The analysis techniques for convexity: CAT-spaces (2)

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ABSTRACT: we consider discrete cocompact isometric action in hadamard space, and the action belongs to a class of groups. and we gives sharp conditions for a warped product of metric spaces to have a given curvature bound in the sense of alexandrov. we show that a number of different notions of dimension coincide for length spaces with curvature bounded above.

INTRODUCTION

We prove a conjecture of Schneider: the spherical caps are the only spherically convex bodies of the sphere which remain spherically convex after any two-point symmetrization.

Thus we have a broad new construction of spaces with curvature bounds, either above (CBA) or below (CBB) . as application, we extend the standard cone and suspension constructions of spaces with curvature bounds.

This paper is concerned with the structure of quasi-isometry between products of symmetric spaces and Euclidean buildings. We recall that Euclidean space, hyperbolic space, and complex hyperbolic space each admits an abundance of self-quasi isometries.

1. The structure of Hadamard space

The definition of Hadamard space is:

\[ d(z, m)^2 + \frac{d(x, y)^2}{4} \leq \frac{d(x, z)^2 + d(z, y)^2}{2} \]  

(1)

Where, \( m = (x + y)/2 \) is the midpoint of \(|xy|\)

and \( d(x, m) = d(y, m) = d(x, y)/2 \)

1.1 Tits metric and Tits boundary

Theorem 1.1: the Tits boundary of a hadamard space is a CAT(1) space.

Firstly, we define the Tits angle at infinity (see [1])

\[ \angle_{\text{Tits}}(\xi, \eta) = \lim_{x \to \xi, y \to \eta} \angle_{\text{v}}(x', y') \]  

(1)

In which,

\[ \angle_{\text{v}}(x', y') = \pi - \angle_{\text{v'}}(v, y') - \angle_{\text{y}}(v, x') \leq \angle_{x'}(\xi, y') \]  

(2)

Then, we introduce isomorphism:

\[ P_{Y} \cong Y \times N_{Y} \cup \{y\} \times N_{Y} = P_{Y} \cap \cap HB \xi(y) \]  

(3)

Using the isomorphism theorem to get:

\[ P_{Y} \cap \cap HB \xi(y)/Y \cong N_{Y} \cong N_{Y} \cap Y \cong N_{Y} \]  

(4)

Which imply that,
\[ x \in P_Y \cap \bigcap \overline{HB} \quad \xi(y) = G \Rightarrow Y = N_Y \quad (5) \]

The above argument yields,

\[ N_Y \triangleleft G = P_Y \cap \bigcap \overline{HB} \quad \xi(y) \quad (6) \]

Apply the relation (6) to Tits angle:

\[ x', y' \in vG, \quad v \in G \]

\[ \Rightarrow \angle_v : vG \times vG \to [0, \pi] \Rightarrow 2x' \cdot y' + (x' + y')v \in [0, \pi] \quad (7) \]

Where, we weaken the metric space \((X, \rho)\) to be a group.

### 1.2 Convex subset and parallel set

**Theorem 1.2**: every pair \(\xi, \eta \in \partial X\) with \(\angle_{\text{Tits}}(\xi, \eta) < \pi\) has a midpoint.

Here, we consider the parameter:

\[ \lambda_{ij} = \frac{|v_{x_i}|}{|v_{x_j}|} > 1 \quad (1) \]

And we introduce the distance induced by norms:

\[ D(f, g) = \|f - g\| < \delta = D(\kappa) = \text{Diam} \left( M^2 \right) \]

We now can define a tuple as:

\[ \left\| f - g \right\| \leq \sum_{n=N+1}^{\infty} a_n < \gamma \frac{r}{2} \quad (3) \]

Equivalently,

\[ \left\| f - x \right\| \leq \max_{0 \leq n \leq N} \left\{ \frac{a_n}{a_k} \right\} \sum_{n=0}^{N} a_k \min \left\{ 1, \left\| f - g \right\|_{\lambda} \right\} \quad (4) \]

Where,

\[ M = \max_{0 \leq n \leq N} \left\{ \frac{a_n}{a_k} \right\}, \quad \lambda = 1 \quad (5) \]

At last, our estimate holds:

\[ \left\| x_i \left( \lambda_{ij} m^j \right) \right\| \quad \left\| y_i \left( \lambda_{ij} m^j \right) \right\| \leq \frac{\lambda_{ij}}{2} M \sum_{n=0}^{N} a_k \min \left\{ 1, \left\| x - y \right\|_{\lambda} \right\} \rightarrow \frac{\lambda_{ij}^2}{2} r \quad (6) \]
1.3 Induced isomorphism and boundary problem

Theorem 1.3: suppose that \( x_i \) is a sequence in metric space which converges to a boundary point. Then \( \Phi \) converge to a boundary point too.

