The least eigenvalues of the signless Laplacian of non-bipartite graphs with fixed diameter

Min Zhu, Yihao Guo, Fenglei Tian, Lingfei Lu

College of Science, China University of Mining and Technology, Xuzhou, Jiangsu, P. R. China

Abstract. Let $\zeta_n(d)$ ( $\mu_n(d)$ ) be the set of connected non-bipartite ( unicyclic ) graphs with $n$ vertices and diameter $d$. In this paper, we first determine the graph whose least eigenvalue of the signless Laplacian attains the minimum in $\mu_n(d)$, then by the eigenvalue interlacing property, the problem of determining the minimizing graph in $\zeta_n(d)$ can be transformed to that of determining the minimizing graph in $\mu_n(d)$. Thus we obtain a lower bound for the least eigenvalue of the signless Laplacian of a non-bipartite graph in terms of the diameter $d$.

Keywords: non-bipartite graph; signless Laplacian; Least eigenvalue; diameter

1 Introduction

Let $G$ be a simple graph with vertices $1, 2, \cdots, n$, of degrees $d_1, d_2, \cdots, d_n$, respectively. Let $A(G)$ be the $(0, 1)$-adjacency matrix of $G$, and let $D(G)$ be the diagonal matrix $\text{diag}(d_1, d_2, \cdots, d_n)$. The matrix $L(G) = D(G) - A(G)$ is the Laplacian of $G$, while $Q(G) = D(G) + A(G)$ is called the signless Laplacian of $G$. We call the eigenvalues of $Q(G)$ the $Q$-eigenvalues of graph $G$, it is known that $Q(G)$ is nonnegative, symmetric and positive semidefinite. So its eigenvalues are all nonnegative real numbers and can be arranged as $q_1(G) \geq q_2(G) \geq \cdots \geq q_n(G) \geq 0$. The least $Q$-eigenvalue is
$q_n(G)$, and the eigenvectors corresponding to $q_n(G)$ are called the first $Q$-eigenvectors of $G$. For the properties of the least $Q$-eigenvalue, we refer the readers to [1-6]. A graph is called minimizing in a class of graphs if its least $Q$-eigenvalue attains the minimum among all graphs in the class. Denote by $\zeta_n(d)$ $(\mu_n(d))$ the set of connected non-bipartite (unicyclic) graphs with $n$ vertices and diameter $d$. Let $\mu_n(g,d)$ denote the set of unicyclic graphs of order $n$ with odd girth $g$ and diameter $d$, $(d \geq \frac{g-1}{2})$.

If $G$ is connected, then $q_n(G) = 0$ if and only if $G$ is bipartite. So, connected non-bipartite graphs are considered here. The investigation on the lower bound of the least $Q$-eigenvalue of a graph is an important topic in the theory of $Q$-spectra. M. Desai, V. Rao discuss the relationship between the least $Q$-eigenvalue and the bipartiteness of graphs in [8]. Cardoso et al. [3] and Fan et al. [10] investigate the least $Q$-eigenvalue of non-bipartite unicyclic graphs. Liu et al. [11] give some bounds for the clique number and independence number of graphs in terms of the least $Q$-eigenvalue. Lima et al. [7] survey the known results and present some new ones for the least $Q$-eigenvalue. Wang et al. [13] investigated how the least $Q$-eigenvalue of a graph changes by relocating a bipartite branch.
from one vertex to another vertex, and minimized the least \( Q \) – eigenvalue among the connected graphs of fixed order which contain a given non-bipartite graph as an induced subgraph. Fan et al. [14] determine the minimizing graph of non-partite graphs in terms of the number of pendant vertices.

In this paper, we first show that \( U_n(g,d) \) (see Fig.1) is the unique minimizing graph in \( \mu_n(g,d) \), and then determine that \( U_n(3,d) \) is the unique minimizing graph in \( \mu_n(d) \). At last, by the eigenvalue interlacing property (see following Lemma 2.6), the problem of determining the minimizing graph in \( \zeta_n(d) \) can be transformed to that of determining the minimizing graph in \( \mu_n(d) \).

2. Preliminaries

We first introduce some notations. Let \( C_n \) and \( P_n \) to denote the cycle and the path, on \( n \) vertices, respectively. We also use \( P = v_1, v_2, \ldots, v_n \) to denote a path on vertices \( v_1, v_2, \ldots, v_n \) with edges \( v_iv_{i+1} \) for \( i = 1, 2, \ldots, n-1 \). Let \( N_G(v) \) be the set of the neighborhood of the vertex \( v \) in graph \( G \). Let \( G \) be a graph, \( G \) is called trivial if it contains only one vertex; otherwise, it is called nontrivial.

