S-Denying of the Signature Conditions Expands General Relativity's Space

Dmitri Rabounski, Florentin Smarandache, Larissa Borissova

Department of Mathematics, University of New Mexico, Gallup, NM 87301, USA

E-mail: rabounski@yahoo.com; fsmarandache@yahoo.com; lborissova@yahoo.com

We apply the S-denying procedure to signature conditions in a four-dimensional pseudo-Riemannian space — i. e. we change one (or even all) of the conditions to be partially true and partially false. We obtain five kinds of expanded space-time for General Relativity. Kind I permits the space-time to be in collapse. Kind II permits the space-time to change its own signature. Kind III has peculiarities, linked to the third signature condition. Kind IV permits regions where the metric fully degenerates: there may be non-quantum teleportation, and a home for virtual photons. Kind V is common for kinds I, II, III, and IV.

1 Einstein's basic space-time

Euclidean geometry is set up by Euclid's axioms: (1) given two points there is an interval that joins them; (2) an interval can be prolonged indefinitely; (3) a circle can be constructed when its centre, and a point on it, are given; (4) all right angles are equal; (5) if a straight line falling on two straight lines makes the interior angles on one side less than two right angles, the two straight lines, if produced indefinitely, meet on that side. Non-Euclidean geometries are derived from making assumptions which deny some of the Euclidean axioms. Three main kinds of non-Euclidean geometry are conceivable — Lobachevsky-Bolyai-Gauss geometry, Riemann geometry, and Smarandache geometry.

In Lobachevsky-Bolyai-Gauss (hyperbolic) geometry the fifth axiom is denied in the sense that there are infinitely many lines passing through a given point and parallel to a given line. In Riemann (elliptic) geometry*, the axiom is satisfied formally, because there is no line passing through a given point and parallel to a given line. But if we state the axiom in a broader form, such as "through a point not on a given line there is only one line parallel to the given line", the axiom is also denied in Riemann geometry. Besides that, the second axiom is also denied in Riemann geometry, because herein the straight lines are closed: an infinitely long straight line is possible but then all other straight lines are of the same infinite length.

In Smarandache geometry one (or even all) of the axioms is false in at least two different ways, or is false and also true [1, 2]. This axiom is said to be Smarandachely denied (S-denied). Such geometries have mixed properties of Euclidean, Lobachevsky-Bolyai-Gauss, and Riemann geometry. Manifolds that support such geometries were introduced by Iseri [3].

Riemannian geometry is the generalization of Riemann geometry, so that in a space of Riemannian geometry:

(1) The differentiable field of a 2nd rank non-degenerate symmetric tensor $g_{\alpha\beta}$ is given so that the distance ds between any two infinitesimally close points is given by the quadratic form

$$ds^2 = \sum_{0 \leqslant \alpha, \beta \leqslant n} g_{\alpha\beta}(x)\, dx^\alpha dx^\beta = g_{\alpha\beta}\, dx^\alpha dx^\beta,$$

known as the Riemann metric†. The tensor $g_{\alpha\beta}$ is called the fundamental metric tensor, and its components define the geometrical structure of the space;

(2) The space curvature may take different numerical values at different points in the space.

Actually, a Riemann geometry space is the space of the Riemannian geometry family, where the curvature is constant and has positive numerical value.

In the particular case where $g_{\alpha\beta}$ takes the diagonal form

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix},$$

the Riemannian space becomes Euclidean.

Pseudo-Riemannian spaces consist of specific kinds of Riemannian spaces, where $g_{\alpha\beta}$ (and the Riemannian metric ds^2) has sign-alternating form so that its diagonal components bear numerical values of opposite sign.

Einstein's basic space-time of General Relativity is a four-dimensional pseudo-Riemannian space having the sign-alternating signature $(+---)$ or $(-+++)$, which reserves one dimension for time $x^0 = ct$ whilst the remaining three are reserved for three-dimensional space, so that the space metric is‡

$$ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta = g_{00}\, c^2 dt^2 + 2g_{0i}\, c\, dt\, dx^i + g_{ik}\, dx^i dx^k.$$

* *Elleipein* — "to fall short"; *hyperballein* — "to throw beyond" (Greek).

† Here is a space of n dimensions.

‡ Landau and Lifshitz in *The Classical Theory of Fields* [4] use the signature $(-+++)$, where the three-dimensional part of the four-dimensional impulse vector is real. We, following Eddington [5], use the signature $(+---)$, because in this case the three-dimensional *observable impulse*, being the projection of the four-dimensional impulse vector on an observer's spatial section, is real. Here $\alpha, \beta = 0, 1, 2, 3$, while $i, k = 1, 2, 3$.

In general the four-dimensional pseudo-Riemannian space is curved, inhomogeneous, gravitating, rotating, and deforming (any or all of the properties may be anisotropic). In the particular case where the fundamental metric tensor $g_{\alpha\beta}$ takes the strictly diagonal form

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

the space becomes four-dimensional pseudo-Euclidean

$$ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta = c^2 dt^2 - dx^2 - dy^2 - dz^2,$$

which is known as Minkowski's space (he had introduced it first). It is the basic space-time of Special Relativity.

2 S-denying the signature conditions

In a four-dimensional pseudo-Riemannian space of signature $(+---)$ or $(-+++)$, the basic space-time of General Relativity, there are *four signature conditions* which define this space as pseudo-Riemannian.

Question: What happens if we S-deny one (or even all) of the four signature conditions in the basic space-time of General Relativity? What happens if we postulate that one (or all) of the signature conditions is to be denied in two ways, or, alternatively, to be true and false?

Answer: If we S-deny one or all of the four signature conditions in the basic space-time, we obtain a new expanded basic space-time for General Relativity. There are five main kinds of such expanded spaces, due to four possible signature conditions there.