Here we take the following limit into account:

\[
\lim_{R \to \infty} \frac{1}{R} \cdot d_H(\Phi(p) \cap B_R(p), \Phi \cap B_R(\Phi)) = 0
\]  

The homomorphism can be seen,

\[
\partial_\infty \Phi = \Phi \partial_\infty X
\]

In section 1.1 we weaken the metric space and hence we can view the map between metric spaces as the map between groups at infinity.

By the above, without loss of generality we may assume the normal group as:

\[
K = \ker \varphi
\]

Since \( N \) is a normal subgroup which contain \( K \) in \( G \), therefore:

\[
\Rightarrow G / N \cong \varphi(G) / \varphi(N) \Rightarrow \varphi(cn^{-1}) = e
\]

Then we can construct a homomorphism:

\[
G \sim \varphi(G) / \varphi(N) \cong G / N \Rightarrow \varphi(n^{-1}) = cn^{-1}
\]

And thus we can write out the map:

\[
\partial_\infty \Phi = \Phi(v) = cv
\]

With, \( v = n^{-1} \)

Another result we used here is from section 1.2:

\[
R \to \infty \Rightarrow \sum_{n=0}^{\infty} a_n < \frac{\epsilon}{2} \to 0 \Rightarrow \| x - y \| \rightarrow 0 \Rightarrow x(v) \in [0, \pi / 2]
\]

\[
\Rightarrow \partial_\infty \Phi \in [0, \frac{\epsilon \sqrt{\pi}}{2}]
\]

1.4 Gromov hyperbolic group

From [2], we can express the distance \( d(v_0, v_k) \) as:

\[
d(v_0, v_k) = k \delta_0 + (1 - k) \sum_{i=0}^{k} m_i \delta g_i > (1 - k - \frac{k}{2^2} - \frac{k}{2^3} - \ldots - \frac{k}{2^n}) \sum_{i=0}^{k} m_i \delta g_i
\]
\begin{align*}
&= (1 - 2k) \sum_{0}^{k} m_{i} \delta_{gi} \\
\end{align*}

1.5 The fixed point in Hadamard space

Theorem 1.5.1 : (see lemma 2.10 in [2])

Theorem 1.5.2 : (see lemma 2.23 in [2])

Firstly, we consider the stabilizer :

\[ x_{V} = v, \quad v \in G_{v} \]

Next, we define a cyclic by applying its centralizer :

\[ s_{1} \ldots s_{k} \equiv 1^{a_{1}} 2^{a_{2}} \ldots k^{a_{k}} \Rightarrow 1a_{1} + 2a_{2} + \ldots + ka_{k} = k \]

\[ \Rightarrow 1 \log s_{1} + 2 \log s_{2} + \ldots + k \log s_{k} = k \]

Since \( s_{i} = \sum \pm 1 \), from the results in section 1.4, we have :

\[ \frac{k}{4} (\frac{k}{2} - 1) + (\frac{k}{2} + 1) \log \frac{k}{2} - 1 + (\frac{k}{2} + 2) \log \frac{k}{2} - 2 + \ldots + k \log \frac{1}{k} = k \]

