Numerical Solution of Time-Dependent Gravitational Schrödinger Equation

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In recent years, there are attempts to describe quantization of planetary distance based on time-independent gravitational Schrödinger equation, including Rubcic & Rubcic's method and also Nottale's Scale Relativity method. Nonetheless, there is no solution yet for time-dependent gravitational Schrödinger equation (TDGSE). In the present paper, a numerical solution of time-dependent gravitational Schrödinger equation is presented, apparently for the first time. This numerical solution leads to gravitational Bohr-radius, as expected. In the subsequent section, we also discuss plausible extension of this gravitational Schrödinger equation to include the effect of phion condensate via Gross-Pitaevskii equation, as described recently by Moffat. Alternatively one can consider this condensate from the viewpoint of Bogoliubov-deGennes theory, which can be approximated with coupled time-independent gravitational Schrödinger equation. Further observation is of course recommended in order to refute or verify this proposition.

1 Introduction

In the past few years, there have been some hypotheses suggesting that quantization of planetary distance can be derived from a gravitational Schrödinger equation, such as Rubcic & Rubcic and also Nottale's scale relativity method [1, 3]. Interestingly, the gravitational Bohr radius derived from this gravitational Schrödinger equation yields prediction of new type of astronomical observation in recent years, i.e. extra-solar planets, with unprecedented precision [2].

Furthermore, as we discuss in preceding paper [4], using similar assumption based on gravitational Bohr radius, one could predict new planetoids in the outer orbits of Pluto which are apparently in good agreement with recent observational finding.. Therefore one could induce from this observation that the gravitational Schrödinger equation (and gravitational Bohr radius) deserves further consideration.

In the meantime, it is known that all present theories discussing gravitational Schrödinger equation only take its time-independent limit. Therefore it seems worth to find out the solution and implication of time-dependent gravitational Schrödinger equation (TDGSE). This is what we will discuss in the present paper.

First we will find out numerical solution of time-independent gravitational Schrödinger equation which shall yield gravitational Bohr radius as expected [1, 2, 3]. Then we extend our discussion to the problem of time-dependent gravitational Schrödinger equation.

In the subsequent section, we also discuss plausible extension of this gravitational Schrödinger equation to include the effect of phion condensate via Gross-Pitaevskii equation, as described recently by Moffat [5]. Alternatively one can consider this phion condensate model from the viewpoint of Bogoliubov-deGennes theory, which can be approximated with coupled time-independent gravitational Schrödinger equation. To our knowledge this proposition of coupled time-independent gravitational Schrödinger equation has never been considered before elsewhere.

Further observation is of course recommended in order to verify or refute the propositions outlined herein.

All numerical computation was performed using Maple. Please note that in all conditions considered here, we use only gravitational Schrödinger equation as described in Rubcic & Rubcic [3], therefore we neglect the scale relativistic effect for clarity.

2 Numerical solution of time-independent gravitational Schrödinger equation and time-dependent gravitational Schrödinger equation

First we write down the time-independent gravitational Schrödinger radial wave equation in accordance with Rubcic & Rubcic [3]:

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \frac{8\pi m^2 E'}{H^2}R + \\ + \frac{2}{r}\frac{4\pi^2 GMm^2}{H^2}R - \frac{\ell(\ell+1)}{r^2}R = 0\,. \tag{1}$$

When H, V, E' represents gravitational Planck constant, Newtonian potential, and the energy per unit mass of the orbiting body, respectively, and [3]:

$$H = h\left(2\pi\, f\,\frac{Mm_n}{m_0^2}\right), \tag{2}$$

$$V(r) = -\frac{GMm}{r}\,, \tag{3}$$

$$E' = \frac{E}{m}\,. \tag{4}$$

By assuming that R takes the form:

$$R = e^{-\alpha\, r} \tag{5}$$

and substituting it into equation (1), and using simplified terms only of equation (1), one gets:

$$\Psi = \alpha^e\, e^{-\alpha\, r} - \frac{2\alpha\, e^{-\alpha\, r}}{r} + \frac{8\pi GMm^2\, e^{-\alpha\, r}}{r\, H^2}\,. \tag{6}$$