Here we are going to consider each of the five kinds of expanded spaces.

Starting from a purely mathematical viewpoint, the signature conditions are derived from sign-alternation in the diagonal terms g_{00}, g_{11}, g_{22}, g_{33} in the matrix $g_{\alpha\beta}$. From a physical perspective, see §84 in [4], the signature conditions are derived from the requirement that the three-dimensional observable interval

$$d\sigma^2 = h_{ik}\, dx^i dx^k = \left(-g_{ik} + \frac{g_{0i}\, g_{0k}}{g_{00}}\right) dx^i dx^k$$

must be positive. Hence the three-dimensional observable metric tensor $h_{ik} = -g_{ik} + \frac{g_{0i}\, g_{0k}}{g_{00}}$, being a 3×3 matrix defined in an observer's reference frame accompanying its references, must satisfy three obvious conditions

$$\det\|h_{11}\| = h_{11} > 0\,,$$

$$\det\begin{Vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{Vmatrix} = h_{11}\, h_{22} - h_{12}^2 > 0\,,$$

$$h = \det\|h_{ik}\| = \det\begin{Vmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{Vmatrix} > 0\,.$$

From here we obtain the signature conditions in the fundamental metric tensor's matrix $g_{\alpha\beta}$. In a space of signature $(+---)$, the signature conditions are

$$\det\|g_{00}\| = g_{00} > 0\,, \qquad \text{(I)}$$

$$\det\begin{Vmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{Vmatrix} = g_{00}\, g_{11} - g_{01}^2 < 0\,, \qquad \text{(II)}$$

$$\det\begin{Vmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{Vmatrix} > 0\,, \qquad \text{(III)}$$

$$g = \det\|g_{\alpha\beta}\| = \det\begin{Vmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{Vmatrix} < 0\,. \qquad \text{(IV)}$$

An expanded space-time of kind I: In such a space-time the first signature condition $g_{00} > 0$ is S-denied, while the other signature conditions remain unchanged. Given the expanded space-time of kind I, the first signature condition is S-denied in the following form

$$\det\|g_{00}\| = g_{00} \geqslant 0\,,$$

which includes two particular cases, $g_{00} > 0$ and $g_{00} = 0$, so $g_{00} > 0$ is partially true and partially false.

Gravitational potential is $\mathrm{w} = c^2(1 - \sqrt{g_{00}})$ [6, 7], so the S-denied first signature condition $g_{00} \geqslant 0$ means that in such a space-time $\mathrm{w} \leqslant c^2$, i. e. two different states occur

$$\mathrm{w} < c^2\,, \qquad \mathrm{w} = c^2.$$

The first one corresponds to the regular space-time, where $g_{00} > 0$. The second corresponds to a special space-time state, where the first signature condition is simply denied $g_{00} = 0$. This is the well-known condition of gravitational collapse.

Landau and Lifshitz wrote, "nonfulfilling of the condition $g_{00} > 0$ would only mean that the corresponding system of reference cannot be accomplished with real bodies" [4].

Conclusion on the kind I: An expanded space-time of kind I ($g_{00} \geqslant 0$) is the generalization of the basic space-time of General Relativity ($g_{00} > 0$), including regions where this space-time is in a state of collapse, ($g_{00} = 0$).

An expanded space-time of kind II: In such a space-time the second signature condition $g_{00}\, g_{11} - g_{01}^2 < 0$ is S-denied, the other signature conditions remain unchanged. Thus, given the expanded space-time of kind II, the second signature condition is S-denied in the following form

$$\det\begin{Vmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{Vmatrix} = g_{00}\, g_{11} - g_{01}^2 \leqslant 0\,,$$

which includes two different cases

$$g_{00}\,g_{11} - g_{01}^2 < 0\,, \qquad g_{00}\,g_{11} - g_{01}^2 = 0\,,$$

whence the second signature condition $g_{00}\,g_{11} - g_{01}^2 < 0$ is partially true and partially false.

The component g_{00} is defined by the gravitational potential $\mathrm{w} = c^2(1 - \sqrt{g_{00}})$. The component g_{0i} is defined by the space rotation linear velocity (see [6, 7] for details)

$$v_i = -c\frac{g_{0i}}{\sqrt{g_{00}}}\,, \quad v^i = -c\,g^{0i}\sqrt{g_{00}}\,, \quad v_i = h_{ik}\,v^k.$$

Then we obtain the S-denied second signature condition $g_{00}\,g_{11} - g_{01}^2 \leqslant 0$ (meaning the first signature condition is not denied $g_{00} > 0$) as follows

$$g_{11} - \frac{1}{c^2}\,v_1^2 \leqslant 0\,,$$

having two particular cases

$$g_{11} - \frac{1}{c^2}\,v_1^2 < 0\,, \qquad g_{11} - \frac{1}{c^2}\,v_1^2 = 0\,.$$

To better see the physical sense, take a case where g_{11} is close to -1.* Then, denoting $v^1 = v$, we obtain

$$v^2 > -c^2, \qquad v^2 = -c^2.$$

The first condition $v^2 > -c^2$ is true in the regular basic space-time. Because the velocities v and c take positive numerical values, this condition uses the well-known fact that positive numbers are greater than negative ones.

The second condition $v^2 = -c^2$ has no place in the basic space-time; it is true as a particular case of the common condition $v^2 \geqslant -c^2$ in the expanded spaces of kind II. This condition means that as soon as the linear velocity of the space rotation reaches light velocity, the space signature changes from $(+---)$ to $(-+++)$. That is, given an expanded space-time of kind II, the transit from a non-isotropic sub-light region into an isotropic light-like region implies change of signs in the space signature.

