ABOUT VERY PERFECT NUMBERS[1]

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A natural number n is called very perfect if $\sigma(\sigma(n)) = 2n$ (see [1]).

Theorem. The square of an odd prime number cannot be very perfect number.

Proof: Let's consider $n = p^2$, where p is an odd prime number, then

$$\sigma(n) = 1 + p + p^2, \quad \sigma(\sigma(n)) = \sigma(1 + p + p^2) = 2p^2.$$

We decompose $\sigma(n)$ in canonical form, from where $1 + p + p^2 = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$. Because $p(p+1)+1$ is odd, in the canonical decompose must be only odd numbers.

$$\sigma(\sigma(n)) = (1 + p_1 + \dots + p_1^{\alpha})\dots(1 + p_k + \dots + p_k^{\alpha_k}) = \frac{p_1^{\alpha_1+1} - 1}{p_1 - 1} \dots \frac{p_k^{\alpha_k+1} - 1}{p_k - 1} = 2p^2$$

Because

$$\frac{p_1^{\alpha_1+1} - 1}{p_1 - 1} > 2, \dots, \frac{p_k^{\alpha_k+1} - 1}{p_k - 1} > 2$$

one obtains that $2p^2$ cannot be decomposed in more than two factors, then each one is > 2, therefore $k \le 2$.

Case 1. For $k = 1$ we find $\sigma(n) = 1 + p + p^2 = p_1^{\alpha_1}$, from where one obtains

$$p_1^{\alpha_1+1} = p_1(1 + p + p^2) \text{ and}$$

$$\sigma(\sigma(n)) = \frac{p_1^{\alpha_1+1} - 1}{p_1 - 1} = 2p^2, \quad p_1(1 + p + p^2) - 1 = 2p^2(p_1 - 1),$$

from where

$$p_1 - 1 = p(pp_1 - 2p - p_1).$$

The right side is divisible by p, thus $p_1 - 1$ is a p multiple. Because $p_1 > 2$ it results

$$p_1 \ge p - 1 \text{ and } p_1^2 \ge (p+1)^2 > p^2 + p + 1 = p_1^{\alpha_1},$$

thus $\alpha_1 = 1$ and

$$\sigma(n) = p^2 + p + 1 = p_1, \quad \sigma(\sigma(n)) = \sigma(p_1) = 1 + p_1.$$

If n is very perfect then $1 + p_1 = 2p^2$ or $p^2 + p + 2 = 2p^2$. The solutions of the equation are $p = -1$, and $p = 2$ which is a contradiction.

Case 2. For $k = 2$ we have $\sigma(n) = p^2 + p + 1 = p_1^{\alpha_1} p_2^{\alpha_2}$.

[1]Together with Mihály Bencze and Florin Popovici

$$\sigma\left(\sigma(n)\right)=\left(1+p_1+\ldots+p_1^{\alpha}\right)\left(1+p_2+\ldots+p_2^{\alpha_2}\right)=\frac{p_1^{\alpha_1+1}-1}{p_1-1}\cdot\frac{p_2^{\alpha_2+1}-1}{p_2-1}=2p^2 .$$

Because

$$\frac{p_1^{\alpha_1+1}-1}{p_1-1}>2 \text{ and } \frac{p_2^{\alpha_2+1}-1}{p_2-1}>2 ,$$

it results

$$\frac{p_1^{\alpha_1+1}}{p_1-1}=p \text{ and } \frac{p_2^{\alpha_2+1}-1}{p_2-1}=2p$$

(or inverse), thus

$$p_1^{\alpha_1+1}-1=p\left(p_1-1\right),\ p_2^{\alpha_2+1}-1=2p\left(p_2-1\right),$$

then

$$p_1^{\alpha_1+1}p_2^{\alpha_2+1}-p_1^{\alpha_1+1}-p_2^{\alpha_2+1}+1=2p^2\left(p_1-1\right)\left(p_2-1\right),$$

thus

$$\sigma(n)=p^2+p+1=p_1^{\alpha_1+1}p_2^{\alpha_2+1}$$

and

$$p_1p_2\left(p^2+p+1\right)=2p^2\left(p_1-1\right)\left(p_2-1\right)+p_1^{\alpha_1+1}+p_2^{\alpha_2+1}-1$$

or

$$p_1p_2p\left(p+1\right)+p_1p_2-1=2p^2\left(p_1-1\right)\left(p_2-1\right)+\left(p_1^{\alpha_1+1}-1\right)+\left(p_2^{\alpha_2+1}-1\right)=$$
$$=2p^2\left(p_1-1\right)\left(p_2-1\right)+p\left(p_1-1\right)+2p\left(p_2-1\right)$$

accordingly p divides p_1p_2-1, thus $p_1p_2>p+1$ and

$$p_1^2p_2^2\geq\left(p+1\right)^2>p^2+p+1=p_1^{\alpha_1}p_2^{\alpha_2} .$$

Hence:

Π_1) If $\alpha_1=1$ and $n=2p^2$,

then

$$\sigma(n)=p^2+p+1=p_1p_2^{\alpha_2} \text{ and } \frac{p_1^2-1}{p_1-1}=p \text{, and } \frac{p_2^{\alpha_2+1}-1}{p_2-1}=2p ,$$

thus $p_1+1=p$ which is a contradiction.

Π_2) If $\alpha_2=1$ and $n=2p^2$,

then

$$\sigma(n)=p^2+p+1=p_1^{\alpha_1}p_2 \text{ and } \frac{p_1^{\alpha_1+1}-1}{p_1-1}=p \text{, and}$$

$$\frac{p_2^2-1}{p_2-1}=2p ,$$

thus

$$p_2+1=2p,\ p_2=2p-1$$

and

$$\sigma(n)=p^2+p+1=p_1^{\alpha_1}(2p+1) ,$$

from where

$$4\sigma(n) = (2p-1)(2p+3) + 7 = 4p_1^{\alpha_1}(2p-1),$$

accordingly 7 is divisible by $2p-1$ and thus p is divisible by 4 which is a contradiction.

REFERENCE:

[1] Suryanarayama – Elemente der Mathematik – 1969.

["Octogon", Braşov, Vol. 5, No. 2, 53-4, October 1997.]