This paper constructs the zeta functions of the quaternion and the sporadic finite simple groups.

Although many researchers have tried the proof of the Riemann hypothesis, they have not resulted in the success. The proof of this Riemann hypothesis has been an important issue of mathematics. In this paper, we construct zeta functions of the quaternion and sporadic finite simple groups as preparation proving the Riemann hypothesis.

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1 Introduction

1.1 Subject

According to the Riemann hypothesis, the nontrivial zeros of the Riemann zeta function all have real part 1/2. We construct zeta functions from sporadic finite simple groups as preparation proving the Riemann hypothesis.
1.2 Importance of subject

The proof of the Riemann hypothesis is one of the most important outstanding problems in mathematics.

For this reason, many mathematicians have tried the proof of the Riemann hypothesis. However, those trials were not successful. One of the methods to solve the problem of mathematics is to express the problem in the other mathematics. Therefore, the mathematician has tried to construct the zeta function from the other mathematics. For this reason, it has been an important issue to construct the zeta function from the other mathematics.

1.3 Past construction method

Leonhard Euler constructed the zeta function of the prime numbers, and made the argument of the zeta function the natural number in 1737. Bernhard Riemann expanded the argument of the zeta function to the complex number in 1859.

Srinivasa Ramanujan constructed the zeta function for automorphic form in 1916. Kornblum constructed the zeta function for prime polynomials on the finite fields in 1919. Atle Selberg constructed the zeta function for prime polynomials on the complex number fields in 1952. Initially Selberg's zeta function was considered as a zeta function for groups; however, it is consisted as a zeta function for prime geodesic lines of the manifolds.

Research trends other than the construction of a zeta function are as follows.

David Hilbert and George Polya suggested that the zeros of a zeta function were probably eigenvalues of a certain operator around 1914. This conjecture is called Hilbert-Polya conjecture.

Shigenobu Kurokawa is studying the zeta function by the polynomial ring of prime numbers, fields with one element $F_1[2, 3, 5, 7, ...]$ around 1996. Christopher Deninger is studying the eigenvalue interpretation of the zeros of a zeta function in 1998. Alain Connes showed the relation between noncommutative geometry and the Riemann hypothesis in 1999. Nick Katz and Peter Sarnak are studying the distribution of zeros in arithmetic quantum chaos theory in 1999.

1.4 New construction method of this paper

We define the polynomial ring of a group in order to add groups and to add the members of a group.

The argument of a zeta function is a complex number. We can interpret a complex number as the subgroup of a quaternion. Therefore, we can expand the argument of a zeta function to a quaternion. On the other hand, a complex number is one-dimensional complex general linear group. And the sporadic finite simple groups are high dimensional complex general linear groups. Therefore, we can expand the argument of a zeta function to sporadic finite simple groups.
2 Explanation of zeta function and sporadic finite simple group

2.1 Euler’s zeta function
Leonhard Euler constructed the zeta function of the prime numbers, and made the argument of the zeta function the natural number in 1737.

(Euler’s zeta function)

\[
\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}
\]
\[n \in \mathbb{N}\]  \hspace{1cm} (2.1)

We express the zeta function by prime number as follows.

(Euler product of the zeta function)

\[
\zeta(n) = \prod_{p: \text{prime}} \frac{1}{1 - p^{-n}}
\]
\hspace{1cm} (2.3)

2.2 Riemann zeta function and Riemann hypothesis
Bernhard Riemann expanded the argument of the zeta function to the complex number in 1859.

(Riemann zeta function)

\[
\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}
\]
\[s \in \mathbb{C}\]  \hspace{1cm} (2.4)

Riemann expected the following thing.

(Riemann hypothesis)
The real part \(\Re(\rho)\) is 1/2 for all nontrivial zeros \(\rho\) of zeta function \(\zeta(s)\).