Conclusion on the kind II: An expanded space-time of kind II ($v^2 \geqslant -c^2$) is the generalization of the basic space-time of General Relativity ($v^2 > -c^2$) which permits the peculiarity that the space-time changes signs in its own signature as soon as we, reaching the light velocity of the space rotation, encounter a light-like isotropic region.

An expanded space-time of kind III: In this space-time the third signature condition is S-denied, the other signature conditions remain unchanged. So, given the expanded space-time of kind III, the third signature condition is

$$\det\begin{Vmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{Vmatrix} \geqslant 0\,,$$

*Because we use the signature $(+---)$.

which, taking the other form of the third signature condition into account, can be transformed into the formula

$$\det\begin{Vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{Vmatrix} = h_{11}\,h_{22} - h_{12}^2 \geqslant 0\,,$$

that includes two different cases

$$h_{11}\,h_{22} - h_{12}^2 > 0\,, \qquad h_{11}\,h_{22} - h_{12}^2 = 0\,,$$

so that the third initial signature condition $h_{11}\,h_{22} - h_{12}^2 > 0$ is partially true and partially false. This condition is not clear. Future research is required.

An expanded space-time of kind IV: In this space-time the fourth signature condition $g = \det\|\,g_{\alpha\beta}\| < 0$ is S-denied, the other signature conditions remain unchanged. So, given the expanded space-time of kind IV, the fourth signature condition is

$$g = \det\|\,g_{\alpha\beta}\| = \det\begin{Vmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{Vmatrix} \leqslant 0\,,$$

that includes two different cases

$$g = \det\|\,g_{\alpha\beta}\| < 0\,, \qquad g = \det\|\,g_{\alpha\beta}\| = 0\,,$$

so that the fourth signature condition $g < 0$ is partially true and partially false: $g < 0$ is true in the basic space-time, $g = 0$ could be true in only he expanded spaces of kind IV.

Because the determinants of the fundamental metric tensor $g_{\alpha\beta}$ and the observable metric tensor h_{ik} are connected as follows $\sqrt{-g} = \sqrt{h}\,\sqrt{g_{00}}$ [6, 7], degeneration of the fundamental metric tensor ($g = 0$) implies that the observable metric tensor is also degenerate ($h = 0$). In such fully degenerate areas the space-time interval ds^2, the observable spatial interval $d\sigma^2 = h_{ik}\,dx^i dx^k$ and the observable time interval $d\tau$ become zero†

$$ds^2 = c^2 d\tau^2 - d\sigma^2 = 0\,, \qquad c^2 d\tau^2 = d\sigma^2 = 0\,.$$

Taking formulae for $d\tau$ and $d\sigma$ into account, and also the fact that in the accompanying reference frame we have $h_{00} = h_{0i} = 0$, we write $d\tau^2 = 0$ and $d\sigma^2 = 0$ as

$$d\tau = \left[1 - \frac{1}{c^2}\left(\mathrm{w} + v_i u^i\right)\right] dt = 0\,, \qquad dt \neq 0\,,$$

$$d\sigma^2 = h_{ik}\,dx^i dx^k = 0\,,$$

where the three-dimensional coordinate velocity $u^i = dx^i/dt$ is different to the observable velocity $\mathrm{v}^i = dx^i/d\tau$.

†Note, $ds^2 = 0$ is true not only at $c^2 d\tau^2 = d\sigma^2 = 0$, but also when $c^2 d\tau^2 = d\sigma^2 \neq 0$ (in the isotropic region, where light propagates). The properly observed time interval is determined as $d\tau = \sqrt{g_{00}}\,dt + \frac{g_{0i}}{c\sqrt{g_{00}}}\,dx^i$, where the coordinate time interval is $dt \neq 0$ [4, 5, 6, 7].

With $h_{ik} = -g_{ik} + \frac{1}{c^2} v_i v_k$, we obtain aforementioned physical conditions of degeneration in the final form

$$\mathrm{w} + v_i u^i = c^2, \qquad g_{ik} u^i u^k = c^2 \left(1 - \frac{\mathrm{w}}{c^2}\right)^2.$$

As recently shown [8, 9], the degenerate conditions permit non-quantum teleportation and also virtual photons in General Relativity. Therefore we expect that, employing an expanded space of kind IV, one may join General Relativity and Quantum Electrodynamics.

Conclusion on the kind IV: An expanded space-time of kind IV ($g \leqslant 0$) is the generalization of the basic space-time of General Relativity ($g < 0$) including regions where this space-time is in a fully degenerate state ($g = 0$). From the viewpoint of a regular observer, in a fully degenerate area time intervals between any events are zero, and spatial intervals are zero. Thus, such a region is observable as a point.

An expanded space-time of kind V: In this space-time all four signature conditions are S-denied, therefore given the expanded space-time of kind V the signature conditions are

$$\det \| g_{00} \| = g_{00} \geqslant 0\,,$$

$$\det \begin{Vmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{Vmatrix} = g_{00}\, g_{11} - g_{01}^2 \leqslant 0\,,$$

$$\det \begin{Vmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{Vmatrix} \geqslant 0\,,$$

$$g = \det \| g_{\alpha\beta} \| = \det \begin{Vmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{Vmatrix} \leqslant 0\,,$$

so all four signature conditions are partially true and partially false. It is obvious that an expanded space of kind V contains expanded spaces of kind I, II, III, and IV as particular cases, it being a common space for all of them.

Negative S-denying expanded spaces: We could also S-deny the signatures with the possibility that say $g_{00} > 0$ for kind I, but this means that the gravitational potential would be imaginary, or, even take into account the "negative" cases for kind II, III, etc. But most of them are senseless from the geometrical viewpoint. Hence we have only included five main kinds in our discussion.