Graph \( G \) is called unicyclic if it is connected and has the same number of vertices and edges (or \( G \) contains exactly one cycle). The girth of \( G \) is the minimum of the lengths of all cycles in \( G \). A pendant vertex of \( G \) is a vertex of degree 1. A path \( P = v_0, v_1, \ldots, v_{t-1}, v_t \) in \( G \) is called a pendant path if \( d_{v_0} \geq 3 \), \( d_{v_1} = d_{v_2} = \cdots = d_{v_{t-1}} = 2 \) and \( d_{v_t} = 1 \). If \( t = 1 \), then \( v_0v_t \) is a pendant edge of \( G \).

Let \( x = (x_1, x_2, \ldots, x_n)' \) be a column vector, and let \( G \) be a graph on vertices \( V(G) = \{v_1, v_2, \ldots, v_n\} \). The vector \( x \) can be viewed as a function defined
on $V(G)$; that is, any vertex $v_i$ is given by the value $x_i = x_{v_i}$. Thus the quadratic form $x'Qx$ can be written as

$$x'Qx = \sum_{uv \in E(G)} [x_u + x_v]^2.$$

One can find that $q$ is a $Q$-eigenvalue of $G$ corresponding to an eigenvector $x$ if and only if $x \neq 0$ and

$$[q - d_v]x_v = \sum_{u \in N_G(v)} x_u,$$

for each $v \in V(G)$. In addition, for an arbitrary unit vector $x$,

$$q_n(G) \leq x'Q(G)x,$$

with equality if and only if $x$ is a first $Q$-eigenvector of $G$.

Let $G_1$ and $G_2$ be two vertex-disjoint graphs, and let $v \in G_1$, $u \in G_2$. The coalescence of $G_1$ and $G_2$ with respect to $v$ and $u$, denoted by $G_1 \bullet G_2$, is obtained from $G_1$ and $G_2$ by identifying $v$ with $u$ and forming a new vertex. Let $G$ be a connected graph, and let $v$ be a cut vertex of $G$. Then $G$ can be expressed in the form $G = H(v) \bullet F(v)$, where $H$ and $F$ are subgraphs of $G$ both containing $v$. Here, we call $H$ (or $F$) a branch of $G$ with root $v$. With respect to a vector $x$ defined on $G$, the branch $H$ is called a zero branch if $x_v = 0$ for all $v \in V(H)$; otherwise, $H$ is called a nonzero branch.

Let $G = G_1(v_2) \bullet G_2(u)$, $G^* = G_1(v_1) \bullet G_2(u)$, where $v_1$ and $v_2$ are two distinct vertices of $G_1$ and $u$ is a vertex of $G_2$. We say that $G^*$ is obtained from $G$ by relocating $G_2$ from $v_2$ to $v_1$. Then, we give some lemmas that will be used in the proof of our result.

**Lemma 2.1** ([13]) Let $H$ be a bipartite branch of a connected graph $G$ with root $u$. Let $x$ be a first $Q$-eigenvector of $G$.
(1) If $x_u = 0$, then $H$ is a zero branch of $G$ with respect to $x$.

(2) If $x_u \neq 0$, then $x_p \neq 0$ for every vertex $p$ of $H$. Furthermore, for every vertex $p$ of $H$, $x_p x_u$ is positive or negative, depending on whether $p$ is or is not in the same part of bipartite graph $H$ as $u$; consequently, $x_p x_q < 0$ for each edge $pq \in E(H)$.

**Lemma 2.2** ([13]) Let $G$ be a connected non-bipartite graph, and let $x$ be a first $Q$-eigenvector of $G$. Let $T$ be a tree with root $u$, which is a nonzero branch with respect to $x$. Then $|x_q| < |x_p|$ whenever $p$ and $q$ are vertices of $T$ such that $q$ lies on the unique path from $u$ to $p$.

**Lemma 2.3** ([13]) Let $G_1$ be a connected graph containing at least two vertices $v_1$, $v_2$, and let $G_2$ be a connected bipartite graph containing a vertex $u$. Let $G = G_1(v_2) \cdot G_2(u)$ and $G' = G_1(v_1) \cdot G_2(u)$. If there exists a first $Q$-eigenvector $x$ of $G$ such that $|x_{v_1}| \geq |x_{v_2}|$, then $q_n(G') \leq q_n(G)$, with equality only if $|x_{v_1}| = |x_{v_2}|$ and $d_{G_1(v_1)} x_u = -\sum_{v \in N_{G_2}(u)} x_v$.

