n - x^*) - (y_n - y^*) \| \| Bu_n - Bx^* \|,
$$

which implies that

$$
(1 - a_n - b_n) \| (u_n - x^*) - (y_n - y^*) \|^2
\leq a_n \| v - x^* \|^2 + (1 - a_n - b_n) 2\mu \| (u_n - x^*) - (y_n - y^*) \| \| Bu_n - Bx^* \|
+ (\| x_n - x^* \| + \| x_{n+1} - x^* \|) \| x_{n+1} - x_n \|.
$$

This together with (C1), (3.11) and (3.16), we obtain

$$
\| (u_n - x^*) - (y_n - y^*) \| \to 0 \quad \text{as} \quad n \to \infty.
$$

(3.17)

From Lemma 2.6 and (2.2), it follows that

$$
\| (y_n - t_n) + (x^* - y^*) \|^2 = \| (y_n - \lambda Ay_n) - (y^* - \lambda Ay^*) \\
- [P_C(y_n - \lambda Ay_n) - P_C(y^* - \lambda Ay^*)] + \lambda (Ay_n - Ay^*) \|.
$$
Let us assume that all \( j \) belong to \( \{ 0, 1 \} \) and \( \alpha_{1}^{n} \rightarrow 0 \) as \( n \rightarrow \infty \) for all \( j \in \{ 1, \ldots, N - 1 \} \) and \( \alpha_{3}^{n} \rightarrow 0 \) as \( n \rightarrow \infty \) for \( j = 1, 2, \ldots, N \). Let \( S \) be the \( S \)-mappings generated by \( T_{1}, T_{2}, \ldots, T_{N} \) and \( \alpha_{1}, \alpha_{2}, \ldots, \alpha_{N} \) where \( \alpha_{j} = (\alpha_{1}^{j}, \alpha_{2}^{j}, \alpha_{3}^{j}) \), for \( j = 1, 2, \ldots, N \). From Lemma 2.8, we have \( \| S_{n} t_{n} - S t_{n} \| \rightarrow 0 \) as \( n \rightarrow \infty \). Since

\[
\| S_{n} t_{n} - t_{n} \| \leq \| S_{n} t_{n} - S t_{n} \| + \| S t_{n} - t_{n} \|
\]
it follows by (3.18) that \( \|St_n - t_n\| \to 0 \) as \( n \to \infty \).
Since \( t_{n_i} \to z \) and \( \|St_n - t_n\| \to 0 \), we obtain by Lemma 2.5 and Lemma 2.7 that \( z \in F(S) = \bigcap_{i=1}^{N} F(T_i) \).

(b) Now, we show that \( z \in GSVI(C, A, B) \).
Since
\[
\|t_n - x_n\| \leq \|S_nt_n - t_n\| + \|S_nt_n - x_n\|
\]
it follows from (3.18) and (3.12) that \( \|t_n - x_n\| \to 0 \) as \( n \to \infty \). Furthermore, by Lemma 3.1, we have \( G : C \to C \) is nonexpansive. Then, we have
\[
\|t_n - G(t_n)\| = \|P_C(y_n - \lambda Ay_n) - G(t_n)\|
= \|P_C[P(u_n - \mu Bu_n) - \lambda AP(u_n - \mu Bu_n)] - G(t_n)\|
= \|G(u_n) - G(t_n)\| \leq \|u_n - t_n\|
\]
which implies \( \|t_n - G(t_n)\| \to 0 \) as \( n \to \infty \). Again by Lemma 2.5, we have \( z \in GSVI(C, A, B) \).

(c) We show that \( z \in MEP(F, \varphi) \). Since \( t_{n_i} \to z \) and \( \|x_n - t_n\| \to 0 \), we obtain that \( x_{n_i} \to z \). From \( \|u_n - x_n\| \to 0 \), we also obtain that \( u_{n_i} \to z \). By using the same argument as that in the proof of [11, Theorem 3.1, pp. 1825], we can show that \( z \in MEP(F, \varphi) \). Therefore there holds \( z \in \Omega \).

On the other hand, it follows from (2.3) and \( S_nt_{n_i} \to z \) as \( i \to \infty \) that
\[
\limsup_{n \to \infty} \langle v - \bar{x}, x_n - \bar{x} \rangle = \limsup_{n \to \infty} \langle v - \bar{x}, S_nt_n - \bar{x} \rangle = \lim_{i \to \infty} \langle v - \bar{x}, S_nt_{n_i} - \bar{x} \rangle
= \langle v - \bar{x}, z - \bar{x} \rangle \leq 0.
\]
Hence, we have
\[
\|x_{n+1} - \bar{x}\|^2 = \langle a_n v + b_n x_n + (1 - a_n - b_n)S_nt_n - \bar{x}, x_{n+1} - \bar{x} \rangle
= a_n \langle v - \bar{x}, x_{n+1} - \bar{x} \rangle + b_n \langle x_n - \bar{x}, x_{n+1} - \bar{x} \rangle
+ (1 - a_n - b_n) \langle S_nt_n - \bar{x}, x_{n+1} - \bar{x} \rangle
\leq a_n \langle v - \bar{x}, x_{n+1} - \bar{x} \rangle + \frac{1}{2} b_n (\|x_n - \bar{x}\|^2 + \|x_{n+1} - \bar{x}\|^2)
+ \frac{1}{2} (1 - a_n - b_n) (\|t_n - \bar{x}\|^2 + \|x_{n+1} - \bar{x}\|^2)
\leq a_n \langle v - \bar{x}, x_{n+1} - \bar{x} \rangle + \frac{1}{2} b_n (\|x_n - \bar{x}\|^2 + \|x_{n+1} - \bar{x}\|^2)
+ \frac{1}{2} (1 - a_n - b_n) (\|x_n - \bar{x}\|^2 + \|x_{n+1} - \bar{x}\|^2)
= a_n \langle v - \bar{x}, x_{n+1} - \bar{x} \rangle + \frac{1}{2} (1 - a_n) (\|x_n - \bar{x}\|^2 + \|x_{n+1} - \bar{x}\|^2),
\]
for all \( x_n \in C \).