3 Classification of the expanded spaces for General Relativity

In closing this paper we repeat, in brief, the main results.

There are currently three main kinds of non-Euclidean geometry conceivable — Lobachevsky-Bolyai-Gauss geometry, Riemann geometry, and Smarandache geometries. A four-dimensional pseudo-Riemannian space, a space of the Riemannian geometry family, is the basic space-time of General Relativity. We employed S-denying of the signature conditions in the basic four-dimensional pseudo-Riemannian space, when a signature condition is partially true and partially false. S-denying each of the signature conditions (or even all the conditions at once) gave an expanded space for General Relativity, which, being an instance of the family of Smarandache spaces, include the pseudo-Riemannian space as a particular case. There are four signature conditions. So, we obtained five kinds of the expanded spaces for General Relativity:

Kind I Permits the space-time to be in collapse;

Kind II Permits the space-time to change its own signature as reaching the light speed of the space rotation in a light-like isotropic region;

Kind III Has some specific peculiarities (not clear yet), linked to the third signature condition;

Kind IV Permits full degeneration of the metric, when all degenerate regions become points. Such fully degenerate regions provide trajectories for non-quantum teleportation, and are also a home space for virtual photons.

Kind V Provides an expanded space, which has common properties of all spaces of kinds I, II, III, and IV, and includes the spaces as particular cases.

The foregoing results are represented in detail in the book [10], which is currently in print.

4 Extending this classification: mixed kinds of the expanded spaces

We can S-deny one axiom only, or two axioms, or three axioms, or even four axioms simultaneously. Hence we may have: $C_4^1 + C_4^2 + C_4^3 + C_4^4 = 2^4 - 1 = 15$ kinds of expanded spaces for General Relativity, where C_n^i denotes combinations of n elements taken in groups of i elements, $0 \leqslant i \leqslant n$. And considering the fact that each axiom can be S-denied in three different ways, we obtain $15 \times 3 = 45$ kinds of expanded spaces for General Relativity. Which expanded space would be most interesting?

We collect all such "mixed" spaces into a table. Specific properties of the mixed spaces follow below.

1.1.1: $g_{00} \geqslant 0$, $h_{11} > 0$, $h_{11}h_{22} - h_{12}^2 > 0$, $h > 0$. At $g_{00} = 0$, we have the usual space-time permitting collapse.

1.1.2: $g_{00} > 0$, $h_{11} \geqslant 0$, $h_{11}h_{22} - h_{12}^2 > 0$, $h > 0$. At $h_{11} = 0$ we have $h_{12}^2 < 0$ that is permitted for imaginary values of h_{12}: we obtain a complex Riemannian space.

1.1.3: $g_{00} > 0$, $h_{11} > 0$, $h_{11}h_{22} - h_{12}^2 \geqslant 0$, $h > 0$. At $h_{11}h_{22} - h_{12}^2 = 0$, the spatially observable metric $d\sigma^2$ permits purely spatial isotropic lines.

1.1.4: $g_{00} > 0$, $h_{11} > 0$, $h_{11}h_{22} - h_{12}^2 > 0$, $h \geqslant 0$. At $h = 0$, we have the spatially observed metric $d\sigma^2$ completely degenerate. An example — zero-space [9], obtained as a completely degenerate Riemannian space. Because $h = -\frac{g}{g_{00}}$, the