After factoring this equation (6) and solving it by equating the factor with zero, yields:

$$RR = -\frac{2\left(4\pi GMm^2 - H^2\alpha\right)}{\alpha^2 H^2} = 0\,, \tag{7}$$

or

$$RR = 4\pi GMm^2 - H^2\alpha = 0\,, \tag{8}$$

and solving for α, one gets:

$$a = \frac{4\pi^2 GMm^2}{H^2}\,. \tag{9}$$

Gravitational Bohr radius is defined as inverse of this solution of α, then one finds (in accordance with Rubcic & Rubcic [3]):

$$r_1 = \frac{H^2}{4\pi^2 GMm^2}\,, \tag{10}$$

and by substituting back equation (2) into (11), one gets [3]:

$$r_1 = \left(\frac{2\pi\, f}{\alpha c}\right)^2 GM\,, \tag{11}$$

which is equivalent with Nottale's result [1, 2], especially when we introduce the quantization number: $r_n = r_1 n^2$ [3]. For complete Maple session of these all steps, see Appendix 1.

Solution of time-dependent gravitational Schrödinger equation is more or less similar with the above steps, except that we shall take into consideration the right hand side of Schrödinger equation and also assuming time dependent form of r:

$$R = e^{-\alpha\, r(t)}\,. \tag{12}$$

Therefore the gravitational Schrödinger equation now reads:

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \frac{8\pi m^2 E'}{H^2}R + {} + \frac{2}{r}\,\frac{4\pi^2 GMm^2}{H^2}R - \frac{\ell\,(\ell+1)}{r^2}R = H\frac{dR}{dt}\,, \tag{13}$$

or by using Leibniz chain rule, we can rewrite equation (15) as:

$$-H\frac{dR}{dr\,(t)}\frac{dr\,(t)}{dt} + \frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \frac{8\pi m^2 E'}{H^2}R + {} + \frac{2}{r}\,\frac{4\pi^2 GMm^2}{H^2}R - \frac{\ell\,(\ell+1)}{r^2}R = 0\,. \tag{14}$$

The remaining steps are similar with the aforementioned procedures for time-independent case, except that now one gets an additional term for RR:

$$RR' = H^3\alpha\left(\frac{d}{dt}r\,(t)\right)r\,(t) - \alpha^2 r\,(t)H^2 + {} + 8\pi GMm^2 - 2H^2\alpha = 0\,. \tag{15}$$

At this point one shall assign a value for $\frac{d}{dt}r\,(t)$ term, because otherwise the equation cannot be solved. We choose $\frac{d}{dt}r\,(t) = 1$ for simplicity, then one gets solution for (17):

$$a2{:=}\left\langle \begin{matrix} \alpha=\alpha,\ \pi=\pi,\ m=m,\ H=H,\ G=G,\ M=M, \\ t=RootOf(r(\_Z)\alpha H^3 - r(\_Z)\alpha^2H^2 + 8\pi^2GMm^2 - 2\alpha H^2) \end{matrix}\right\rangle,$$
$$\{\alpha=0,\ t=t,\ m=m,\ H=H,\ G=G,\ M=M,\ \pi=0\},$$
$$\{\alpha=0,\ \pi=\pi,\ t=t,\ m=m,\ H=H,\ M=M,\ G=0\},$$
$$\left\{\pi=\pi,\ t=t,\ m=m,\ H=H,\ M=M,\ \alpha=H,\ G=\tfrac{H^3}{4\pi^2Mm^2}\right\},$$
$$\{\alpha=\alpha,\ H=0,\ \pi=\pi,\ t=t,\ m=m,\ M=M,\ G=G\},$$
$$\{\alpha=0,\ \pi=\pi,\ t=t,\ m=m,\ H=H,\ G=G,\ M=0\},$$
$$\{\alpha=0,\ \pi=\pi,\ t=t,\ H=H,\ G=G,\ M=M,\ m=0\},$$
$$\{\alpha=\alpha,\ H=0,\ \pi=\pi,\ t=t,\ m=m,\ G=G,\ M=0\},$$
$$\{\alpha=\alpha,\ H=0,\ \pi=\pi,\ t=t,\ G=G,\ M=M,\ m=0\}$$