We now complete our proof and write out the fixed point :

\[ v = 1 \cdot 2 \cdot \ldots \cdot \frac{k}{2} \cdot (\frac{k}{2} - 1) \cdot (\frac{k}{2} - 2) \cdot \ldots \cdot 1 \]

\[ \Phi \left( \frac{1}{1 \cdot 2 \cdot \ldots \cdot \frac{k}{2} \cdot (\frac{k}{2} - 1) \cdot (\frac{k}{2} - 2) \cdot \ldots \cdot 1} \right) = e = c^{k} (1 \cdot 2 \cdot \ldots \cdot \frac{k}{2} \cdot (\frac{k}{2} - 1) \cdot (\frac{k}{2} - 2) \cdot \ldots \cdot 1) \]

\[ \Rightarrow c^{k} = \left( \frac{1}{1 \cdot 2 \cdot \ldots \cdot \frac{k}{2} \cdot (\frac{k}{2} - 1) \cdot (\frac{k}{2} - 2) \cdot \ldots \cdot 1} \right)^{2} \]

\[ \Rightarrow \Phi (\xi) = \frac{\sqrt{\pi}}{2^{\frac{1}{k}} / k} \sim \frac{\sqrt{\pi}}{n^{\frac{1}{k}} / k} \]

Here, we give a method to find the fixed point in Hadamard space, by constructing the normal subgroup which contain the kernel to search this fixed point.

2. Spherical building and Euclid building

2.1 The asymptotic cone in symmetric space

Theorem 2.1 : (see theorem 2.4.6 in [1])

The Cauchy sequence in theorem 2.1 gives :

\[ \left| d_{ij}(x^{k} \cdot x^{l}) - d_{w}(x^{k} \cdot x^{l}) \right| < \frac{1}{2} \]

(1)
Then we introduce the quotient space:
\[
\| y_{j+1} \| = \left| d_i(x^k_i \cdot x^l_i) - d_w(x^k_i, x^l_i) \right|
\]  \quad (2)

In which,
\[
y_{j+1} \sim x_n \quad \quad \quad \quad \quad \quad (3)
\]

\[
x_{n_k} \sim \quad \text{is the subsequence of } x_n.
\]

2.2 Mostow rigidity and lie group homomorphism (see [3] [4])

Theorem 2.2: (see lemma 2 in [4])

We need the expansion of lie group homomorphism:
\[
G \rightarrow G' \Rightarrow \text{Isom}(X) \rightarrow \text{Isom}(X')
\]  \quad (1)

The property of quotient space lead:
\[
M = G / H
\]  \quad (2)

By theorem 2.2, we can define the quotient map as:
\[
\phi : T \rightarrow \text{Ad}_G(T)
\]  \quad (3)

Assume now that the product space in (2) is:
\[
M (G) = T \times \phi G
\]  \quad (4)

Then, we introduce the lifting to handle this quotient map, such that:
\[
\varphi : G \rightarrow M \rightarrow G'
\]  \quad (5)

With,
\[
\varphi : M \rightarrow G' \quad \phi : T \rightarrow \text{Ad}_G(T)
\]  \quad (6)

Consider the inner automorphism:
\[
G \rightarrow G' \Rightarrow G \equiv G / H \equiv \text{Inn}G \quad \Leftrightarrow \quad x \rightarrow axa^{-1}
\]  \quad (7)

Calculating the order of the groups above,
\[
M = G / H = GU
\]  \quad (8)

Where,
\[
|G| = p^{\alpha}a, \quad |U| = p^{\beta}b, \quad |G \cap U| = p^{\gamma}c
\]  \quad (9)

Which lead that:
\[
|M| = |G / H| = |GU| = p^{\alpha + \beta - \gamma} \cdot \frac{ab}{c} \quad \text{ (} p \text{ is prime)}
\]  \quad (10)

The assertion follows:
\[
\frac{c}{ab}p^\gamma - \alpha - \beta \quad \Rightarrow \quad ax = ae^{\frac{c}{ab}p^\gamma - \alpha - \beta} (11)
\]

\[
ax^{-1} = ae^{\frac{c}{ab}p^\gamma - \alpha - \beta}a^{-1} (12)
\]

2.3 Gromov hyperbolic space

Theorem 2.3: suppose that there exists a visual gromov hyperbolic metric space. Then this space with its convex boundary are roughly similar.