**Lemma 2.4** ([13]) Let $G_1$ be a connected non-bipartite graph containing two vertices $v_1$, $v_2$, and let $P$ be a nontrivial path with $u$ as an end vertex. Let $G = G_1(v_2) \cdot P(u)$, and let $G' = G_1(v_1) \cdot P(u)$. If there exists a first $Q$-eigenvector $x$ of $G$ such that $|x_{v_1}| > |x_{v_2}|$ or $|x_{v_1}| = |x_{v_2}| > 0$, then $q_n(G') \leq q_n(G)$.

**Lemma 2.5** ([14]) Let $U_g(g, d)$ be the graph with some vertices labeled as in Fig.1, where $v_1, v_2, \cdots v_g$ are the vertices of the unique cycle $C_g$ labeled in an
anticlockwise way. Let $x$ be a first $Q-$eigenvector of $U_n(g,d)$. Then, the following hold:

1. $x_v = x_{v+1}$ for $i = 1,2,\ldots, \frac{n-1}{2}$.

2. $x_v > 0$, and $x_v x_w < 0$ for every edges $vw$ of $U_n(g,d)$ except $v_{\frac{n}{2}}, v_{\frac{n}{2}+1}$.

3. $x_1 > x_2 > \cdots > x_{\frac{n}{2}} > 0$.

**Lemma 2.6** ([3]) Let $G$ be a graph of order $n$ containing an edge $e$. Let $q_1, q_2, \cdots, q_n$ ($q_1 \geq q_2 \geq \cdots \geq q_n$) and $s_1, s_2, \cdots, s_n$ ($s_1 \geq s_2 \geq \cdots \geq s_n$) be the $Q-$eigenvalues of $G$ and $G - e$. Then

$$0 \leq s_n \leq q_n \leq \cdots \leq s_2 \leq q_2 \leq s_1 \leq q_1.$$ 

**3. Characterization of the extremal graph**

**Lemma 3.1** Let $U$ be the minimizing graph in $\mu_n(g,d)$ and $P$ be a diameter-path of $U$ , then $P$ must encounters the unique cycle $C$, and $|V(P) \cap V(C)| = \frac{n+1}{2}$.

**Proof.** Let $P = u_1, u_2, \cdots, u_{d+1}$ be a diameter-path of $U$ and $C = v_1, v_2, \cdots, v_g, v_1$ be the unique cycle of $U$. Suppose that $|V(P) \cap V(C)| = \phi$. Since $U$ is connected, then suppose that there exists a shortest path $v_g, w_1, w_2, \cdots, w_k, u_1$ connecting $C$ and $P$, where $w_1, w_2, \cdots, w_k \in V(U) \setminus (V(P) \cup V(C))$. Let $x$ be an eigenvector of $Q(U)$ corresponding to $q_n(U)$ and define graph $U_1$.

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The corresponding author. E-mail address: 741856964@qq.com
Then in either case (indeed, they are isomorphic, without loss of generality, we choose $U_1$ be the graph with vertices labels as the latter), $P$ is still a diameter-path of $U_1$, $U_1 \in \mu_n(g,d)$ and $|V(P) \cap V(C)| = 1$. And by Lemma 2.2, $q_n(U) \geq q_n(U_1)$, a contradiction. Hence, $|V(P) \cap V(C)| \neq \phi$, so, the diameter-path $P$ encounters the cycle $C$ in $U$.

Then we continue to define graph $U_i$, ($i = 1, 2, \ldots, \frac{\varepsilon+1}{2}$)

$$U_i = \begin{cases} U_{i-1} - \{u_{k+i}, v_{k+i}\} + \{v_{2-i}u_{k+i}\}, & \text{if } x_{v_{2-i}} \geq x_{u_{k+i}}; \\ U_{i-1} - \{v_{2-i}, v_{1-i}\} + \{u_{k+i}, v_{k+i}\}, & \text{if } x_{v_{2-i}} < x_{u_{k+i}}. \end{cases}$$

In the graph $U_i$, we can easily see that $P$ is still a diameter-path of $U_i$, $U_i \in \mu_n(g,d)$ and $|V(P) \cap V(C)| = i$. And by Lemma 2.2, $q_n(U_1) \geq q_n(U_2) \geq q_n(U_3) \geq \cdots \geq q_n(U_{\frac{\varepsilon+1}{2}})$.