Positive S-denying spaces, $N \geqslant 0$		Negative S-denying spaces, $N \leqslant 0$		S-denying spaces, where $N > 0 \cup N < 0$	
Kind	Signature conditions	Kind	Signature conditions	Kind	Signature conditions
One of the signature conditions is S-denied					
1.1.1	$\text{I} \geqslant 0,\ \text{II} > 0,\ \text{III} > 0,\ \text{IV} > 0$	1.2.1	$\text{I} \leqslant 0,\ \text{II} > 0,\ \text{III} > 0,\ \text{IV} > 0$	1.3.1	$\text{I} \gtrless 0,\ \text{II} > 0,\ \text{III} > 0,\ \text{IV} > 0$
1.1.2	$\text{I} > 0,\ \text{II} \geqslant 0,\ \text{III} > 0,\ \text{IV} > 0$	1.2.2	$\text{I} > 0,\ \text{II} \leqslant 0,\ \text{III} > 0,\ \text{IV} > 0$	1.3.2	$\text{I} > 0,\ \text{II} \gtrless 0,\ \text{III} > 0,\ \text{IV} > 0$
1.1.3	$\text{I} > 0,\ \text{II} > 0,\ \text{III} \geqslant 0,\ \text{IV} > 0$	1.2.3	$\text{I} > 0,\ \text{II} > 0,\ \text{III} \leqslant 0,\ \text{IV} > 0$	1.3.3	$\text{I} > 0,\ \text{II} > 0,\ \text{III} \gtrless 0,\ \text{IV} > 0$
1.1.4	$\text{I} > 0,\ \text{II} > 0,\ \text{III} > 0,\ \text{IV} \geqslant 0$	1.2.4	$\text{I} > 0,\ \text{II} > 0,\ \text{III} > 0,\ \text{IV} \leqslant 0$	1.3.4	$\text{I} > 0,\ \text{II} > 0,\ \text{III} > 0,\ \text{IV} \gtrless 0$
Two of the signature conditions are S-denied					
2.1.1	$\text{I} \geqslant 0,\ \text{II} \geqslant 0,\ \text{III} > 0,\ \text{IV} > 0$	2.2.1	$\text{I} \leqslant 0,\ \text{II} \leqslant 0,\ \text{III} > 0,\ \text{IV} > 0$	2.3.1	$\text{I} \gtrless 0,\ \text{II} \gtrless 0,\ \text{III} > 0,\ \text{IV} > 0$
2.1.2	$\text{I} \geqslant 0,\ \text{II} > 0,\ \text{III} \geqslant 0,\ \text{IV} > 0$	2.2.2	$\text{I} \leqslant 0,\ \text{II} > 0,\ \text{III} \leqslant 0,\ \text{IV} > 0$	2.3.2	$\text{I} \gtrless 0,\ \text{II} > 0,\ \text{III} \gtrless 0,\ \text{IV} > 0$
2.1.3	$\text{I} \geqslant 0,\ \text{II} > 0,\ \text{III} > 0,\ \text{IV} \geqslant 0$	2.2.3	$\text{I} \leqslant 0,\ \text{II} > 0,\ \text{III} > 0,\ \text{IV} \leqslant 0$	2.3.3	$\text{I} \gtrless 0,\ \text{II} > 0,\ \text{III} > 0,\ \text{IV} \gtrless 0$
2.1.4	$\text{I} > 0,\ \text{II} \geqslant 0,\ \text{III} > 0,\ \text{IV} \geqslant 0$	2.2.4	$\text{I} > 0,\ \text{II} \leqslant 0,\ \text{III} > 0,\ \text{IV} \leqslant 0$	2.3.4	$\text{I} > 0,\ \text{II} \gtrless 0,\ \text{III} > 0,\ \text{IV} \gtrless 0$
2.1.5	$\text{I} > 0,\ \text{II} \geqslant 0,\ \text{III} \geqslant 0,\ \text{IV} > 0$	2.2.5	$\text{I} > 0,\ \text{II} \leqslant 0,\ \text{III} \leqslant 0,\ \text{IV} > 0$	2.3.5	$\text{I} > 0,\ \text{II} \gtrless 0,\ \text{III} \gtrless 0,\ \text{IV} > 0$
2.1.6	$\text{I} > 0,\ \text{II} > 0,\ \text{III} \geqslant 0,\ \text{IV} \geqslant 0$	2.2.6	$\text{I} > 0,\ \text{II} > 0,\ \text{III} \leqslant 0,\ \text{IV} \leqslant 0$	2.3.6	$\text{I} > 0,\ \text{II} > 0,\ \text{III} \gtrless 0,\ \text{IV} \gtrless 0$
Three of the signature conditions are S-denied					
3.1.1	$\text{I} > 0,\ \text{II} \geqslant 0,\ \text{III} \geqslant 0,\ \text{IV} \geqslant 0$	3.2.1	$\text{I} > 0,\ \text{II} \leqslant 0,\ \text{III} \leqslant 0,\ \text{IV} \leqslant 0$	3.3.1	$\text{I} > 0,\ \text{II} \gtrless 0,\ \text{III} \gtrless 0,\ \text{IV} \gtrless 0$
3.1.2	$\text{I} \geqslant 0,\ \text{II} > 0,\ \text{III} \geqslant 0,\ \text{IV} \geqslant 0$	3.2.2	$\text{I} \leqslant 0,\ \text{II} > 0,\ \text{III} \leqslant 0,\ \text{IV} \leqslant 0$	3.3.2	$\text{I} \gtrless 0,\ \text{II} > 0,\ \text{III} \gtrless 0,\ \text{IV} \gtrless 0$
3.1.3	$\text{I} \geqslant 0,\ \text{II} \geqslant 0,\ \text{III} > 0,\ \text{IV} \geqslant 0$	3.2.3	$\text{I} \leqslant 0,\ \text{II} \leqslant 0,\ \text{III} > 0,\ \text{IV} \leqslant 0$	3.3.3	$\text{I} \gtrless 0,\ \text{II} \gtrless 0,\ \text{III} > 0,\ \text{IV} \gtrless 0$
3.1.4	$\text{I} \geqslant 0,\ \text{II} \geqslant 0,\ \text{III} \geqslant 0,\ \text{IV} > 0$	3.2.4	$\text{I} \leqslant 0,\ \text{II} \leqslant 0,\ \text{III} \leqslant 0,\ \text{IV} > 0$	3.3.4	$\text{I} \gtrless 0,\ \text{II} \gtrless 0,\ \text{III} \gtrless 0,\ \text{IV} > 0$
All the signature conditions are S-denied					
4.1.1	$\text{I} \geqslant 0,\ \text{II} \geqslant 0,\ \text{III} \geqslant 0,\ \text{IV} \geqslant 0$	4.2.1	$\text{I} \leqslant 0,\ \text{II} \leqslant 0,\ \text{III} \leqslant 0,\ \text{IV} \leqslant 0$	4.3.1	$\text{I} \gtrless 0,\ \text{II} \gtrless 0,\ \text{III} \gtrless 0,\ \text{IV} \gtrless 0$

Table 1: The expanded spaces for General Relativity (all 45 mixed kinds of S-denying). The signature conditions are denoted by Roman numerals

metric ds^2 is also degenerate.

1.2.1: $g_{00} \leqslant 0,\ h_{11} > 0,\ h_{11}h_{22} - h_{12}^2 > 0,\ h > 0$. At $g_{00} = 0$, we have kind 1.1.1. At $g_{00} < 0$ physically observable time becomes imaginary $d\tau = \frac{g_{0i}\,dx^i}{c\sqrt{g_{00}}}$.

1.2.2: $g_{00} > 0,\ h_{11} \leqslant 0,\ h_{11}h_{22} - h_{12}^2 > 0,\ h > 0$. At $h_{11} = 0$, we have kind 1.1.2. At $h_{11} < 0$, distances along the axis x^1 (i. e. the values $\sqrt{h_{11}}dx^1$) becomes imaginary, contradicting the initial conditions in General Relativity.