Firstly, we introduce the metric (in [5])

In the convex boundary \(Cov(\partial X)\), the theorem 2.3 imply that:

\[
\left| f(z, h) - f(z', h') \right| = \frac{\log(D / h) + \log(D / h')}{\varepsilon} - 2 \min\{ -\log(d(z, z')), \frac{\log(D / h)}{\varepsilon}, \frac{\log(D / h')}{\varepsilon} \}
\]

\[
= -\frac{\log h + \log h'}{\varepsilon} + 2 \min\{ -\log(d(z, z')), \frac{\log h}{\varepsilon}, \frac{\log h'}{\varepsilon} \} (1)
\]

The algebraic property of symmetric space gives (see [6])

\[
\pi_i: X \rightarrow X_i
\]

We thus can define the road map as:

\[
H(t, s) = \begin{cases} 
\alpha \left( \frac{4t}{1 + s} \right), & 0 \leq t \leq \frac{s + 1}{4} \\
\beta \left( \frac{4t - s - 1}{4} \right), & \frac{s + 1}{4} \leq t \leq \frac{s + 2}{4} \\
\gamma \left( \frac{4t - s - 2}{2 - s} \right), & \frac{s + 2}{4} \leq t \leq 1
\end{cases} (3)
\]

Where,

\[
\frac{c}{ab}p^\gamma - \alpha - \beta
\]

\[
\varepsilon \in [0, \frac{a + 1}{4}] \cup [\frac{a + 1}{4}, \frac{a + 2}{4}] \cup [\frac{a + 2}{4}, 1] (4)
\]

Lastly, we introduce the measure (in [7])

\[
\Rightarrow -\frac{\log h + \log h'}{\varepsilon} + 2 \min\{ -\log(d(z, z')), \frac{\log h}{\varepsilon}, \frac{\log h'}{\varepsilon} \} \leq \|y\| \leq \varepsilon \sim \frac{\ln 2}{n - 1} (5)
\]

We discuss (1) as the following three cases:

\[
\pm \frac{\log h - \log h'}{\varepsilon} \in [0, \frac{a + 1}{4}] \cup [\frac{a + 2}{4}, 1] \sim \frac{\ln 2}{n - 1}
\]
\[-\frac{\log h + \log h}{\epsilon} - 2 \log( d (z, z') \in [\frac{a+1}{4}, \frac{a+2}{4}]) \sim \ln \frac{2}{n-1} \quad (6)\]

### 2.4 Coxeter complex

Now we go back to [1] [2], further study the coxeter complex:

We define the map first,

\[ \theta_S : S \rightarrow S \setminus W \land \theta_B : B \rightarrow S \setminus W \land \theta \circ \sigma = \text{id}_{\Delta} \quad (1) \]

Consequently we can introduce homotopy for (1),

\[
\begin{align*}
F(x,4t) &= \frac{(1-4t)f(x) + 4t \cdot g(x)}{[1-4t]f(x) + 4t \cdot g(x)} \\
F(x,4t-1) &= \frac{4tf(x) + (4t-1) \cdot g(x)}{[4tf(x) + (4t-1) \cdot g(x)]} \\
F(x,2t-1) &= \frac{2tf(x) + (2t-1) \cdot g(x)}{[2tf(x) + (2t-1) \cdot g(x)]} \quad (2)
\end{align*}
\]

### 2.5 Moufang building and convex set

Here, we define the map as:

\[ \phi : x \rightarrow axa^{-1} = aeab -1 \in [0, \frac{a+1}{4}] \cup [\frac{a+1}{4}, \frac{a+2}{4}] \cup [\frac{a+2}{4}, 1] \]

### 2.6 The topology of building building

The bijection is:

\[ A \subseteq X \leftrightarrow \partial_{\text{Tits}} A \subseteq \partial_{\text{Tits}} X \quad (1) \]

We introduce boundary isomorphism and construct exact sequence:

\[ \sigma = \sum_{q+1}^{\text{a}} \frac{1}{q+1} a_i = \frac{p+1}{q+1} \sum_{0}^{p} \frac{1}{p+1} a_i + \frac{q-p}{q+1} \sum_{p+1}^{q} \frac{1}{q-p} a_i \quad (2) \]

\[ \cdots \rightarrow H_{n-1}(X, Y) \rightarrow H_n(X, Y) \rightarrow \cdots \quad (3) \]

