Arithmetic Progression of more Squares

Dr.Muneer Jebreel Karama

Assistance Professor

College of Applied Science, University Graduates Union
Palestine Polytechnic University

PO Box 198, Palestine, West Bank, Hebron

Tel: +972(or +970)- 2238442 E-mail: muneerk@ppu.edu

Abstract

The aim of this paper is presenting new arithmetic progressions among squares i.e. more than three squares, moreover to introduce arithmetic progression of higher power.

Keywords: Arithmetic Progression, squares, higher power

1. Introduction

Referring to my research [1] it is possible to produce a chine of triplets of the type (a,b,c) of positive integers so that a^2, b^2, c^2 are in Arithmetic Progression (AP) such that :
$b^2 – a^2 = c^2 – b^2$.

In 2008 [2] Peth″o and Ziegler have found an arithmetic progression of length 4 that lies on some curve $X^2−dY^2 = m$ and they have found an arithmetic progression such that there does not exist such a curve.

In 2010 [3] Enrique Gonz and J. Äorn Steuding, they gave a partial answer to this question: Let d be a squarefree integer. Does there exist four squares in arithmetic progression over $\mathbb{Q}(\sqrt{d})$? depending on the value of d. In the affirmative case, they construct explicit arithmetic progressions consisting of four squares over $\mathbb{Q}(\sqrt{d})$.

2. Background

It is well known that the right triangles whose sides are integers *X,Y,Z* (a “Pythagorean triple”) formed Arithmetical Progressions (AP) among three squares such that the following equation holds:

$$(X+Y)^2+(X-Y)^2=2Z^2 \tag{1}$$

For example the Pythagorean triple (3,4,5) represents the AP (1,5,7), i.e. $7^2+1^2=2(5)^2$, or more familiars $5^2-1^2=7^2-5^2$, in fact equation (1) has infinitely many solutions, and chains with common difference 24.

3. Extension of equation (1)

With slight modification of equation (1), we got the following new equation:

$$(X+Y)^2+(X-Y)^2+(-X+Y)^2+(-X-Y)^2=4Z^2 \tag{2}$$

Proof: assume that $X=2mn$, $Y=m^2-n^2$, and $Z= m^2+n^2$, then by direct substitutions we got $(2mn+ m^2-n^2)^2+(2mn-(m^2-n^2))^2 +(-2mn+ m^2-n^2)^2 +(-2mn-(m^2-n^2))^2= 4(m^2+n^2)^2=4 Z^2$.

I show the first sixteen consecutive AP of equation (2) in table 1, for examples:

1) $7^2+1^2+(-7)^2+(-1)^2=100=4(25)=4(5)^2$, i.e. $7^2+(-7)^2-(5)^2-(5)^2= (5)^2+(5)^2-(-1)^2 -1^2$, so we have the AP (-7,-1,1,5,7).
2) $137^2+7^2+(-7)^2+(-137)^2=37636=4(9409)=4(97)^2$, i.e. $137^2+(-137)^2-(97)^2-(97)^2= (97)^2+(97)^2-(-7)^2 -7^2$, so we have the AP (-137,-7,7,97,137).

Table 1 The first sixteen AP of equation (2)

Pythagorean triples			$(X+Y)$	$(X-Y)$	$(-X+Y)$	$(-X-Y)$	$4(Z)$
X	Y	Z					
3	4	5	7	1	-1	-7	4(5)
5	12	13	17	7	-7	-17	4(13)
7	24	25	31	17	-17	-31	4(25)
8	15	17	23	7	-7	-23	4(17)
9	40	41	49	31	-31	-49	4(41)
11	60	61	71	49	-49	-71	4(61)
12	35	37	47	23	-23	-47	4(37)
13	84	85	97	71	-71	-97	4(85)
16	63	65	79	47	-47	-79	4(65)
20	21	29	41	1	-1	-41	4(29)
28	45	53	73	17	-17	-73	4(53)
33	56	65	89	23	-23	-89	4(65)
36	77	85	113	41	-41	-113	4(85)
39	80	89	119	41	-41	-119	4(89)
48	55	73	103	7	-7	-103	4(73)
65	72	97	137	7	-7	-137	4(97)

Insight looking at Table 2, suggests more, and deep studding of the chains formulas that connected many AP's with each others. Also finding formula of the common difference which will be multiple of 24.