Therefore one can conclude that there is time-dependent modification factor to conventional gravitational Bohr radius solution. For complete Maple session of these steps, see Appendix 2.

3 Gross-Pitaevskii effect. Bogoliubov-deGennes approximation and coupled time-independent gravitational Schrödinger equation

At this point it seems worthwhile to take into consideration a proposition by Moffat, regarding modification of Newtonian acceleration law due to phion condensate medium, to include Yukawa type potential [5, 6]:

$$a\,(r) = -\frac{G_\infty M}{r^2} + K\frac{\exp(-\mu_\phi r)}{r^2}(1+\mu_\phi r)\,. \tag{16}$$

Therefore equation (1) can be rewritten to become:

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \frac{8\pi m^2E'}{H^2}R + {} + \frac{2}{r}\,\frac{4\pi^2\big(GM - K\exp(-\mu_\phi r)(1+\mu_\phi r)\big)m^2}{H^2}R - {} - \frac{\ell\,(\ell+1)}{r^2}R = 0\,, \tag{17}$$

or by assuming $\mu = 2\mu_0 = \mu_0 r$ for the exponential term, equation (17) can be rewritten as:

$$\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}+\frac{8\pi m^2E'}{H^2}R+ \\ +\frac{2}{r}\frac{4\pi^2\left(GM-Ke^{-2\mu_0}(1+\mu_0 r)\right)m^2}{H^2}R- \\ -\frac{\ell(\ell+1)}{r^2}R=0\,. \tag{18}$$

Then instead of equation (8), one gets:

$$RR''=8\pi GMm^2-2H^2\alpha-8\pi^2m^2Ke^{-\mu_0}(1+\mu)=0. \tag{19}$$

Solving this equation will yield a modified gravitational Bohr radius which includes Yukawa effect:

$$r_1=\frac{H^2}{4\pi^2(GM-Ke^{-2\mu_0})m^2} \tag{20}$$

and the modification factor can be expressed as ratio between equation (20) and (11):

$$\chi=\frac{GM}{(GM-Ke^{-2\mu_0})}\,, \tag{21}$$

for complete Maple session of these steps, see Appendix 3.

A careful reader may note that this "Yukawa potential effect" as shown in equation (21) could be used to explain the small discrepancy (around ±8%) between the "observed distance" and the computed distance based on gravitational Bohr radius [4, 6a]. Nonetheless, in our opinion such an interpretation remains an open question, therefore it may be worth to explore further.

There is, however, an alternative way to consider phion condensate medium, i.e. by introducing coupled Schrödinger equation, which is known as Bogoliubov-deGennes theory [7]. This method can be interpreted also as generalisation of assumption by Rubcic-Rubcic [3] of subquantum structure composed of positive-negative Planck mass. Therefore, taking this proposition seriously, then one comes to hypothesis that there shall be coupled Newtonian potential, instead of only equation (3).