Also note that the synthetic maps is 0, so

\[ F^{p+i} : \Delta_{n-1} \rightarrow \Delta_n \quad (4) \]

with, \[ F^{p+i} : \Delta_{n-1} \rightarrow \Delta_n \]
2.7 Affine weyl group

Here we write down the metric (in [1])

1. \[ d(\theta(xy), \theta(xz)) \leq \sum_x d(y, z), \quad d_{\Delta \text{mod}}(\theta(xw, yw), \theta(xw, zw)) \leq \sum_x w(y, zw) \]

2. \[ d(x, p) \geq \frac{d(p, F)}{\sin(\Delta \text{mod}(\partial x, \partial \Delta \text{mod}))} \]

3. \[ |y\ z| \leq \frac{1}{\sin \alpha_o} \cdot d(y\ , yy\ ) \]

2.8 Isomorphism, quasi-isometry and similar approximation

In this section we give an approximation to the radius:

From section 2.7, we can use \[ \frac{d(p, F)[y\ z]}{d(y\ , yy\ )} \] to approximate the radius.

The chain maps and chain homotopy gives:

\[ \phi : x \rightarrow axa^{-1} \Rightarrow F^r: \Delta_{n-1} \rightarrow \Delta_n = (\frac{1}{n} - \frac{1}{n+1})^r \quad (1) \]

On the other hand, the property of Hadamard space imply:

\[ \frac{d(p, F)[y\ z]}{d(y\ , yy\ )} = d(p, F)[yz]\sqrt{\frac{d(x, z)^2 + d(z, y)^2}{2} - d(z, m)^2} \quad (2) \]

Next we apply the results in section 2.3 and 2.4 to handle the limit of the metric in section 2.7.

\[ |yz| = 2^{n-1}, \quad \sqrt{\frac{d(x, z)^2 + d(z, y)^2}{2} - d(z, m)^2} = \frac{e^e}{\sqrt{2}} \quad (3) \]

We can choose,

\[ F^r : \Delta_{n-1} \rightarrow \Delta_n = (\frac{1}{n} - \frac{1}{n+1})^r \rightarrow 0 \quad (4) \]

And we get our limit estimate at last:

\[ \|x\| \rightarrow R : \frac{e^e}{\sqrt{2}n+1} \quad (5) \]

Meanwhile,

\[ F(x, 4t) = \frac{(1 - 4t)f(x) + 4t \cdot g(x)}{\|1 - 4t f(x) + 4t \cdot g(x)\|} \]
We now finish our proof with the following relation formula:

\[ F(x) = 0 \quad \text{for} \quad t \in [0, \frac{R}{\sqrt{2}} + 1, \frac{R}{\sqrt{2}} + 2, \frac{R}{\sqrt{2}} + 2], \quad \text{and for} \quad t \in [\frac{R}{\sqrt{2}} + 2, 4, 4, 4, 4], + 1 \]

Here, we study the convex function in the Gromov space. Firstly, we construct an auto-homomorphism; then we define the road map and divide this problem into three cases; then we use exact sequence to approximate the limit location of 2, and we divide the function \( F(x, t) \) into three cases too.

### 3. CBA space and CBB space (see [8] [9])

#### 3.1 Barycentric simplex

Theorem 3.1: (see section 4 in [8])

Firstly, we write out the chain maps:

\[ H_n(\alpha_k) : H_n(B(0, \delta), \partial B(0, \delta)) \to H_n(E^n, E^n \setminus \{0\}) \]

\[ H_n(\psi_k) : H_n(B(0, \delta R_k), \partial B(0, \delta R_k)) \to H_n(X, V_k) \]

We introduce the Poincaré characteristic and consider the infinite group:

\[ H_n(B(0, \delta), \partial B(0, \delta)) \cong H_{n-1}(S^{n-1}) \xrightarrow{i} H_{n-1}(S^{n-1}, E^{n-1}) \]

\[ H_n(E^n, S^{n-1}) \xrightarrow{\partial} H_{n-1}(S^{n-1}) \xrightarrow{i} H_{n-1}(E^{n-1}, S^{n-1}) \]

Which imply that:

\[ \rho(E^n) = k, \quad \rho(E^n) = l = k + 1 \]

