Derivation of two-valuedness and angular momentum of spin-1/2 from rotation of 3-sphere

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Abstract

We derive the two-valuedness and the angular momentum of a spin-1/2 from a rotation of 3-dimensional surface of a sphere, in this paper.

We will derive the two-valuedness of the spin as follows.
We interpret the angle of rotation of the 3-sphere as the phase of a wave function. We interpret the 3-sphere as the absolute value of a wave function.

We can express 3-sphere as the manifold with a constant sum of squares of the radius of two circles. When one circle's radius becomes the maximum, the other circle's radius becomes zero. Therefore, we can turn the circle inside out naturally. If we combine the circle turned inside out with the original circle, the manifold becomes a torus with a node. If we rotate the node of the torus by 360 degrees, we can turn the torus inside out. If we rotate the node of the torus 720 degrees, we can return the torus to the original state. This property is consistent with the property of the spin.

We derive the angular momentum of the spin as follows.
We make 3-dimensional solid sphere by removing one point from 3-sphere. On the other hand, we can make boundary like a 3-sphere by removing one point from 3-dimensional space. We combine the boundaries of them. By repeating this, we can construct 3-dimensional helical space.

The angle of rotation of the 3-sphere is the angle of rotation of 3-dimensional helical space. In addition, we can interpret the angle of the rotation in the helical space as the coordinates of the normal 3-dimensional space. Therefore, we can interpret the angular momentum of the 3-sphere as the angular momentum of normal space.

CONTENTS

1 Introduction

1.1 Subject

In order to quantize gravity, we constructed a particle as a string that is 1-dimensional manifold. In this paper, we construct the wave function of a spin-1/2 as 3-dimensional surface of a sphere (3-sphere) that is 3-dimensional manifold.

1.2 Importance of Subject

Although many researchers have tried quantization of gravity, they have not resulted in the success. Quantization of the gravity has been an important issue of physics.

One method for quantizing gravity was to interpret a point particle as a string that is 1-dimensional manifold. Therefore, we can deduce that it is effective to interpret a wave function of a particle as a manifold.

1.3 History of research

1.3.1 History of research of spin

George Eugene Uhlenbeck and Samuel Abraham Goudsmit discovered the spin of the electron in 1925. Wolfgang Pauli formulated the spin by Pauli matrices in 1927. Paul Adrien Maurice Dirac derived the spin by the Dirac equation in 1928.

1.3.2 History of research of manifold

Albert Einstein constructed the general theory of relativity by 4-dimensional Riemann manifold in 1916. Theodor Kaluza[2] and Oskar Klein[3] constructed proposed in the Kaluza–Klein theory by 1-dimensional circle in 1926.

1.4 New construction method of this paper

We derive the angular momentum of the spin as follows.

We make 3-dimensional solid sphere by removing one point from 3-sphere. On the other hand, we can make boundary like a 3-sphere by removing one point from 3-dimensional space. We combine the boundaries of them. By repeating this, we can construct 3-dimensional helical space.

The angle of rotation of the 3-sphere is the angle of rotation of 3-dimensional helical space. In addition, we can interpret the angle of the rotation in the helical space as the coordinates of the normal 3-dimensional space. Therefore, we can interpret the angular momentum of the 3-sphere as the angular momentum of normal space.

2 Confirmation of the traditional research of the spin

2.1 Pauli matrices and quaternion

In order to express the spin, Pauli defined the following Pauli matrices in 1927.

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.1}$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.2}$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{2.3}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.4}$$

The products are shown below.

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = -i\sigma_1\sigma_2\sigma_3 = \sigma_0 \tag{2.5}$$

We define the following matrices.

$$E = \sigma_0 \tag{2.6}$$

$$I = i\sigma_3 \tag{2.7}$$

$$J = i\sigma_2 \tag{2.8}$$

$$K = i\sigma_1 \tag{2.9}$$

The products are shown below.

$$I^2 = J^2 = K^2 = IJK = -E \tag{2.10}$$

These are the representation of the following quaternion.

$$i^2 = j^2 = k^2 = ijk = -1 \tag{2.11}$$

William Rowan Hamilton discovered the quaternion in 1843.

2.2 Confirmation of the experiment of two-valuedness and angular momentum of spin

2.2.1 Verification of two-valuedness of the spin by experiment

H. Rauch[4] and S. A. Werner[5] verified the two-valuedness of the spin by the Neutron Interference Experiment in 1975. In this section, we confirm the two-valuedness of the spin.

We can express the wave function of a particle rotating about the *z*-axis by Pauli matrices as follows.

$$\psi(\theta) = \exp(-\frac{i\sigma_3\theta}{2}) \tag{2.12}$$

The rotating the angle of the rotation θ by 360 degrees does not bring it back to the same state, but to the state with the opposite phase. The rotating the angle of the rotation θ by 720 degrees brings it back to the original state.