4. Theorem1. *If X,Y,Z are the sides of right triangle then;*

$$(X+Y+Z)^2+(X-Y+Z)^2+(X+Y-Z)^2+(-X+Y+Z)^2=8Z^2 \qquad (3)$$

assume that $X=2mn$, $Y=m^2-n^2$, and $Z=m^2+n^2$, then by direct substitutions we got:

$(2mn+m^2-n^2+m^2+n^2)^2+(2mn-m^2+n^2+m^2+n^2)^2+(2mn+m^2-n^2-m^2-n^2)^2+(-2mn+m^2-n^2+m^2+n^2)^2=8(m^2+n^2)^2=8Z^2$.

For example the Pythagorean triple (3,4,5) represents the AP (2,4,6,12), i.e. $2^2+4^2+6^2+12^2=8(5)^2$, hence: $12^2+6^2-5^2-5^2-5^2-5^2=5^2+5^2+5^2+5^2-2^2-4^2$,if we cancel 2 from the AP (2,4,6,12), then we get (1,2,3,6), which is of course AP.($1^2+2^2+3^2+6^2=2(5)^2$) , this example has new ideas that maybe provoke for more new results which is open for Mathematicians .

Table 2 shows the first sixteen AP of equation 3Table 2 The first sixteen AP of equation (3)

Pythagorean triples			$(X+Y+Z)$	$(X-Y+Z)$	$(X+Y-Z)$	$(-X+Y+Z)$	$8(Z)$
X	Y	Z					
3	4	5	12	4	2	6	8(5)
5	12	13	30	6	4	20	8(13)
7	24	25	56	8	6	42	8(25)
8	15	17	40	10	6	24	8(17)
9	40	41	90	10	8	72	8(41)
11	60	61	132	12	10	110	8(61)
12	35	37	84	14	10	60	8(37)
13	84	85	182	14	12	156	8(85)
16	63	65	144	18	14	112	8(65)
20	21	29	70	28	12	30	8(29)
28	45	53	126	36	20	70	8(53)
33	56	65	154	42	24	88	8(65)
36	77	85	198	44	28	126	8(85)
39	80	89	208	48	30	130	8(89)
48	55	73	176	66	30	80	8(73)
65	72	97	234	90	40	104	8(97)

Table 2 needs more studying to fined chains of AP , and common difference.

5. Generalization of theorem 1. *If X,Y,Z are the sides of right triangle then;*

$$(X+Y+Z)^2+(X-Y+Z)^2+(X+Y-Z)^2+(-X+Y+Z)^2+(-X-Y-Z)^2+(-X+Y-Z)^2+(X-Y-Z)^2+(-X-Y+Z)^2=16Z^2 \qquad (4)$$

assume that $X=2mn$, $Y=m^2-n^2$, and $Z=m^2+n^2$, then by direct substitutions we got:

$(2mn+m^2-n^2+m^2+n^2)^2+(2mn-m^2+n^2+m^2+n^2)^2+(2mn+m^2-n^2-m^2-n^2)^2+(-2mn+m^2-n^2+m^2+n^2)^2+(-2mn-m^2+n^2-m^2-n^2)+(-2mn+m^2-n^2-m^2-n^2)+(2mn-m^2+n^2-m^2-n^2)+(-2mn-m^2+n^2+m^2+n^2)^2=16(m^2+n^2)^2=16Z^2$.

Table 3 shows selected solutions of equation 4.

Table 3 solutions of equation 4.