The characteristic of the Abel group gives, that is, the Poincaré number:

\[ \chi(G) = \sum (-1)^q \rho(G_q) = -\lambda_1 h_1 + \lambda_2 h_2 - \ldots + \lambda_k h_k - \mu_1 g_1 + \mu_2 g_2 - \ldots - \mu_{k+1} g_{k+1} \]
Where, \( h_i \) and \( g_i \) are the maximal linear independent of \( E_+^n \) and \( E_-^n \).

\[
\lambda_1 h_1 + \lambda_2 h_2 + \cdots + \lambda_k h_k + \mu_1 g_1 + \mu_2 g_2 + \cdots + \mu_{k+1} g_{k+1} = 0
\]

\[
\Rightarrow \lambda_1 = \lambda_2 = \cdots = \mu_1 = \mu_2 = \cdots = 0 \quad (7)
\]
The condition above is for finite group, to the infinite group, the condition above may not be true.

### 3.2 Subset homomorphism

Theorem 3.2: (see section 6 in [8])

From above, in section 3.1, the uniqueness of the element in \( G \) can be expressed as \( a^i b^j \)

Which imply,

\[
b^{-1} ab = a^r, \quad b^s = a^t, \quad 1 \leq r, t \leq m \quad (1)
\]

We now write down 4 equivalent conditions for (1)

a) \((m, r) = 1, \quad m | r(r-1), \quad a | r, \quad r^s - 1 = 0 (\text{mod} \ m)\)

b) \( G \)'s derived group \( G' \) equal to \( < a^{r-1} > \), and \( |G'| = \frac{m}{d} \), with \( d = (r-1, m) \)

c) representation \( \rho^G_i(a)v_i = \zeta^i v_i \), \( \zeta \) is \( m \)'th unit origin root

d) \( \rho^G_i \) is irreducible \( \iff -r^j i = i (\text{mod} \ m) \)

Consequently,

\[
\Rightarrow r^s - 1 = t(r - 1) \Rightarrow r^{k+1} - 1 = t(r - 1) \quad (2)
\]

We compute its representation to get:

\[
b^{k+1} = a^t \Rightarrow b^{k+1} = a^{1 + r + r^2 + \cdots + r^k} \quad (3)
\]

\[
\rho^i = e^k \quad (4)
\]

Where,

\[
\Rightarrow n = 1 + r + r^2 + \cdots + r^k \quad (5)
\]

In addition, the results in section 3.1 deduce that:

\[
\sum \lambda_i a_i + \sum \mu_j b_j = 0 \Rightarrow \sum e^k = 0 \Rightarrow \sum a^k = 0 \quad (6)
\]

With, \( \exists \lambda_i, \mu_j \neq 0 \)
3.3 Finite group and infinite group

Theorem 3.3: (see section 9 in [8])

The section 6 in [8] also gives:

\[ \sum \lambda_k s_k = \lambda_1 s_1 + (1 - \lambda_1) \sum \frac{\lambda_k}{1 - \lambda_1} s_k \]  \hspace{1cm} (1)

From section 3.1 we can choose \( \mu_{k+1} \) to satisfy:

\[ \mu_{k+1} g = \sum \frac{\lambda_k}{1 - \lambda_1} g_k \Rightarrow \sum \lambda_k s_k = \lambda_1 \sum \frac{\lambda_k}{1 - \lambda_1} s_k = \lambda_{k+1} \sum a_k \rightarrow \varepsilon \sum a_k \]  \hspace{1cm} (2)

We now can write out our last estimate:

\[ (\phi \circ f)(x) \geq (\phi \circ f)(x) + \frac{C}{2} d^2 (x, x') \]  \hspace{1cm} (3)

\[ \phi(s - s') \geq \frac{C}{2L^2} \|s - s\|_\infty \Rightarrow \frac{C}{2L^2} \|s - s\|_\infty = \varepsilon \sum a_k \]  \hspace{1cm} (4)

3.4 Convex function and stripe space

Theorem 3.4.1: (see theorem 1.1 in [9])

Theorem 3.4.2: (see theorem 1.2 in [9])

Firstly, we write out the definition of stripe space:

\[ W_\varepsilon = \{ (p, u) : -\varepsilon f(p) \leq u \leq \varepsilon f(p) \} \]  \hspace{1cm} (1)