We express the examples of nontrivial zeros \(\rho_1\) and \(\rho_2\) in the following figure and formula. The black circles are zeros and the white circle means a pole.
Figure 2.1: Nontrivial zeros of zeta function

\[ \rho_1 = \frac{1}{2} + i(14.13 \ldots) \]  \hspace{1cm} (2.6)  

\[ \rho_2 = \frac{1}{2} + i(21.02 \ldots) \]  \hspace{1cm} (2.7)

### 2.3 Monstrous moonshine

Evariste Galois published the group in the 1830s. Then, the classification of the finite simple groups was achieved in 2004. This classification is called the classification theorem of the finite simple group.

According to the theorem, any finite simple group is the isomorphism of one of the following groups.

1. The cyclic groups of prime order
2. The 5th more than alternating groups
3. Lee type simple groups
4. 26 sporadic simple groups

Although the number of the first to the third groups is infinite, the number of the fourth simple group is 26. One of the simple groups, **monster group** \( M \) has the following order.
\[ |M| = 2^{46} \cdot 3^{30} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \sim 8 \times 10^{53} \]

The monster group was discovered in 1973. The group has 194 irreducible representations. The dimensions of the irreducible representations are shown below.

\[
\begin{align*}
    r_1 &= 1 \\
    r_2 &= 196883 \\
    r_3 &= 21296876
\end{align*}
\]

On the other hand, Felix Klein\(^3\) found **Klein's \(j\)-invariant** in 1877. (Klein's \(j\)-invariant)

\[
    j(z) = \frac{(E_4(z))^3}{\Delta(z)}
\]

\[
    E_4(z) = 1 + \frac{2}{\zeta(1-4)} \sum_{k=1}^{\infty} k^{4-1} \frac{q^k}{1 - q^k}
\]

\[
    \Delta(z) = q \prod_{k=1}^{\infty} (1 - q^k)^{24}
\]

\[
    q = \exp(2\pi i z)
\]

\[
    z \in H
\]

\[
    H = \{ z \in \mathbb{C} \mid \text{Im}(z) > 0 \}
\]

The \(q\)-expansion of Klein's \(j\)-invariant \(j(z)\) is shown below.

\[
    j(z) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \cdots
\]

John McKay found the following relation in 1978.

\[
\begin{align*}
    1 &= r_1 \\
    196884 &= r_1 + r_2 \\
    21493760 &= r_1 + r_2 + r_3
\end{align*}
\]

John McKay and John G. Thompson\(^3\) suggested the next conjectures in 1979.

(McKay - Thompson conjecture)

Module \(V_n\) with some natural meanings on which the simple group monster \(M\) acts exists. Then, we obtain the following formula.
\[ c(k) = \dim(V_k) \quad (2.18) \]
\[ V_0 = 0 \quad (2.19) \]
\[ k \geq 0 \quad (2.20) \]
\[ j(z) = 744 + \sum_{k=0}^{\infty} c(k)q^{k-1} \quad (2.21) \]

John Conway\(^4\) named the above conjecture \textbf{monstrous moonshine} in 1979. Richard Borcherds constructed the moonshine vertex operator algebra \( V \) from vertex operator algebra in 1992. The vertex operator algebra has the starting point in the string theory of modern physics.

3 Definition in this paper

3.1 Polynomial ring

The \textbf{polynomial} is like the following formula.
\[ x^2 + 2xy \quad (3.1) \]

A \textbf{variable} is a character. The characters \( x \) is a variable.

An \textbf{exponent} is a number of same variables. The exponent of the following formula is 2.
\[ x^2 \quad (3.2) \]

A \textbf{monomial} is a product of variables. The above formula is a monomial.

A \textbf{coefficient} is the number of the same monomial. The coefficient of the following formula is 2.
\[ 2xy \quad (3.3) \]

A \textbf{term} is a product of a coefficient and a monomial. The above formula is a term.