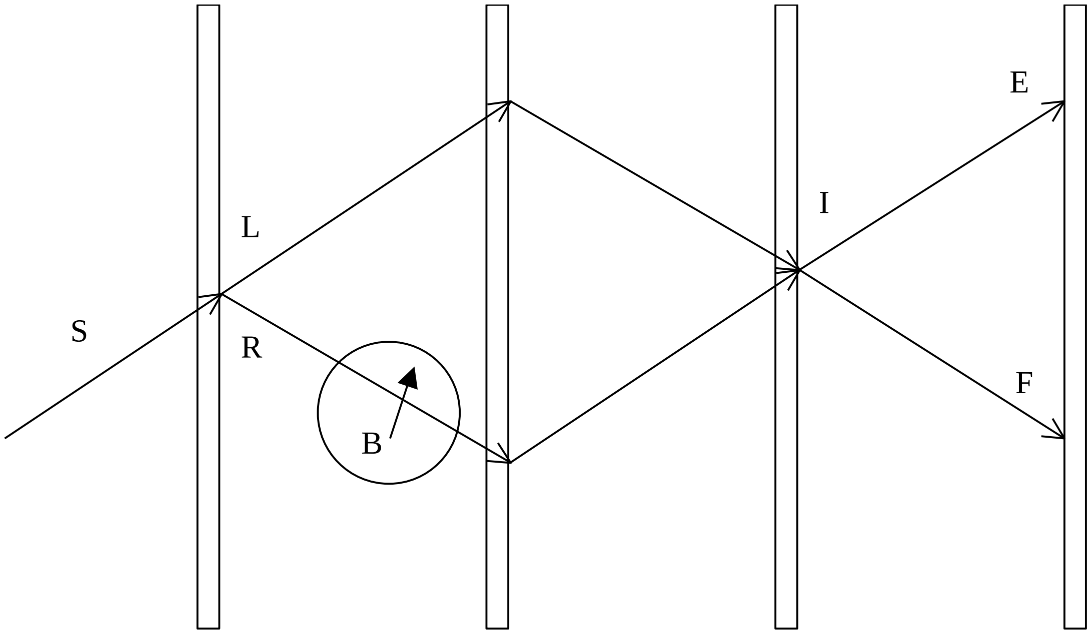

Figure 2.1: Experiment for verification of two-valuedness of spin

We divide Neutron to path *L* and path *R*. Neutron of the path *L* goes through a domain without magnetic field. Neutron of the path *R* goes through a domain with magnetic field. As a result, the magnetic field changes the phase of the neutron of path *R*. Quantity of the change of the phase $\Delta\phi$ is as follows.

$$\Delta\phi = \exp(i\omega T) \tag{2.13}$$

$$\omega = \frac{g_n e B}{mc} \tag{2.14}$$

Here, variable ω is the angular frequency of precession of the spin of the neutron. Variable T is the time neutron passes through the magnetic field. Variable g_n the magnetic moment. Constant e is the elementary charge. Variable B is the strength of the magnetic field. Variable m is the mass of the neutron.

The neutron that passed along the path *L* and path *R* joins at the position *I*. We can observe it at position *E* or position *F*.

Since superposition of a wave function occurs when it joins at position *E* or position *F*, we can observe the phase shift. The phase shift was observed as the result of the experiment actually.

It has been clarified that a spin has two-valuedness by this experiment.

2.2.2 Verification of angular momentum of the spin by experiment

Albert Einstein and Wander Johannes de Haas[6] verified the angular momentum of the spin by the following experiment in 1915.

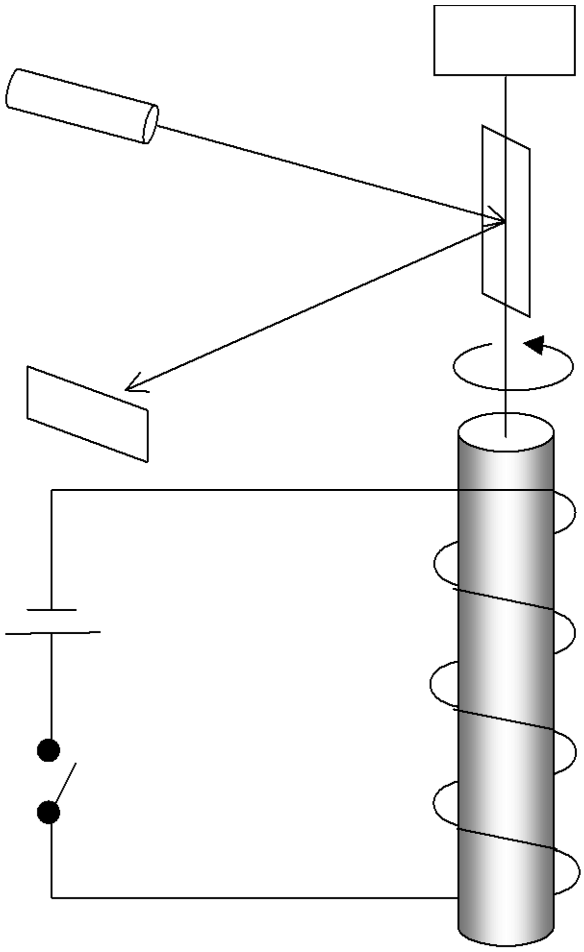

Figure 2.2: The experiment to verify the angular momentum of the spin

The experiment was performed as follows.
We apply the magnetic field to the disk of the magnetic material. Then we make the disk stationary state. After that, we stop the magnetic field. Then disk begins to turn around. This effect is called "Einstein–de Haas effect." It has been clarified that a spin has angular momentum by this experiment.