To simplify Bogoliubov-deGennes equation, we neglect the time-dependent case, therefore the wave equation can be written in matrix form [7, p. 4]:

$$\left[A\right]\left[\Psi\right]=0\,, \tag{22}$$

where $\left[A\right]$ is 2×2 matrix and $\left[\Psi\right]$ is 2×1 matrix, respectively, which can be represented as follows:

$$\left[A\right]=\begin{pmatrix} \frac{8\pi GMm^2e^{-\alpha r}}{r\hbar^2} & \alpha e^{-\alpha r}-\frac{2\alpha e^{-\alpha r}}{r} \\ \alpha e^{-\alpha r}-\frac{2\alpha e^{-\alpha r}}{r} & -\frac{8\pi GMm^2\,e^{-\alpha r}}{r\hbar^2} \end{pmatrix} \tag{23}$$

and

$$\left[\Psi\right]=\begin{pmatrix} f(r) \\ g(r) \end{pmatrix}. \tag{24}$$

Numerical solution of this matrix differential equation can be found in the same way with the previous methods, however we leave this problem as an exercise for the readers.

It is clear here, however, that Bogoliubov-deGennes approximation of gravitational Schrödinger equation, taking into consideration phion condensate medium will yield nonlinear effect, because it requires solution of matrix differential equation* (22) rather than standard ODE in conventional Schrödinger equation. This perhaps may explain complicated structure beyond Jovian Planets, such as Kuiper Belt, inner and outer Oort Cloud etc. which of course these structure cannot be predicted by simple gravitational Schrödinger equation [1, 2, 3]. In turn, from the solution of (22) one could expect that there are multitude of celestial objects not found yet in the Oort Cloud.

Further observation is also recommended in order to verify and explore further this proposition.

4 Concluding remarks

In the present paper, a numerical solution of time-dependent gravitational Schrödinger equation is presented, apparently for the first time. This numerical solution leads to gravitational Bohr-radius, as expected.

In the subsequent section, we also discuss plausible extension of this gravitational Schrödinger equation to include the effect of phion condensate via Gross-Pitaevskii equation, as described recently by Moffat. Alternatively one can consider this condensate from the viewpoint of Bogoliubov-deGennes theory, which can be approximated with coupled time-independent gravitational Schrödinger equation.

It is recommended to conduct further observation in order to verify and also to explore various implications of our propositions as described herein.

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*For recent articles discussing analytical solution of matrix differential equations, the reader is referred to Electronic Journal of Differential Equations (free access online on many mirrors as http://ejde.math.txstate.edu, http://ejde.math.unt.edu, http://www.emis.de/journals/EJDE etc.).

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Appendix 1 Time-independent gravitational Schrödinger equation

> restart;
> with (linalg);
> R: = exp (−(alpha*r));

$$R := e^{-\alpha r}$$

> D1R:=diff (R,r); D2R:=diff (D1R,r);

$$D1R := -\alpha\, e^{-\alpha r}$$
$$D2R := -\alpha^2\, e^{-\alpha r}$$

> SCHEQ1:=D2R+D1R*2/r+8*piˆ2*m*E*R/hˆ2+8*piˆ2*G*M*mˆ2*R/(r*hˆ2)−l*(l+1)*R/rˆ2=0;
> XX1:=factor (SCHEQ1);
> #Using simplified terms only from equation (A*8, of Rubcic & Rubcic, 1998)
> ODESCHEQ:=D2R+D1R*2/r+8*piˆ2*G*M*mˆ2*R/(r*hˆ2)=0;

$$ODESCHEQ := \alpha^2\, e^{-\alpha r} - \frac{2\alpha\, e^{-\alpha r}}{r} + \frac{8\pi^2 GMm^2 e^{-\alpha r}}{rH^2} = 0$$

> XX2:=factor (SCHEQ2);

$$XX2 := \frac{e^{-\alpha r}\left(\alpha^2 rH^2 - 2H^2\alpha + 8\pi^2 GMm^2\right)}{rH^2} = 0$$