It doesn't continue to define graphs according to the above method, otherwise, it contradicts to that $P$ is the diameter-path, so $P$ must encounters the unique cycle $C$ in $U$, and $|V(P) \cap V(C)| = \frac{\varepsilon+1}{2}$.

**Lemma 3.2** Among all graphs in $\mu_n(g,d)$, $U_n(g,d)$ is the unique minimizing graph.

**Proof.** Let $G$ be a minimizing graph in $\mu_n(g,d)$, and let $C_g$ be the unique cycle of $G$ on vertices $v_1, v_2, \cdots v_k$. Graph $G$ can be considered as one obtained from $C_g$ by identifying each $v_i$ with one vertex of some tree $T_i$ of

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The corresponding author. E-mail address: 741856964@qq.com
order \( n_i \) for each \( i = 1,2,\cdots,g \), where \( \sum_{i=1}^{d} n_i = n \). Note that some trees \( T_i \) may be trivial.

Let \( x \) be a unit first \( Q \)-eigenvector of \( G \), then there exists at least one \( i \), \((1 \leq i \leq g)\), such that \( x_{v_i} \neq 0 \). Otherwise, by Lemma 2.1(1), each \( T_i \), \((1 \leq i \leq g)\), is a zero branch of \( G \) with respect to \( x \), and it follows that \( x \) is the zero vector, a contradiction. We claim that each nontrivial tree \( T_j \) is a nonzero branch with respect to \( x \). Otherwise, there exists a nontrivial tree \( T_j \) attached at \( v_j \), \((1 \leq j \leq g)\), such that \( x_{v_j} = 0 \). By Lemma 2.3, relocating the tree \( T_j \) from \( v_j \) to \( v_i \) for some \( i \) for which \( x_{v_i} \neq 0 \), we obtain a graph in \( \mu_n(g,d) \) with smaller least \( Q \)-eigenvalue, a contradiction. We also claim that there is only one nontrivial tree \( T \). If not, there exist two nontrivial trees, say \( T_i \), \( T_j \) attached at \( v_i \), \( v_j \), respectively. By Lemma 2.2 and 2.4, relocating the tree \( T_j \) from \( v_j \) to one vertex of tree \( T_i \) (if \( \left| x_{v_j} \right| \geq \left| x_{v_i} \right| \)), or relocating the tree \( T_i \) from \( v_i \) to one vertex of tree \( T_j \), (if \( \left| x_{v_i} \right| \geq \left| x_{v_j} \right| \)), we can obtain a graph in \( \mu_n(g,d) \) with smaller least \( Q \)-eigenvalue, it contradicts to the minimum of \( G \).

We assume that \( P' = u_0, u_1, \cdots, u_{d'} \) (let \( u_0 = v_g \) and \( d' = d - \frac{s-1}{2} \)) is the diameter-path of the only nontrivial tree \( T \). We claim that any vertex \( x \in V(G \setminus \{C_g \cup P'\}) \) is a pendant vertex, and if exists, it attached to the unique vertex \( u_{d'-1} \). First, we suppose that there exits a pendant path \( P'' = w_0, w_1, \cdots, w_s \) \((2 \leq s \leq d')\) attached to the path \( P' \).

Case 1. \( s = d' \), then we can see that \( w_0 = u_0 \).
If $|x_{u_{r+1}}| \geq |x_{u_{r-1}}|$, then replacing edge $w_{r-1}w_s$ by $u_{d'-1}w_s$, (otherwise, replacing $u_{d'-1}w_s$ by $w_{r-1}u_{d'}$). We can obtain a new graph $G'$, and $G' \in \mu_n(g,d)$, by lemma 2.4, it followed that $G'$ has smaller least $Q$-eigenvalue, a contradiction.

Case 2. $2 \leq s < d'$, then $w_0 \in \{u_0, u_1, \cdots u_{d'-1}\}$, as the $P'$ is the diameter-path of $T$. As the assumption that $G$ is a minimizing graph and $|x_{u_{d'-1}}| \geq |x_{u_0}|$ (by lemma 2.2), by lemma 2.4, we can see that $w_0 = u_{d'-1}$. Then we compare $|x_{u_{d'-1}}|$ with $|x_{u_{d'-1}}|$, by the same discussion as the Case 1, We can obtain a new graph $G'$, and $G' \in \mu_n(g,d)$, by lemma 2.4, it followed that $G'$ has smaller least $Q$-eigenvalue, a contradiction.

So, any vertex $x \in V(G \setminus C \cup P')$ is a pendant vertex.