3 Derivation of the two-valuedness and angular momentum of spin

3.1 Derivation of the two-valuedness of spin

The point particle cannot rotate, because point particle has the radius of rotation 0. We need infinite momentum to get finite angular momentum by the radius of rotation 0.
We can express angular momentum L by using the radius r and momentum p as follows. The operator × is outer product.

$$L = r \times p \tag{3.1}$$

If L is finite and radius r is 0, momentum p becomes infinite.
On the other hand, we cannot derive the two-valuedness of the spin by a rotation of 2-dimensional surface of a sphere (2-sphere). Therefore, we consider the rotation of 3-dimensional surface of a sphere (3-sphere).

3.1.1 Consideration of 3-sphere

We can express 3-sphere S^3 by combining two 3-dimensional solid sphere $B^3{}_1$ and $B^3{}_2$ in the following figure.

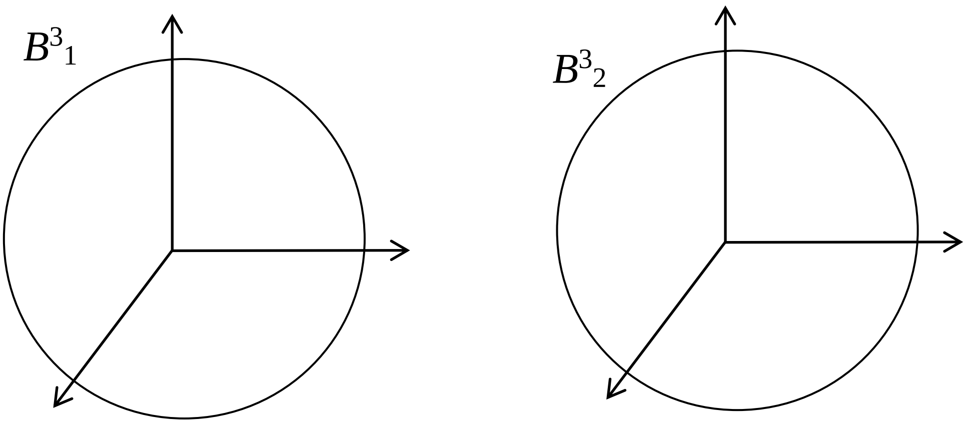

Figure 3.1: 3-sphere

3-sphere has 6 kinds of spin R_1, *R2*, *R3*, *R4*, *R5*, *and R6* like the following figure.

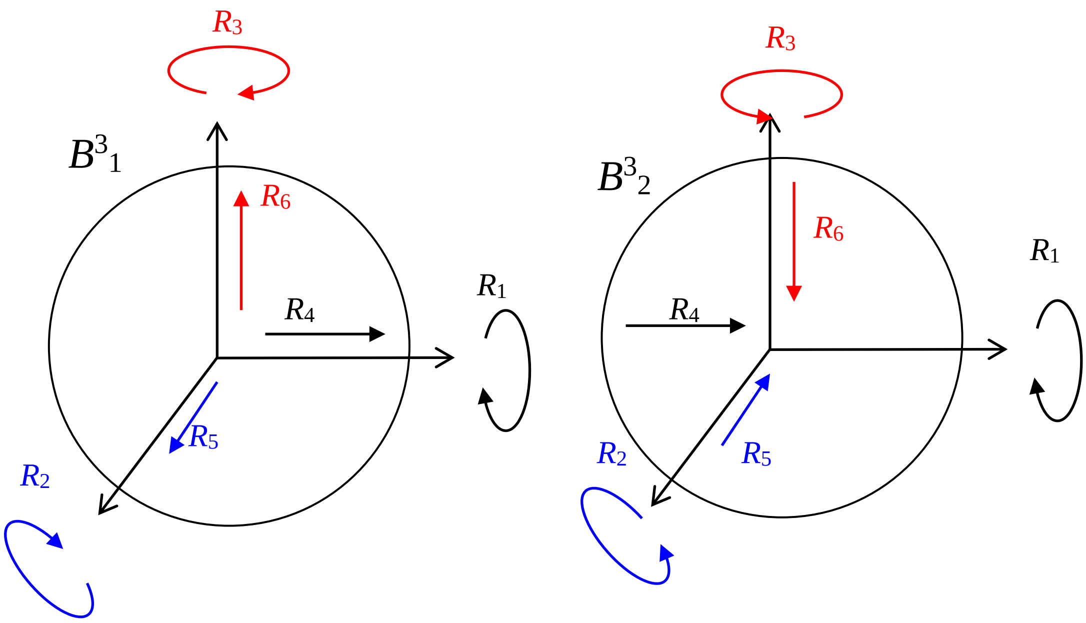

Figure 3.2: Rotation of 3-sphere

It is not difficult to consider the spin R_1, *R2*, *R3*. However, it is difficult to consider spin R_4, R_5, R_6.

It is difficult to consider 3-sphere because the 3-sphere exists in the 4-dimensional space. Then we try to consider the 3-sphere by taking a view of two sections of 3-sphere simultaneously. We call the method to take a view of the two sections simultaneously like this "simultaneous sections method."

First, we try to apply "simultaneous sections method" to 2-sphere because it is easier to consider 2-sphere than 3-sphere.