> RR:= solve (XX2, r);

$$RR := -\frac{2(4\pi^2 GMm^2 - H^2\alpha)}{\alpha^2 H^2}$$

> #Then solving for RR=0, yields:
> SCHEQ3:= 4*piˆ2*G*M*mˆ2−hˆ2*alpha=0;

$$SCHEQ3 := 4\pi^2 GMm^2 - H^2\alpha = 0$$

> a:= solve (SCHEQ3, alpha);

$$a := \frac{4\pi^2 GMm^2}{H^2}$$

> #Gravitational Bohr radius is defined as inverse of alpha:
> gravBohrradius:=1/a;

$$r_{gravBohr} := \frac{H^2}{4\pi^2 GMm^2}$$

Appendix 2 Time-dependent gravitational Schrödinger equation

> #Solution of gravitational Schrodinger equation (Rubcic, Fizika 1998);
> restart;
> #with time evolution (Hagendorn's paper);
> S:=r (t); R:=exp (−(alpha*S)); R1:=exp (−(alpha*r));

$$S := r(t)$$
$$R := e^{-\alpha r}$$

> D4R:= diff (S,t); D1R:=−alpha*exp (−(alpha*S)); D2R:=−alphaˆ2*exp (−(alpha*S)); D5R:=D1R*D4R;

$$D4R := \frac{d}{dt}\, r\,(t)$$
$$D1R := -\alpha\, e^{-\alpha\, r(t)}$$
$$D2R := -\alpha^2\, e^{-\alpha\, r(t)}$$
$$D1R := -\alpha\, e^{-\alpha\, r(t)}\,\frac{d}{dt}\, r\,(t)$$

> #Using simplified terms only from equation (A*8)
> SCHEQ3:= −h*D5R+D2R+D1R*2/S+8*piˆ2*G*M*mˆ2*R/(S*hˆ2);
> XX2:=factor (SCHEQ3);

$$XX2 := \frac{e^{-\alpha r(t)}\left(H^3\alpha\frac{dr(t)}{dt}r(t) - \alpha^2 r(t)H^2 - 2H^2\alpha + 8\pi^2 GMm^2\right)}{r\,(t)H^2} = 0$$

> #From standard solution of gravitational Schrodinger equation, we know (Rubcic, Fizika 1998):
> SCHEQ4:= 4*piˆ2*G*M*mˆ2−hˆ2*alpha;

$$SCHEQ4 := 4\pi^2 GMm^2 - H^2\alpha$$

> #Therefore time-dependent solution of Schrodinger equation may introduce new term to this gravitational Bohr radius.
> SCHEQ5:=(XX2*(S*hˆ2)/(exp (−(alpha*S))))−2*SCHEQ4;

$$ODESCHEQ5 := H^3\alpha\frac{dr(t)}{dt}r(t) - \alpha^2 r(t)H^2$$

> #Then we shall assume for simplicity by assigning value to d[r (t)]/dt:
> D4R:=1;
> Then we can solve again SCHEQ5 similar to solution of SCHEQ4:
> a2:=solve((hˆ3*alpha*(D4R)*S−alphaˆ2*S*hˆ2)+2*SCHEQ4);

$$a2 := \left\langle \begin{matrix} \alpha=\alpha,\ \pi=\pi,\ m=m,\ H=H,\ G=G,\ M=M, \\ t = RootOf(r(\_Z)\alpha H^3 - r(\_Z)\alpha^2 H^2 + 8\pi^2 GMm^2 - 2\alpha H^2) \end{matrix} \right\rangle,$$
$$\{\alpha=0,\ t=t,\ m=m,\ H=H,\ G=G,\ M=M,\ \pi=0\},$$
$$\{\alpha=0,\ \pi=\pi,\ t=t,\ m=m,\ H=H,\ M=M,\ G=0\},$$
$$\left\{\pi=\pi,\ t=t,\ m=m,\ H=H,\ M=M,\ \alpha=H,\ G=\frac{H^3}{4\pi^2 Mm^2}\right\},$$
$$\{\alpha=\alpha,\ H=0,\ \pi=\pi,\ t=t,\ m=m,\ M=M,\ G=G\},$$
$$\{\alpha=0,\ \pi=\pi,\ t=t,\ m=m,\ H=H,\ G=G,\ M=0\},$$
$$\{\alpha=0,\ \pi=\pi,\ t=t,\ H=H,\ G=G,\ M=M,\ m=0\},$$
$$\{\alpha=\alpha,\ H=0,\ \pi=\pi,\ t=t,\ m=m,\ G=G,\ M=0\},$$
$$\{\alpha=\alpha,\ H=0,\ \pi=\pi,\ t=t,\ G=G,\ M=M,\ m=0\}$$

> #Therefore one could expect that there is time-dependent change of gravitational Bohr radius.