A polynomial is the sum of terms. The following formula is a polynomial.
\[ x^2 + 2xy \quad (3.4) \]

A variable and a coefficient are member of the natural number \( N \) and so on. We express the set of the polynomials with the variables and the coefficients are members of the natural number \( N \) as follows.
\[ \mathbb{N}[x] \quad (3.5) \]

A \textbf{polynomial ring} is a set of polynomial. The above formula is a polynomial ring.

3.2 Polynomial ring of a members of group

We interpret the variables of polynomials as members of a group.

We express the \textbf{polynomial ring} of a member \( s \) of a group \( G \) as follows.
\[ G[s] \quad (3.6) \]
3.3 Polynomial ring of groups
We interpret the variables of polynomials as groups.

We express the polynomial ring of a group \( G \) as a member of a set \( K \) as follows.

\[ K[G] \quad (3.7) \]

3.4 Representation of group
A representation of a group \( G \) on a vector space \( V \) is a map from a group \( G \) to a general linear matrix \( \text{GL}(V) \).

\[ \rho: G \to \text{GL}(V) \quad (3.8) \]

\[ \rho(gh) = \rho(g)\rho(h) \quad (3.9) \]

\[ g, h \in G \quad (3.10) \]

\[ \rho(g), \rho(h) \in \rho(G) \quad (3.11) \]

In this paper, we write the representation \( \rho(g) \) of the member of the group as \( g \) shortly. We write the set \( \rho(G) \) of the representations of the member of the group as \( G \) shortly.

3.5 Definition of product of group
We define the product of groups as the direct product of a group. We define the product of groups \( AB \) as follows.

\[ AB = A \times B \quad (3.12) \]

We define the exponent of the group \( A \) as follows.

\[ A^2 = A \times A \quad (3.13) \]

We define the multiplier of the unitary group \( U(1) \) as follows.

\[ U^2(1) = U(1) \times U(1) \quad (3.14) \]

3.6 Definition of sum of group
We number each member of the group. We call the number, the group member number.

We express the member of the group member number \( \omega \) of a group as follows.

\[ g(\omega) \in G \quad (3.15) \]

We define the sum of the groups which are isomorphism.

\[ G \sim H \quad (3.16) \]

We define the sum of the groups as the sum of representations per group member number.
\[ F = G + H \]  
(3.17)

\[ \rho(f(\omega)) = \rho(g(\omega)) + \rho(h(\omega)) \]  
(3.18)

We write the above formula shortly as follows.

\[ f(\omega) = g(\omega) + h(\omega) \]  
(3.19)

We define the coefficient of the group as follows.

\[ 2A = A + A \]  
(3.20)

We define the coefficient of the unitary group U(1) as follows.

\[ 2U(1) = U(1) + U(1) \]  
(3.21)

4 Expansion of the argument of the zeta function to sporadic finite simple group

4.1 Expansion of the argument of the zeta function to the quaternion

The argument \( s \) of zeta function \( \zeta(s) \) is a member of the complex number field \( \mathbb{C} \). We interpret the complex number field \( \mathbb{C} \) as the subfield of the quaternion field \( \mathbb{H} \). Then, we expand the argument of a zeta function to the member of a quaternion field.

\[ \zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} \]  
(4.1)

\[ s \in \mathbb{H} \]  
(4.2)

4.2 Expansion of the argument of zeta function to the general linear matrix

The argument of zeta function is a member of the complex number field \( \mathbb{C} \). The complex number field \( \mathbb{C} \) is a subgroup of general linear matrix \( \text{GL}(V) \). Then, we expand the argument \( s \) of a zeta function to the member of general linear matrix \( \text{GL}(V) \).

\[ \zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} \]  
(4.3)

\[ s \in \text{GL}(V) \]  
(4.4)

4.3 Expansion to sporadic finite simple group of argument of zeta function

There are 26 sporadic finite simple groups. We express the \( k \)-th simple group \( S_k \).

We define the group \( T^{26} \) by simple groups \( S_k \) as follows.

\[ T^{26} = S_1 \times S_2 \times S_3 \times \cdots \times S_{26} \]  
(4.5)

We expand the argument of a zeta function to member of the the group \( T^{26} \).
\[ \zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} \]  
\[ s \in T^{26} \]  

5Conclusion
In this paper we got the following results.