3.1.2 Taking a view of 2-sphere by simultaneous sections method

We suppose that 2-sphere S^2 in 3-dimensional space specified by the coordinates *x*, *y*, and *z*. If the radius of the 2-sphere is 1, 2-sphere satisfies the following equation.

$$x^2 + y^2 + z^2 = 1 \tag{3.2}$$

We can express this sphere by sectional view of *x*-*y* plane and the position on *z*-axis.

$$x^2 + y^2 = \sin^2\theta \tag{3.3}$$

$$z^2 = \cos^2\theta \tag{3.4}$$

Here, angle θ satisfies the following equation.

$$\cos^2\theta + \sin^2\theta = 1 \tag{3.5}$$

We show the 2-sphere that is applied "simultaneous sections method" to in the following figure.

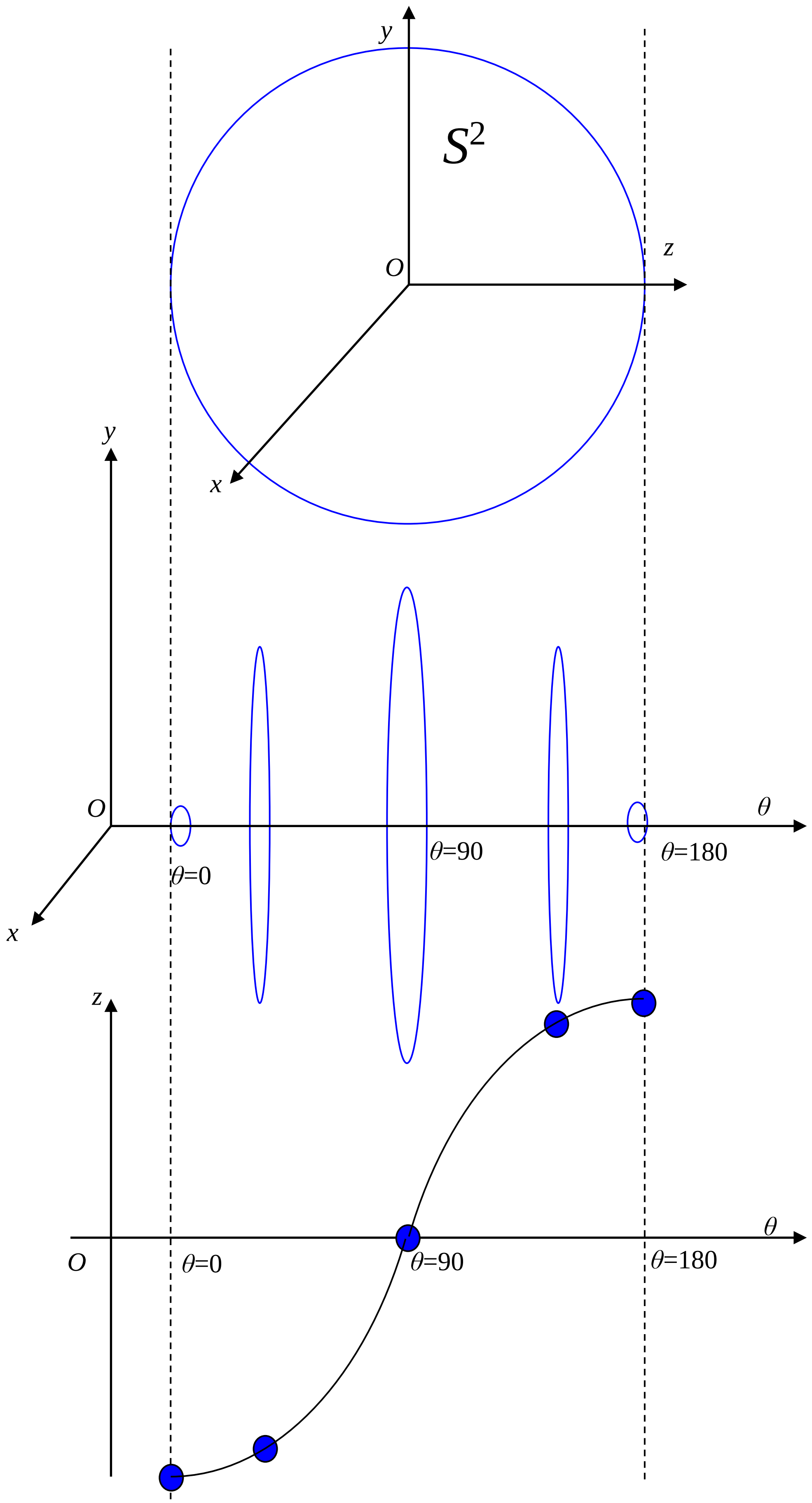

Figure 3.3: Simultaneous sections of 2-sphere

We express the radius of the circle in x-y plane and position z at the angle θ.

Table 3.1: The radius of the circle in x-y plane and position z at the angle θ

Angle θ	Radius of the circle in x-y plane	Position z
0	0	-1
90	1	0
180	0	1

We can consider the structure of the 2-sphere by taking a view of the radius of the circle in x-y plane and position z simultaneously, like this.

Then, we apply "simultaneous sections method" to 3-sphere.