which implies that
\[\|x_{n+1} - \overline{x}\|^2 \leq (1 - a_n)\|x_n - \overline{x}\|^2 + 2a_n\langle v - \overline{x}, x_{n+1} - \overline{x}\rangle.\]

It follows from Lemma 2.2 and (3.19) that \(\{x_n\}\) converges strongly to \(\overline{x}\). This completes the proof. \(\square\)

If \(\alpha_{n,j}^2 = 0\) for all \(j \in \{1,2,\ldots,N\}\) and all \(n \in \mathbb{N}\) in Theorem 3.2, then the mapping \(S_n\) reduces to the mapping \(W_n\), so the following result is obtained directly from Theorem 3.2.

**Corollary 3.3.** Let \(C\) be a nonempty closed and convex subset of a real Hilbert space \(H\). Let \(F\) be a function from \(C \times C\) to \(\mathbb{R}\) satisfying (A1)-(A5) and \(\varphi : C \to \mathbb{R} \cup \{+\infty\}\) be a proper lower semicontinuous and convex function. Let \(A : C \to H\) be \(\alpha\)-inverse-strongly monotone mapping and let \(B : C \to H\) be a \(L\)-Lipschitzian and relaxed \((c',d')\)-cocoercive mapping. Let \(\{T_i\}_{i=1}^N\) be a finite family of nonexpansive self-mappings of \(C\) such that \(\Omega = \bigcap_{i=1}^N F(T_i) \cap GSVI(C,A,B) \cap MEP(F,\varphi) \neq \emptyset\).

For all \(n \in \mathbb{N}\), let \(\{\alpha_{n,j}^1\}_{j=1}^N \subset (0,1)\) with \(0 < b < 1\). Let \(W_n\) be the \(W\)-mapping generated by \(T_1,T_2,\ldots,T_N\) and \(\alpha_{1,1}^1,\alpha_{1,2}^2,\ldots,\alpha_{1,N}^N\). Assume that either (B1) or (B2) holds and that \(v\) is an arbitrary point in \(C\). Let \(x_1 \in C\) and \(\{x_n\},\{u_n\},\{y_n\}\) be the sequences generated by

\[
\begin{cases}
F(u_n,y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C, \\
y_n = P_C(u_n - \mu Bu_n), \\
x_{n+1} = a_n v + b_n x_n + (1 - a_n - b_n)W_n P_C(y_n - \lambda Ay_n), \quad n \geq 1.
\end{cases}
\]

If \(\lambda \in (0,2\alpha_1)\), \(0 < \mu < \frac{2(d' - d^2)}{L^2}\) and the sequences \(\{r_n\},\{a_n\},\{b_n\}\) are as in Theorem 3.2, then \(\{x_n\}\) converges strongly to \(\overline{x} = P_{\Omega} v\) and \((\overline{x},\overline{y})\) is a solution of GSVI (1.2), where \(\overline{y} = P_C(\overline{x} - \mu Bu_n)\).

If \(N = 1\), \(T_1 = S\), \(\varphi = 0\) and \(\alpha_{2,j}^1 = 0\), \(\alpha_{3,1}^1 = 0\) \(\forall n \in \mathbb{N}\) in Theorem 3.2, then we obtain the following result.

**Corollary 3.4.** Let \(C\) be a nonempty closed and convex subset of a real Hilbert space \(H\). Let \(F\) be a function from \(C \times C\) to \(\mathbb{R}\) satisfying (A1)-(A5). Let \(A : C \to H\) be \(\alpha\)-inverse-strongly monotone mapping and let \(B : C \to H\) be a \(L\)-Lipschitzian and relaxed \((c',d')\)-cocoercive mapping. Let \(S\) be a nonexpansive self-mappings of \(C\) such that \(\Omega = F(S) \cap GSVI(C,A,B) \cap EP(F) \neq \emptyset\). Assume that \(v\) is an arbitrary
point in $C$. Let $x_1 \in C$ and $\{x_n\}, \{u_n\}, \{y_n\}$ be the sequences generated by

$$
\begin{align*}
F(u_n, y) + \frac{1}{r_n} (y - u_n, u_n - x_n) &\geq 0, \quad \forall y \in C, \\
y_n &= P_C(u_n - \mu Bu_n), \\
x_{n+1} &= a_nv + b_n x_n + (1 - a_n - b_n) SP_C(y_n - \lambda Ay_n), \quad n \geq 1.
\end{align*}
$$

If $\lambda \in (0, 2\alpha)$, $0 < \mu < \frac{2(d-c' L^2)}{L^2}$ and the sequences $\{r_n\}, \{a_n\}, \{b_n\}$ are as in Theorem 3.2, then $\{x_n\}$ converges strongly to $\bar{x} \in P_{\Omega} v$ and $(\bar{x}, \bar{y})$ is a solution of GSVI (1.2), where $\bar{y} = P_C(\bar{x} - \mu B\bar{x})$.

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