1.2.3: $g_{00} > 0,\ h_{11} > 0,\ h_{11}h_{22} - h_{12}^2 \leqslant 0,\ h > 0$. This is a common space built on a particular case of kind 1.1.3 where $h_{11}h_{22} - h_{12}^2 = 0$ and a subspace where $h_{11}h_{22} - h_{12}^2 < 0$. In the latter subspace the spatially observable metric $d\sigma^2$ becomes sign-alternating so that the space-time metric has the signature $(+-+-)$ (this case is outside the initial statement of General Relativity).

1.2.4: $g_{00} > 0,\ h_{11} > 0,\ h_{11}h_{22} - h_{12}^2 > 0,\ h \leqslant 0$. This space is built on a particular case of kind 1.1.2 where $h = 0$ and a subspace where $h < 0$. At $h < 0$ we have the spatial metric $d\sigma^2$ sign-alternating so that the space-time metric has the signature $(+--+)$ (this case is outside the initial statement of General Relativity).

1.3.1: $g_{00} \gtrless 0,\ h_{11} > 0,\ h_{11}h_{22} - h_{12}^2 > 0,\ h > 0$. Here we have the usual space-time area ($g_{00} > 0$) with the signature $(+---)$, and a sign-definite space-time ($g_{00} < 0$) where the signature is $(----)$. There are no intersections of the areas in the common space-time; they exist severally.

1.3.2: $g_{00} > 0,\ h_{11} \gtrless 0,\ h_{11}h_{22} - h_{12}^2 > 0,\ h > 0$. Here we have a common space built on two separated areas where $(+---)$ (usual space-time) and a subspace where $(++--)$. The areas have no intersections.

1.3.3: $g_{00} > 0,\ h_{11} > 0,\ h_{11}h_{22} - h_{12}^2 \gtrless 0,\ h > 0$. This is a common space built on the usual space-time and a particular space-time of kind 1.2.3, where the signature is $(+-+-)$. The areas have no intersections.

1.3.4: $g_{00} > 0,\ h_{11} > 0,\ h_{11}h_{22} - h_{12}^2 > 0,\ h \gtrless 0$. This is a common space built on the usual space-time and a particular space-time of kind 1.2.4, where the signature is $(+--+)$. The areas have no intersections.

2.1.1: $g_{00} \geqslant 0,\ h_{11} \geqslant 0,\ h_{11}h_{22} - h_{12}^2 > 0,\ h > 0$. This is a complex Riemannian space with a complex metric $d\sigma^2$, permitting collapse.

2.1.2: $g_{00} \geqslant 0,\ h_{11} > 0,\ h_{11}h_{22} - h_{12}^2 \geqslant 0,\ h > 0$. This space permits collapse, and purely spatial isotropic directions.

2.1.3: $g_{00} \geqslant 0,\ h_{11} > 0,\ h_{11}h_{22} - h_{12}^2 > 0,\ h \geqslant 0$. This space permits complete degeneracy and collapse. At $g_{00} = 0$ and $h = 0$, we have a collapsed zero-space.

2.1.4: $g_{00} > 0,\ h_{11} \geqslant 0,\ h_{11}h_{22} - h_{12}^2 > 0,\ h \geqslant 0$. Here we have a complex Riemannian space permitting complete degeneracy.

2.1.5: $g_{00} > 0,\ h_{11} \geqslant 0,\ h_{11}h_{22} - h_{12}^2 \geqslant 0,\ h > 0$. At

$h_{11} = 0$, we have $h_{12}^2 = 0$: a partial degeneration of the spatially observable metric $d\sigma^2$.

2.1.6: $g_{00} > 0$, $h_{11} > 0$, $h_{11}h_{22} - h_{12}^2 \geqslant 0$, $h \geqslant 0$. This space permits the spatially observable metric $d\sigma^2$ to completely degenerate: $h = 0$.

2.2.1: $g_{00} \leqslant 0$, $h_{11} \leqslant 0$, $h_{11}h_{22} - h_{12}^2 > 0$, $h > 0$. At $g_{00} = 0$ and $h_{11} = 0$, we have a particular space-time of kind 2.1.1. At $g_{00} < 0$, $h_{11} < 0$ we have a space with the signature $(----)$ where time is like a spatial coordinate (this case is outside the initial statement of General Relativity).

2.2.2: $g_{00} \leqslant 0$, $h_{11} > 0$, $h_{11}h_{22} - h_{12}^2 \leqslant 0$, $h > 0$. At $g_{00} = 0$ and $h_{11}h_{22} - h_{12}^2 = 0$, we have a particular space-time of kind 2.1.2. At $g_{00} < 0$ and $h_{11}h_{22} - h_{12}^2 < 0$, we have a space with the signature $(-+-+)$ (it is outside the initial statement of General Relativity).

2.2.3: $g_{00} \leqslant 0$, $h_{11} > 0$, $h_{11}h_{22} - h_{12}^2 > 0$, $h \leqslant 0$. At $g_{00} = 0$ and $h = 0$, we have a particular space-time of kind 2.1.3. At $g_{00} < 0$ and $h_{11}h_{22} - h_{12}^2 < 0$, we have a space-time with the signature $(---+)$ (it is outside the initial statement of General Relativity).

2.2.4: $g_{00} > 0$, $h_{11} \leqslant 0$, $h_{11}h_{22} - h_{12}^2 \leqslant 0$, $h > 0$. At $h_{11} = 0$ and $h_{11}h_{22} - h_{12}^2 = 0$, we have a particular space-time of kind 2.1.5. At $h_{11} < 0$ and $h_{11}h_{22} - h_{12}^2 < 0$, we have a space-time with the signature $(++-+)$ (outside the initial statement of General Relativity).