3.1.3 Taking a view of 3-sphere by simultaneous sections method

We suppose that 3-sphere S^3 in 4-dimensional time space specified by the coordinates t, x, y, and z. If the radius of the 3-sphere is 1, 3-sphere satisfies the following equation.

$$t^2 + x^2 + y^2 + z^2 = 1 \tag{3.6}$$

We can express this sphere by sectional view of t-x-y space and position on z-axis.

$$t^2 + x^2 + y^2 = \sin^2 \theta \tag{3.7}$$

$$z^2 = \cos^2 \theta \tag{3.8}$$

We show the 3-sphere that is applied "simultaneous sections method" to in the following figure.

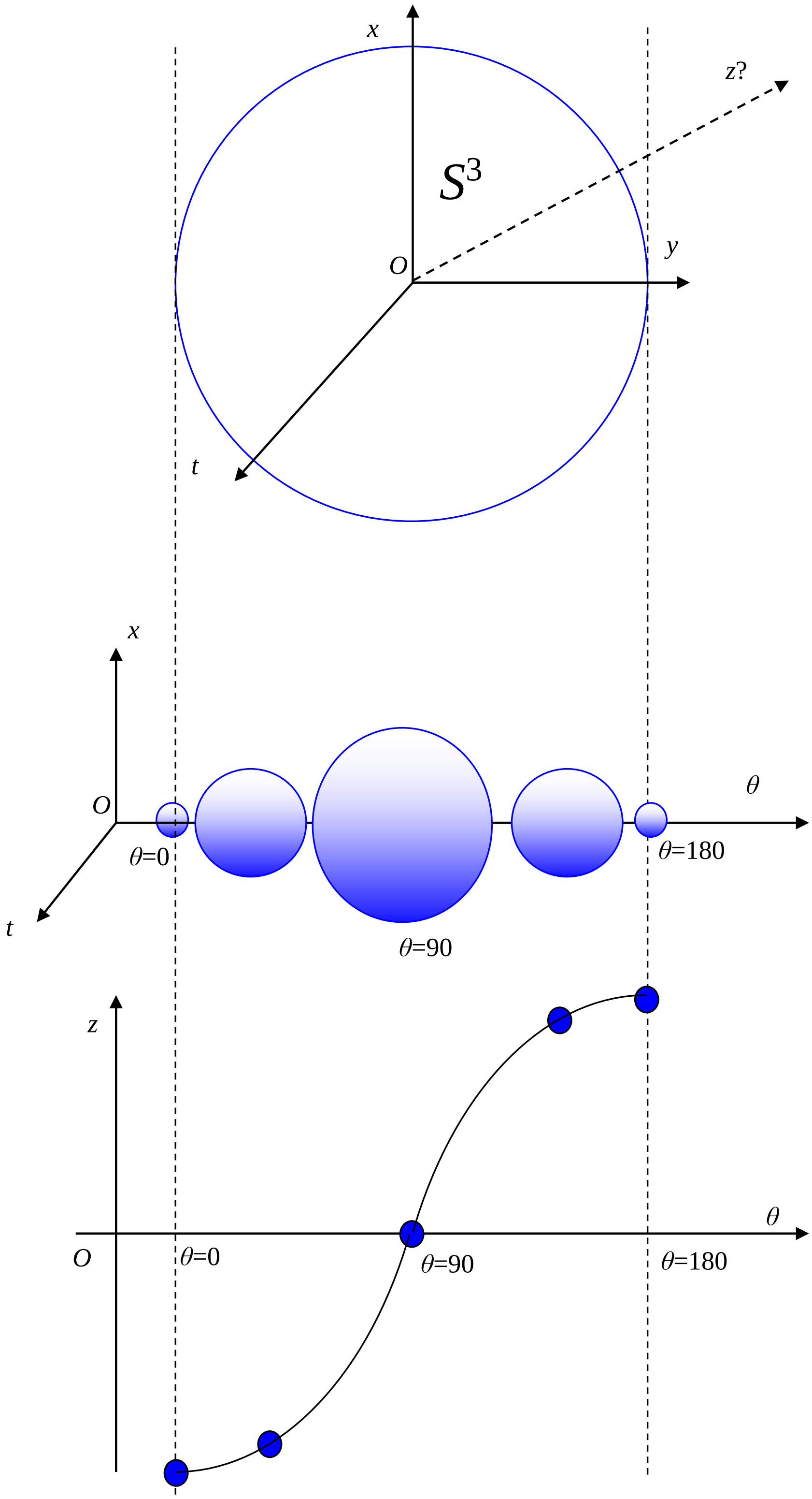

Figure 3.4: Simultaneous sections of 3-sphere

We express the radius of the sphere in *t*-*x*-*y* spaceplane and position *z* at the angle *θ*.

Table 3.2: The radius of the sphere in *t*-*x*-*y* space and position *z* at the angle θ

Angle θ	Radius of the sphere in *t*-*x*-*y* space	Position *z*
0	0	-1
90	1	0
180	0	1

In this section, we divided 3-sphere to 2-sphere and position on an axis. However, we can divide 3-sphere by the other way, too. We consider the way in the next section.

3.1.4 Taking a view of 3-sphere by simultaneous sections method (The other way)

We suppose that 3-sphere S^3 in 4-dimensional time space specified by the coordinates *t*, *x*, *y*, and *z*. If the radius of the 3-sphere is 1, 3-sphere satisfies the following equation.

$$t^2 + x^2 + y^2 + z^2 = 1 \tag{3.9}$$

We can express this sphere by sectional view of *t*-*x* plane and *y*-*z* plane.

$$t^2 + x^2 = \sin^2 \theta \tag{3.10}$$

$$y^2 + z^2 = \cos^2 \theta \tag{3.11}$$

This is Hopf fibration which Heinz Hopf found in 1931.