Apply the Steinhaus theorem to get:

\[ f(x) = m(A \cap (B + \{x\})) \]  \hspace{1cm} (2)

Therefore,

\[ \left| f(x) - f(y) \right| \leq \left| m(G_A \cap (G_B + \{x\})) - m(A \cap (B + \{x\})) \right| + \left| m(G_A \cap (G_B + \{y\})) - m(A \cap (B + \{y\})) \right| \]  \hspace{1cm} (3)

Relate section 3.3 with (3), we obtain:

\[ \Rightarrow \left\| f(s) - f(s') \right\| \leq 6L^2 \varepsilon \sum a_k \leq 6L^2 \varepsilon \sum a_k \]  \hspace{1cm} (4)

with, \( p = x - y \)
3.5 Fibre

The proof of theorem 3.4.1 allow us to define:

\[ \Psi = D \times _\Phi I \rightarrow B \times _f I \quad (1) \]

And we introduce the laplacian operator for cylindrical coordinate:

\[ \nabla ^2 \Phi = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \Phi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2} \quad (2) \]

3.6 Energy equation

The translation invariant of measure imply:

\[ - \varepsilon m(A \cap B) \leq u \leq \varepsilon m(A \cap B) \quad (1) \]

We now define a measure function on intersection:

\[ S_E (a, b) = \{(r \cos \theta, r \sin \theta)|a < r < b\} \quad (2) \]

Yields that:

\[ \Rightarrow \, m^*(S_E (a, b)) \leq \frac{1}{2} (b^2 - a^2)(b - a) \]

\[ \Rightarrow \, \rho (\sin \theta - \cos \theta) \leq \frac{1}{2} (b^2 - a^2)(b - a) \quad (3) \]

Let us combine (3) with the laplacian operator in section 3.5:

\[ \nabla ^2 \Phi = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \Phi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2} \]

\[ \frac{\sin \theta - \cos \theta}{\frac{1}{2} (b^2 - a^2)(b - a)} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \Phi}{\partial \rho} \right) + \left( \frac{\sin \theta - \cos \theta}{\frac{1}{2} (b^2 - a^2)(b - a)} \right)^2 \frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2} \quad (4) \]

At last we compute (4) and show our estimate for energy equation:

\[ b_{\pm} = \frac{\sinh(\sqrt{k} \varepsilon f(p)) \cosh(\sqrt{k} (h - t_{\pm}))}{\sqrt{\sinh^2(\sqrt{k} h) + \sinh^2(\sqrt{k} \varepsilon f(p)) \sinh^2(\sqrt{k} (h - t_{\pm}))}} \]

\[ = \frac{\sinh(\sqrt{k} u) \cosh(\sqrt{k} (h - t_{\pm}))}{\sqrt{\sinh^2(\sqrt{k} h) + \sinh^2(\sqrt{k} u) \sinh^2(\sqrt{k} (h - t_{\pm}))}} \]

\[ \leq \frac{\sinh(\sqrt{k} u) \cosh(\sqrt{k} (h - t_{\pm}))}{\sqrt{\sinh^2(\sqrt{k} u)(1 + \sinh^2(\sqrt{k} (h - t_{\pm}))}} = \frac{\cosh(\sqrt{k} (h - t_{\pm}))}{\sqrt{1 + \sinh^2(\sqrt{k} (h - t_{\pm})}} \quad (5) \]

By observation,
\[
\n\Rightarrow \nabla^2 \Phi \geq \frac{b}{2(b^2 - a^2)} \left( \frac{\partial}{\partial \rho} \left( \frac{\partial \Phi}{\partial \rho} \right) + \frac{b}{2} \frac{\partial^2 \Phi}{\partial \epsilon^2} \right)
\]

We select the parameter \[\frac{b}{2(b^2 - a^2)(b - a)}\] to ensure:

\[
\Rightarrow \nabla^2 \Phi \geq \frac{b}{2(b^2 - a^2)(b - a)} \left( \frac{\partial}{\partial \rho} \left( \frac{\partial \Phi}{\partial \rho} \right) \right) = \frac{b}{2(b^2 - a^2)(b - a)} (f' + pf'')
\]

Our last conclusion holds,

\[
\Rightarrow \lambda_i = \frac{f''(1 - C\rho)}{C\rho}
\]

In which,

\[
C = \frac{b}{2(b^2 - a^2)(b - a)}, \quad \sum \lambda_i = 1, \quad n = 1 + r + r^2 + \ldots + r^k
\]

Firstly, we study the representation of infinite group; then we use the steinhaus theorem to approximate the convex function in stripe space; at last, we introduce the laplacian operator and get a strong result for convexity.