2.2.5: $g_{00} > 0$, $h_{11} \leqslant 0$, $h_{11}h_{22} - h_{12}^2 \geqslant 0$, $h \leqslant 0$. At $h_{11} = 0$ and $h = 0$, we have a particular space-time of kind 2.1.4. At $h_{11} < 0$ and $h < 0$, a space-time with the signature $(+--+)$ (outside the initial statement of General Relativity).

2.2.6: $g_{00} > 0$, $h_{11} > 0$, $h_{11}h_{22} - h_{12}^2 \leqslant 0$, $h \leqslant 0$. At $h_{11}h_{22} - h_{12}^2 = 0$ and $h = 0$, we have a particular space-time of kind 2.1.6. At $h_{11}h_{22} - h_{12}^2 < 0$ and $h < 0$, we have a space-time with the signature $(+-++)$ (outside the initial statement of General Relativity).

2.3.1: $g_{00} \gtrless 0$, $h_{11} \gtrless 0$, $h_{11}h_{22} - h_{12}^2 > 0$, $h > 0$. This is a space built on two areas. At $g_{00} > 0$ and $h_{11} > 0$, we have the usual space-time. At $g_{00} < 0$ and $h_{11} < 0$, we have a particular space-time of kind 2.2.1. The areas have no intersections: the common space is actually built on non-intersecting areas.

2.3.2: $g_{00} \gtrless 0$, $h_{11} > 0$, $h_{11}h_{22} - h_{12}^2 \gtrless 0$, $h > 0$. This space is built on two areas. At $g_{00}>0$ and $h_{11}h_{22}-h_{12}^2>0$, we have the usual space-time. At $g_{00} < 0$ and $h_{11}h_{22} - h_{12}^2 < 0$, we have a particular space-time of kind 2.2.2. The areas, building a common space, have no intersections.

2.3.3: $g_{00} \gtrless 0$, $h_{11} > 0$, $h_{11}h_{22} - h_{12}^2 > 0$, $h \gtrless 0$. This space is built on two areas. At $g_{00} > 0$ and $h_{11} > 0$, we have the usual space-time. At $g_{00} < 0$ and $h_{11} < 0$, a particular space-time of kind 2.2.3. The areas, building a common space, have no intersections.

2.3.4: $g_{00} > 0$, $h_{11} \gtrless 0$, $h_{11}h_{22} - h_{12}^2 > 0$, $h \gtrless 0$. This space is built on two areas. At $h_{11} > 0$ and $h > 0$, we have the usual space-time. At $h_{11} < 0$ and $h < 0$, a particular space-time of kind 2.2.4. The areas, building a common space, have no intersections.

2.3.5: $g_{00} > 0$, $h_{11} \gtrless 0$, $h_{11}h_{22} - h_{12}^2 \gtrless 0$, $h > 0$. This space is built on two areas. At $h_{11}>0$ and $h_{11}h_{22}-h_{12}^2>0$, we have the usual space-time. At $h_{11} < 0$ and $h_{11}h_{22} - h_{12}^2 < 0$, a particular space-time of kind 2.2.5. The areas, building a common space, have no intersections.

2.3.6: $g_{00} > 0$, $h_{11} > 0$, $h_{11}h_{22} - h_{12}^2 \gtrless 0$, $h \gtrless 0$. This space is built on two areas. At $h_{11}h_{22} - h_{12}^2 > 0$ and $h > 0$, we have the usual space-time. At $h_{11}h_{22} - h_{12}^2 < 0$ and $h < 0$, a particular space-time of kind 2.2.6. The areas, building a common space, have no intersections.

3.1.1: $g_{00} > 0$, $h_{11} \geqslant 0$, $h_{11}h_{22} - h_{12}^2 \geqslant 0$, $h \geqslant 0$. This space permits complete degeneracy. At $h_{11} > 0$, $h_{11}h_{22} - h_{12}^2 > 0$, $h > 0$, we have the usual space-time. At $h_{11} = 0$, $h_{11}h_{22} - h_{12}^2 = 0$, $h = 0$, we have a particular case of a zero-space.

3.1.2: $g_{00} \geqslant 0$, $h_{11} > 0$, $h_{11}h_{22} - h_{12}^2 \geqslant 0$, $h \geqslant 0$. At $g_{00} > 0$, $h_{11}h_{22} - h_{12}^2 > 0$, $h > 0$, we have the usual space-time. At $g_{00} = 0$, $h_{11}h_{22} - h_{12}^2 = 0$, $h = 0$, we have a particular case of a collapsed zero-space.

3.1.3: $g_{00} \geqslant 0$, $h_{11} \geqslant 0$, $h_{11}h_{22} - h_{12}^2 > 0$, $h \geqslant 0$. At $g_{00} > 0$, $h_{11} > 0$, $h > 0$, we have the usual space-time. At $g_{00} = 0$, $h_{11} = 0$, $h = 0$, we have a collapsed zero-space, derived from a complex Riemannian space.

3.1.4: $g_{00} \geqslant 0$, $h_{11} \geqslant 0$, $h_{11}h_{22} - h_{12}^2 \geqslant 0$, $h > 0$. At $g_{00} > 0$, $h_{11} > 0$, $h_{11}h_{22} - h_{12}^2 > 0$, we have the usual space-time. At $g_{00} = 0$, $h_{11} = 0$, $h_{11}h_{22} - h_{12}^2 = 0$, we have the usual space-time in a collapsed state, while there are permitted purely spatial isotropic directions $\sqrt{h_{11}}dx^1$.

3.2.1: $g_{00} > 0$, $h_{11} \leqslant 0$, $h_{11}h_{22} - h_{12}^2 \leqslant 0$, $h \leqslant 0$. At $h_{11} = 0$, $h_{11}h_{22} - h_{12}^2 = 0$ and $h = 0$, we have a particular space-time of kind 3.1.1. At $h_{11} < 0$, $h_{11}h_{22} - h_{12}^2 < 0$ and $h < 0$, we have a space-time with the signature $(++++)$ (outside the initial statement of General Relativity).