Appendix 3 Time-independent gravitational Schrödinger equation with Yukawa potential [5]

> #Extension of gravitational Schrodinger equation (Rubcic, Fizika 1998);
> restart;
> #departure from Newton potential;
> R:=exp (−(alpha*r));

$$R := e^{-\alpha r}$$

> D1R:= diff (R,r); D2R:= diff (D1R,r);

$$D1R := -\alpha\, e^{-\alpha r}$$
$$D2R := -\alpha^2\, e^{-\alpha r}$$

> SCHEQ2:=D2R+D1R*2/r+8*piˆ2*(G*M−K*exp(−2*mu)*(1+mu*r))*mˆ2*R/(r*hˆ2)=0;

$$ODESCHEQ := \alpha^2 e^{-\alpha r} - \frac{2\alpha e^{-\alpha r}}{r} + \\ + \frac{8\pi^2 (GM - Ke^{-2\mu}(1+\mu r)) m^2 e^{-\alpha r}}{rH^2} = 0$$

> XX2:=factor(SCHEQ2);
> RR1:=solve(XX2,r);

$$RR1 := -\frac{2(-H^2\alpha + 4\pi^2 GMm^2 - 4\pi^2 m^2 Ke^{-2\mu})}{-\alpha^2 H^2 + 8\pi^2 m^2 Ke^{-2\mu}}$$

> #from standard gravitational Schrodinger equation we know:
> SCHEQ3:=4*piˆ2*G*M*mˆ2−hˆ2*alpha=0;
> a:=solve(SCHEQ3, alpha);
> #Gravitational Bohr radius is defined as inverse of alpha:
> gravBohrradius:=1/a;

$$r_{gravBohr} := \frac{H^2}{4\pi^2 GMm^2}$$

> #Therefore we conclude that the new terms of RR shall yield new terms (YY) into this gravitational Bohr radius:
> PI:=(RR*(alphaˆ2*hˆ2)−(−8*piˆ2*G*M*mˆ2+2*hˆ2*alpha));
> #This new term induced by pion condensation via Gross-Pitaevskii equation may be observed in the form of long-range potential effect. (see Moffat J., arXiv: astro-ph/0602607, 2006; also Smarandache F. and Christianto V. Progress in Physics, v. 2, 2006, & v. 1, 2007, www.ptep-online.com)
> #We can also solve directly:
> SCHEQ5:=RR*(alphaˆ2*hˆ2)/2;

$$SCHEQ5 := \frac{\alpha^2 H^2(-H^2\alpha + 4\pi^2 GMm^2 - 4\pi^2 m^2 Ke^{-2\mu})}{-\alpha^2 H^2 + 8\pi^2 m^2 Ke^{-2\mu}}$$

> a1:=solve(SCHEQ5, alpha);

$$a1 := 0, 0, \frac{4\pi^2 m^2 (GM - Ke^{-2\mu})}{H^2}$$

> #Then one finds modified gravitational Bohr radius in the form:
> modifgravBohrradius:=1/(4*piˆ2*(G*M−K*exp(−2*mu))*mˆ2/hˆ2);

$$r_{modified.gravBohr} := \frac{H^2}{4\pi^2 m^2 (GM - Ke^{-2\mu})}$$

> #This modification can be expressed in chi-factor:
> chi:=modifgravBohrradius/gravBohrradius;

$$\chi := \frac{GM}{GM - Ke^{-2\mu}}$$