4. Use random measure to handle convex differential function

Theorem 4: Let \(0 < \lambda < 1\) and let \(f, g, h\) be nonnegative integrable functions on \(R^n\) satisfying:

\[h((1 - \lambda)x + \lambda y) \geq f^{1-\lambda}g^\lambda\]

Then,

\[\int h \geq (\int f dx)^{1-\lambda}(\int g dx)^\lambda\]

The results for semigroup in [10] is:

\[
T_t \psi(x) = \int \psi e^{-t} x + \sqrt{1 - e^{-2t}} y \mu(dy)
\]

(1)

\[
\int x + th f(t) dt = t(f(x) - f(0)) + \frac{t^2}{2} T_x + \epsilon
\]

(2)

Next we introduce the random measure for this semi operator (see [11] [12] [13])

Firstly, we write \(\psi(e^{-t} x + \sqrt{1 - e^{-2t}} y)\) in dual form:

\[
\psi(e^{-t} x + \sqrt{1 - e^{-2t}} y) = f(x - y)g(y) \leq \frac{e^{-2t}(x - y)^2 + (\sqrt{1 - e^{-2t}} y + e^{-t} y)^2}{2}
\]

(3)
Theorem 4 gives:
\[
\int_x^{x+\varepsilon} f(t) dt = F + G 
\] (4)
\[
(\int h(x) dx)^2 \geq FG 
\] (5)

Substitute (4) and (5) into (2)

\[
t(f(x) - f(0)) + \frac{t^2}{2} T_x + \varepsilon \geq 2 \int h(x) dx
\] (6)

Apply the random measure to the support function, we have:

\[
\int_{R^d \setminus \partial A} f(x) d\mu = \sum_0^{d-1} w_{d-i} \int t^{d-1-i} f(x + tu) dt(\mu)
\] (7)

Nextly, we consider the convolution:

\[
\int \nu (e^{-t} x + \sqrt{1 - e^{-2t}} y) \mu (dy) = \sum_0^{d-1} w_{d-i} \int t^{d-1-i} f(x + tu) dt(\mu) \int \chi (x - s, y - t) ds dt =
\] (8)

Using (4) and (5) to compute (8)

\[
\int \nu (e^{-t} x + \sqrt{1 - e^{-2t}} y) \mu (dy) \leq \int \int \frac{e^{-2t} (x - y)^2 + (\sqrt{1 - e^{-2t}} y + e^{-t} y)^2}{2}
\] (9)

\[
\Rightarrow \exists \frac{t(f(x) - f(0)) + \frac{t^2}{2} T_x + \varepsilon}{2} \cdot \sum_0^{d-1} w_{d-i} \int t^{d-1-i} f(x + tu) dt(\mu)
\]

\[
= \int \frac{e^{-2t} (x - y)^2 + (\sqrt{1 - e^{-2t}} y + e^{-t} y)^2}{2}
\]

\[
\Rightarrow \frac{t^2}{4} FG \left( \sum_0^{d-1} w_{d-i} \int t^{d-1-i} f(x + tu) dt(\mu) \right)^2 = FG
\] (10)

Now the existence is clearly:

\[
\left( \sum_0^{d-1} \rho_{d-i} \int t^{d-1-i} f(tu) dt(\mu) \right)^2 = \frac{4}{t^2}
\]

\[
\Rightarrow \sum_0^{d-1} \rho_{d-i} \int t^{d-1-i} f(tu) dt(\mu) = \frac{2}{t}
\] (11)

with, \( t \to \varepsilon \)

Here we use random measure to divide the integral domain, and we separate the boundary, and then we apply convexity to approximate its boundary space and prove the existence of an equality for convex function.
Reference:

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