3.2.2: $g_{00} \leqslant 0$, $h_{11} > 0$, $h_{11}h_{22} - h_{12}^2 \leqslant 0$, $h \leqslant 0$. At $g_{00} = 0$, $h_{11}h_{22} - h_{12}^2 = 0$ and $h = 0$, we have a particular space-time of kind 3.1.2. At $h_{11} < 0$, $h_{11}h_{22} - h_{12}^2 < 0$ and $h < 0$, we have a space-time with the signature $(--++)$ (outside the initial statement of General Relativity).

3.2.3: $g_{00} \leqslant 0$, $h_{11} \leqslant 0$, $h_{11}h_{22} - h_{12}^2 > 0$, $h \leqslant 0$. At $g_{00} = 0$, $h_{11} = 0$ and $h = 0$, we have a particular space-time of kind 3.1.3. At $h_{11} < 0$, $h_{11}h_{22} - h_{12}^2 < 0$ and $h < 0$, we have a space-time with the signature $(-+-+)$ (outside the initial statement of General Relativity).

3.2.4: $g_{00} \leqslant 0$, $h_{11} \leqslant 0$, $h_{11}h_{22} - h_{12}^2 \leqslant 0$, $h > 0$. At $g_{00} = 0$, $h_{11} = 0$ and $h_{11}h_{22} - h_{12}^2 = 0$, we have a particular space-time of kind 3.1.4. At $g_{00} < 0$, $h_{11} < 0$ and $h_{11}h_{22} - h_{12}^2 < 0$, we have a space-time with the signature $(-++-)$ (outside the initial statement of General Relativity).

3.3.1: $g_{00} > 0$, $h_{11} \gtrless 0$, $h_{11}h_{22} - h_{12}^2 \gtrless 0$, $h \gtrless 0$. This is a space built on two areas. At $h_{11} > 0$, $h_{11}h_{22} - h_{12}^2 > 0$ and $h_{11} > 0$, we have the usual space-time. At $h_{11} < 0$, $h_{11}h_{22} - h_{12}^2 < 0$ and $h_{11} < 0$, we have a particular space-time of kind 3.2.1. The areas have no intersections: the common space is actually built on non-intersecting areas.

3.3.2: $g_{00} \gtrless 0$, $h_{11} > 0$, $h_{11}h_{22} - h_{12}^2 \gtrless 0$, $h \gtrless 0$. This space is built on two areas. At $g_{00} > 0$, $h_{11}h_{22} - h_{12}^2 > 0$ and $h > 0$, we have the usual space-time. At $g_{00} < 0$, $h_{11}h_{22} - h_{12}^2 < 0$ and $h < 0$, we have a particular space-time of kind 3.2.2. The areas, building a common space, have no intersections.

3.3.3: $g_{00} \gtrless 0$, $h_{11} \gtrless 0$, $h_{11}h_{22} - h_{12}^2 > 0$, $h \gtrless 0$. This space is built on two areas. At $g_{00} > 0$, $h_{11} > 0$ and $h > 0$, we have the usual space-time. At $g_{00} < 0$, $h_{11} < 0$ and $h < 0$, we have a particular space-time of kind 3.2.3. The areas, building a common space, have no intersections.

3.3.4: $g_{00} \gtrless 0$, $h_{11} \gtrless 0$, $h_{11}h_{22} - h_{12}^2 \gtrless 0$, $h > 0$. This space is built on two areas. At $g_{00} > 0$, $h_{11} > 0$ and $h_{11}h_{22} - h_{12}^2 > 0$, we have the usual space-time. At $g_{00} < 0$, $h_{11} < 0$ and $h_{11}h_{22} - h_{12}^2 < 0$, a particular space-time of kind 3.2.4. The areas, building a common space, have no intersections.

4.4.1: $g_{00} \geqslant 0$, $h_{11} \geqslant 0$, $h_{11}h_{22} - h_{12}^2 \geqslant 0$, $h \geqslant 0$. At $g_{00} > 0$, $h_{11} > 0$, $h_{11}h_{22} - h_{12}^2 \geqslant 0$ and $h \geqslant 0$, we have the usual space-time. At $g_{00} = 0$, $h_{11} = 0$, $h_{11}h_{22} - h_{12}^2 = 0$ and $h = 0$, we have a particular case of collapsed zero-space.

4.4.2: $g_{00} \leqslant 0$, $h_{11} \leqslant 0$, $h_{11}h_{22} - h_{12}^2 \leqslant 0$, $h \leqslant 0$. At $g_{00} = 0$, $h_{11} = 0$, $h_{11}h_{22} - h_{12}^2 = 0$ and $h = 0$, we have a particular case of space-time of kind 4.4.1. At $g_{00} < 0$, $h_{11} < 0$, $h_{11}h_{22} - h_{12}^2 < 0$ and $h < 0$, we have a space-time with the signature $(----)$ (outside the initial statement of General Relativity). The areas have no intersections.

4.4.3: $g_{00} \gtrless 0$, $h_{11} \gtrless 0$, $h_{11}h_{22} - h_{12}^2 \gtrless 0$, $h \gtrless 0$. At $g_{00} > 0$, $h_{11} > 0$, $h_{11}h_{22} - h_{12}^2 > 0$ and $h > 0$, we have the usual space-time. At $g_{00} < 0$, $h_{11} < 0$, $h_{11}h_{22} - h_{12}^2 < 0$ and $h < 0$, we have a space-time with the signature $(----)$ (outside the initial statement of General Relativity). The areas have no intersections.

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