Now, suppose that $G$ contains at least one such star $S_{u_k}$, which has center $u_k$, $(k = 0, 1, 2, \cdots, d'-2)$, as $|x_{u_{d'-1}}| \geq |x_{u_k}|$ (by lemma 2.2), denote by $G'$ the graph

$$G - \sum_{w \in N_{S_{u_k}}(u_k)} wu_k + \sum_{w \in N_{S_{u_k}}(u_k)} wu_{d'-1}$$

and $G' \in \mu_n(g,d)$, by lemma 2.4, $G'$ has smallest least $Q$-eigenvalue, a contradiction.

So, we can easily conclude that $U_n(g,d)$ is the unique minimizing graph.

Denote by $t_n(g,d)$ the minimum of the least $Q$-eigenvalues of graphs in \(\mu_n(g,d)\), that is, the least $Q$-eigenvalue of $U_n(g,d)$.
Lemma 3.3 \( t_n(g,d) \) is strictly increasing with respect to odd \( g \), \( g \geq 3 \).

Proof. Let \( U_n(g,d) \) with some vertices labeled as in Fig.1, and \( x \) be a unit first \( Q \)-eigenvector of \( U_n(g,d) \). Suppose that \( g \geq 5 \), as \( x_{v_{g+1}} = x_{v_{g+1}} \) by lemma 2.5, replacing edge \( v_{g-2}v_{g-1} \) by edge \( v_{g-2}v_{g+1} \), we obtain a new graph \( G' \in \mu_n(g-2,d) \), which satisfies that \( x'Q(G')x = x'Q(U_n(g,d))x = t_n(g,d) \). So, \( q_n(G') \leq t_n(g,d) \), and hence, \( t_n(g-2,d) \leq q_n(G') \leq t_n(g,d) \). The result follows.

Corollary 3.4 Among all graphs in \( \mu_n(d) \), \( U_n(3,d) \) is the unique minimizing graph.

By the lemma 2.6 and lemma 3.1-3.4, we arrive at the main Theorem of this paper.

Theorem 3.5 Among all graphs in \( \zeta_n(d) \), \( U_n(3,d) \) is the unique minimizing graph.

Proof. Let \( G \) be a minimizing graph in \( \zeta_n(d) \). Then \( G \) contains at least an induced odd cycle, say \( C_g \). Let \( G' \) be a unicyclic spanning subgraph of \( G \), which obtained by deleting an edge in every cycle except for \( C_g \) and maintain that \( G' \in \mu_n(g,d) \). By lemma 2.6 and Corollary 3.4, we can see that

\[
q_n(U_n(3,d)) = t_n(3,d) \leq t_n(g,d) \leq q_n(G') \leq q_n(G)
\]

(3.1)

As \( G \) is a minimizing graph in \( \zeta_n(d) \), all inequalities in (3.1) hold as equalities, by Lemma 3.2 and 3.3, which implies that \( g = 3 \), \( G' = U_n(3,d) \) and \( q_n(G) = q_n(U_n(3,d)) \).

The corresponding author. E-mail address: 741856964@qq.com
Now, we prove that \( G = U_n(3,d) \). Suppose that \( E(G) \setminus E(U_n(3,d)) \neq \emptyset \).

Recalling the definition of \( G' \) and \( G' = U_n(3,d) \), the set \( E(G) \setminus E(U_n(3,d)) \) consists of some edges joining the vertices of \( C_3 \) and the vertices of \( T \) or some edges within the vertices of \( T \). So, for each edge \( uv \in E(G) \setminus E(U_n(3,d)) \), if \( x \) is a first \( Q \) - eigenvector of \( U_n(3,d) \), then by Lemma 2.2 and Lemma 2.5(3) we can see that \( x_u + x_v \neq 0 \).

Let \( x \) be a unit first \( Q \) - eigenvector of \( G \). Then

\[
q_n(G) = \sum_{uv \in E(G)} [x_u + x_v]^2 = \sum_{uv \in E(U_n(3,d))} [x_u + x_v]^2 + \sum_{uv \in E(G) \setminus E(U_n(3,d))} [x_u + x_v]^2 \\
\geq \sum_{uv \in E(U_n(3,d))} [x_u + x_v]^2 \geq q_n(U_n(3,d))
\]

Since \( q_n(G) = q_n(U_n(3,d)) \), \( x \) is also a first \( Q \) - eigenvector of \( U_n(3,d) \), so for each edge \( uv \in E(G) \setminus E(U_n(3,d)) \), \( x_u + x_v = 0 \), a contradiction. Hence, \( E(G) \setminus E(U_n(3,d)) = \emptyset \), the result follows.

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