(1) We expand the argument of the zeta function to the quaternion.
(2) We expand the argument of the zeta function to the general linear matrix.
(3) We expand the argument of the zeta function to the sporadic finite simple group.

6Supplement
6.1 Construction of the zeta function by natural group and prime group

We define a natural group \(N_n\) by a cyclic group \(Z_n\) for natural number \(n\). We define a prime group \(P_p\) by a cyclic group \(Z_p\) for prime number \(p\).

We define the zeta function as follows.
(Zeta function of natural groups)
\[ \zeta(s) = \sum_{k=1}^{\infty} \frac{1}{N_k^s} \]  
\[ s \in \mathbb{C} \]

We express the zeta function by prime group as follows.
(Zeta function of prime groups)
\[ \zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p_p^{-s}} \]  

7Appendix
7.1 Orthogonal group
Any orthogonal group \(O(n)\) is a subgroup of a certain unitary group \(U(n)\).

The orthogonal group \(O(2)\) is a subgroup of unitary group \(U(2)\).
\[ O(2) \subseteq U(2) \]  

The orthogonal group \(O(3)\) is a subgroup of unitary group \(U(3)\).
\[ O(3) \subseteq U(3) \]  

The orthogonal group \(O(4)\) is a subgroup of unitary group \(U(4)\).
\[ O(4) \subseteq U(4) \]  

The orthogonal group \(O(n)\) is a subgroup of unitary group \(U(n)\).
\[ O(n) \subseteq U(n) \] \hspace{1cm} (7.4)

### 7.2 Unitary group

Any unitary group \( U(n) \) is a subgroup of a certain product of unitary group \( U(1) \).

The unitary group \( U(2) \) is a subgroup of a product of unitary group \( U^4(1) \).
\[ U(2) \subseteq U^4(1) \] \hspace{1cm} (7.5)

The unitary group \( U(3) \) is a subgroup of a product of unitary group \( U^6(1) \).
\[ U(3) \subseteq U^6(1) \] \hspace{1cm} (7.6)

The unitary group \( U(4) \) is a subgroup of a product of unitary group \( U^8(1) \).
\[ U(4) \subseteq U^8(1) \] \hspace{1cm} (7.7)

The unitary group \( U(n) \) is a subgroup of a product of unitary group \( U^{2n}(1) \).
\[ U(n) \subseteq U^{2n}(1) \] \hspace{1cm} (7.8)

### 7.3 Finite group

Any natural group \( N_n \) is a product of certain prime groups \( P_p \).
\[ N_n = p_2^{a(2)} \times p_3^{a(3)} \times \cdots \times p_p^{a(p)} \] \hspace{1cm} (7.9)

Here, \( a(p) \) is a function of \( p \).

Any finite group \( F \) is a subgroup of a certain natural group \( N_n \).
\[ F \subseteq N_n \] \hspace{1cm} (7.10)

### 8 Term arrangement

Table 8.1: Prime group etc.

<table>
<thead>
<tr>
<th>Classification</th>
<th>Prime</th>
<th>Natural</th>
<th>Finite</th>
<th>Simple</th>
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<td>Prime group ( P )</td>
<td>Natural group ( N )</td>
<td>Finite group ( F )</td>
<td>Simple group ( S )</td>
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<td>Order of prime group (</td>
<td>P</td>
<td>)</td>
<td>Order of natural group (</td>
</tr>
<tr>
<td>Number</td>
<td>Prime number ( p )</td>
<td>Natural number ( n )</td>
<td>Finite number ( f )</td>
<td>Simple number ( s )</td>
</tr>
</tbody>
</table>

### 9 References

1. Mail: sugiyama_xs@yahoo.co.jp, Site: [http://www.geocities.jp/x_seek/index_e.html](http://www.geocities.jp/x_seek/index_e.html).