We show the 3-sphere that is applied "simultaneous sections method" to in the following figure.

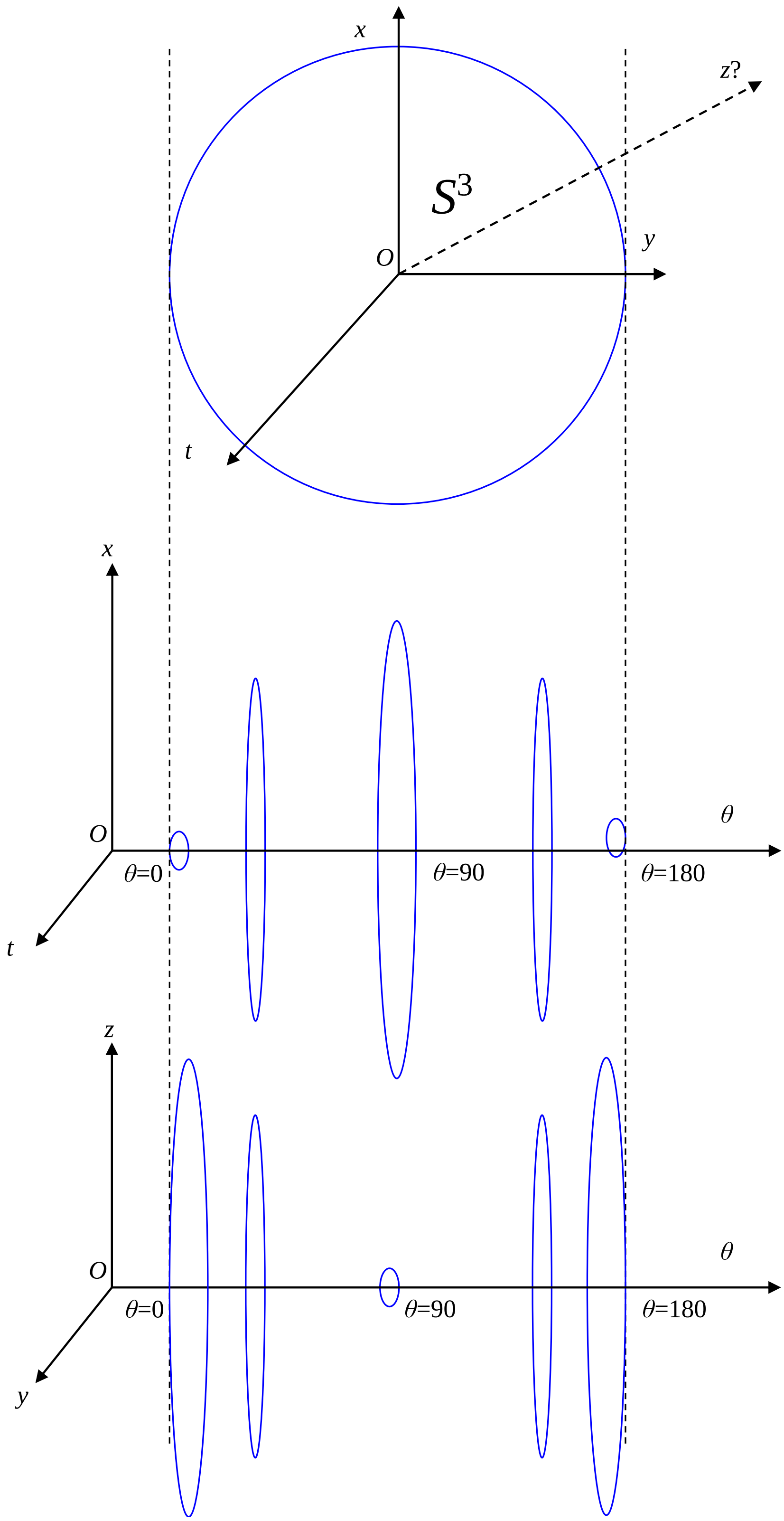

Figure 3.5: Simultaneous sections of 3-sphere (The other way)

We express the radius of the circle in t-x plane and circle in y-z plane at the angle θ.

Table 3.3: The radius of the circle in *t*-*x* plane and circle in *y*-*z* plane at the angle θ

Angle θ	Radius of the circle in *t*-*x* plane	Radius of the circle in *y*-*z* plane
0	0	1
90	1	0
180	0	1

Here we can connect the circle in the *t*-*x* plane at the angle θ=0 and the circle in the *t*-*x* plane at the angle θ=360 because they have same radius 0. In addition, we can connect the circle in the *y*-*z* plane at the angle θ=0 and the circle in the *y*-*z* plane at the angle θ=360 because they have same radius 1.

Therefore, we can interpret the angle θ as a rotation angle of the manifold.

This rotation turns the circle inside out. For example, the circle in *y*-*z* plane is turned inside out at the rotation angle θ=180. Therefore, this rotation is strange spin that is different from the normal spin.

We call the strange spin "Toric spin." In addition, we call normal spin "spheric spin."