Pythagorean Triples X	Y	Z	$(X+Y+Z)$	$(X-Y+Z)$	$(X+Y-Z)$	$(-X+Y+Z)$	$(-X-Y-Z)$	$(-X+Y-Z)$	$(X-Y-Z)$	$(-X-Y+Z)$	$16(Z)$
3	4	5	12	4	2	6	-12	-4	-6	-2	16(5)
5	12	13	30	6	4	20	-30	-6	-20	-4	16(13)
7	24	25	56	8	6	42	-56	-8	-42	-6	16(25)
8	15	17	40	10	6	24	-40	-10	-24	-6	16(17)
9	40	41	90	10	8	72	-90	-10	-72	-8	16(41)
11	60	61	132	12	10	110	-132	-12	-110	-10	16(61)
12	35	37	84	14	10	60	-84	-14	-60	-10	16(37)
13	84	85	182	14	12	156	-182	-14	-156	-12	16(85)
16	63	65	144	18	14	112	-144	-18	-112	-14	16(65)
20	21	29	70	28	12	30	-70	-28	-30	-12	16(29)
28	45	53	126	36	20	70	-126	-36	-70	-20	16(53)
33	56	65	154	42	24	88	-154	-42	-88	-24	16(65)
36	77	85	198	44	28	126	-198	-44	-126	-28	16(85)
39	80	89	208	48	30	130	-208	-48	-130	-30	16(89)
48	55	73	176	66	30	80	-176	-66	-80	-30	16(73)
65	72	97	234	90	40	104	-234	-90	-104	-40	16(97)

It seems that equation 4 is the general form of the previous equations, so we may plug some variables by zero to get them.

6. Higher power of AP. *If X,Y,Z are the sides of right triangle then;*

$$X^8+Y^8+Z^8=2W^2 \qquad (5)$$

Let m,n integers, with (m,n)=1, then the solution of equation 5:
$X=2mn$, $Y=m^2-n^2$, $Z= m^2 + n^2$, *and*
$W=(m^4+2m^3n+2m^2n^2-2mn^3+n^4)(m^4-2m^3n+2m^2n^2+2mn^3+n^4)$
It can be proved by direct substitution. Table 4 shows selected solutions of equation 5

Table 4 selected solutions of equation 5

Pythagorean triples X	Y	Z	*(X)^8*	*(Y)^8*	*(Z)^8*	*2W^2*
3	4	5	6,561	65,536	390,625	$2(481)^2$
5	12	13	390,625	429,981,696	815,730,721	$2(24961)^2$
7	24	25	5,764,801	110,075,314,176	152,587,890,625	$2(362401)^2$
8	15	17	16,777,216	2,562,890,625	6,975,757,441	$2(69121)^2$
9	40	41	43,046,721	6,553,600,000,000	7,984,925,229,121	$2(2696161)^2$
11	60	61	214,358,881	167,961,600,000,000	191,707,312,997,281	$2(13410241)^2$
12	35	37	429,981,696	2,251,875,390,625	3,512,479,453,921	$2(1697761)^2$
13	84	85	815,730,721	2,478,758,911,082,500	2,724,905,250,390,620	$2(51008161)^2$
16	63	65	4,294,967,296	248,155,780,267,521	318,644,812,890,625	$2(16834561)^2$
20	21	29	25,600,000,000	37,822,859,361	500,246,412,961	$2(530881)^2$
28	45	53	377,801,998,336	16,815,125,390,625	62,259,690,411,361	$2(6302881)^2$
33	56	65	1,406,408,618,241	96,717,311,574,016	318,644,812,890,625	$2(14435521)^2$
36	77	85	2,821,109,907,456	1,235,736,291,547,680	2,724,905,250,390,620	$2(44516641)^2$
39	80	89	5,352,009,260,481	1,677,721,600,000,000	3,936,588,805,702,080	$2(53007841)^2$
48	55	73	28,179,280,429,056	83,733,937,890,625	806,460,091,894,081	$2(21428641)^2$
65	72	97	318,644,812,890,625	722,204,136,308,736	7,837,433,594,376,960	$2(66626881)^2$

7. Conclusion.

From the equations above we can conclude such a sequence of patterns that began from equation 1 with $2Z^2$, continuo in equation 2 with $4Z^2$, same of equation 3 with $8Z^2$, and $16Z^2$ with equation 4, so what about $32Z^2$, $64Z^2$, and so on. Are there common pattern connected AP with 2^nZ^2?

References

[1] Muneer Jebreel, Arithmetic progression among squares, *Mathematical Spectrum* , 37(1): 37-37, 2005.

[2] A.Peth˝o and V. Ziegler, *Arithmetic progressions on Pell equations*, Elsevier Science, 5 March 2008.

[3] Enrique Gonz and J.Äorn Steuding, Arithmetic progressions of four squares over quadratic fields, *Publ. Math.* Debrecen , 77/1-2 , 125-138, 2010.