3.1.5 Odd torus and even torus

Here we express 3-sphere as follows.

$$t^2 + x^2 = \sin^2\frac{\theta}{2} \tag{3.12}$$

$$y^2 + z^2 = \cos^2\frac{\theta}{2} \tag{3.13}$$

We can express the 3-sphere by the radius *t*, *x*, and the angle θ in the following figure.

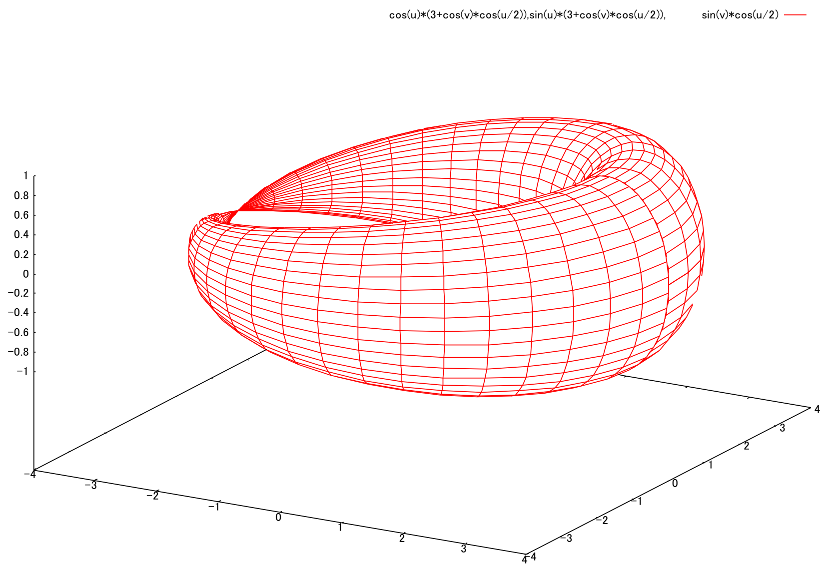

Figure 3.6: Wave function of spin-1/2 particle

We can interpret the above torus as a wave function of spin-1/2 particle. We can express it by the complex function as follows.

$$\psi(\theta) = \exp\left(\frac{i\theta}{2}\right) \tag{3.14}$$

Next, we express 3-sphere as follows.

$$t^2 + x^2 = \sin^2 \frac{2\theta}{2} \tag{3.15}$$

$$y^2 + z^2 = \cos^2 \frac{2\theta}{2} \tag{3.16}$$

We can express the 3-sphere by the radius *t*, *x*, and the angle θ in the following figure.

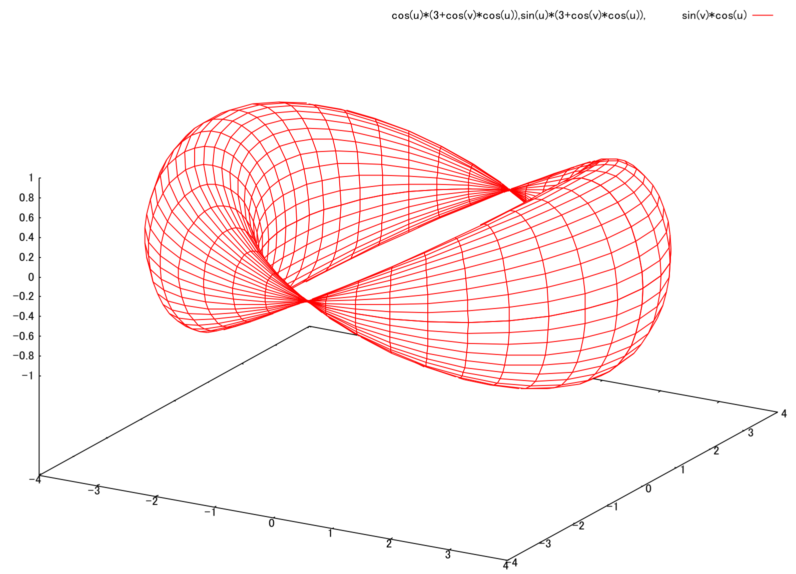

Figure 3.7: Wave function of spin-1 particle

We can interpret the above torus as a wave function of spin-1 particle. We can express it by the complex function as follows.

$$\psi(\theta) = \exp\left(\frac{2i\theta}{2}\right) \tag{3.17}$$

Here we express 3-sphere as follows.

$$t^2 + x^2 = \sin^2 \frac{n\theta}{2} \tag{3.18}$$

$$y^2 + z^2 = \cos^2 \frac{n\theta}{2} \tag{3.19}$$

Variable n is an integer. We call the torus that has even n "even torus." We call the torus that has odd n "odd torus."

3.2 Derivation of angular momentum of spin

In this paper, we interpret spin as a rotation of 3-sphere. Why does the rotation of 3-sphere have same angular momentum as the angular momentum in the 3-dimensional normal space.

In this section, we consider the possibility that the 3-sphere connects to the 3-dimensional normal space.

3.2.1 Construction of 1-dimensional helical space

We can construct 1-dimensional helical space as follows.

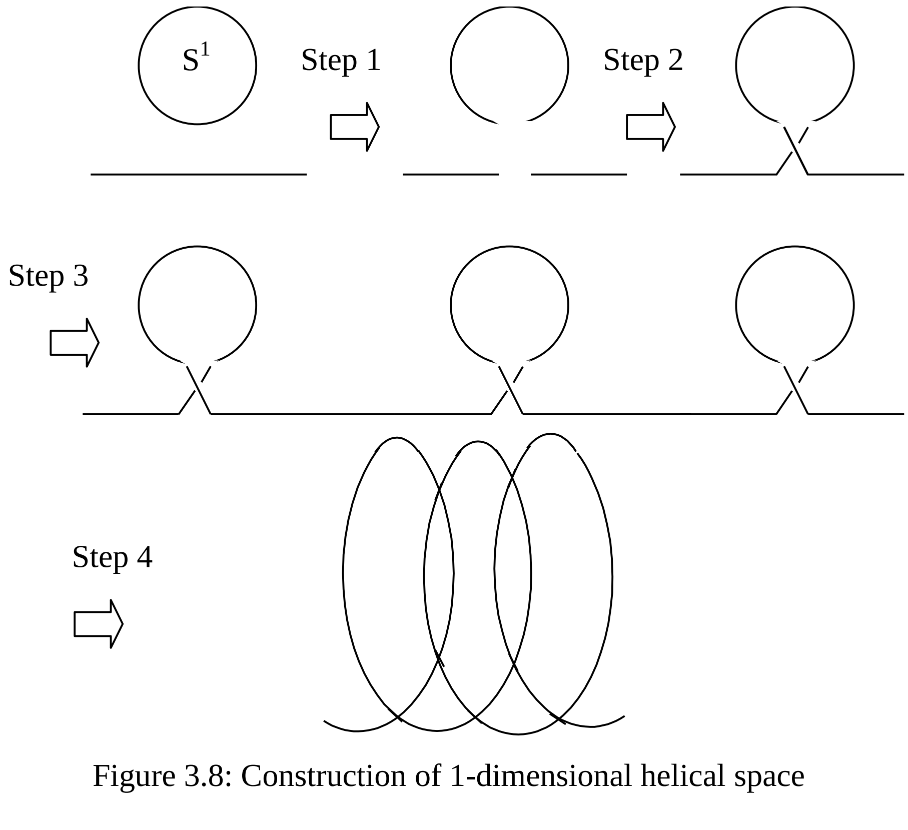

Figure 3.8: Construction of 1-dimensional helical space

We explain the transformation of each step in the following table.

Table 3.4: Construction of 1-dimensional helical space

Step	Method of construction
1	If we remove one point from a circle, we can get an arc. On the other hand, if we remove one point from a segment of a line, we can get two boundaries.
2	We connect their boundaries.
3	If we repeat this process, we can connect many circles.
4	If we change the orientation of the circle, we can construct 1-dimensional helical space.

We can express 1-dimensional helical space by the trigonometric functions as follows.

$$t = r\cos(\theta) \tag{3.20}$$

$$x = r\sin(\theta) \tag{3.21}$$

$$y = p\theta \tag{3.22}$$

Here, θ is the rotation angle of the helical space. r and p are a positive real number.

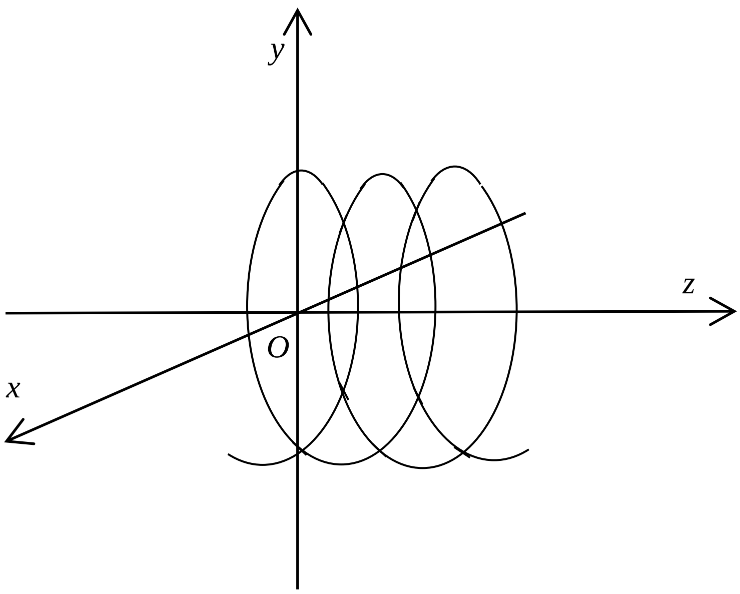

Figure 3.9: 1-dimensional helical space

1 spiral dimensional space can be expressed as follows in the complex. We can express 1-dimensional helical space by the complex function as follows.

$$t + ix = r\exp(i\theta) \tag{3.23}$$

$$y = p\theta \tag{3.24}$$

Here, θ is the rotation angle of the helical space. r and p are a positive real number.

If we combine the both ends, we can get 1-dimensional helical circle.

We can express 1-dimensional helical circle as follows.

$$t = (r\cos(n\phi) + R)\cos(\phi) \tag{3.25}$$

$$x = r\sin(n\phi) \tag{3.26}$$

$$y = (r\cos(n\phi) + R)\sin(\phi) \tag{3.27}$$

Here, n is an integer. ϕ is the rotation angle of the major radius of helical circle. R is major radius of helical circle. r is minor radius of helical circle. We can express the 1-dimensinal helical circle in the following figure.

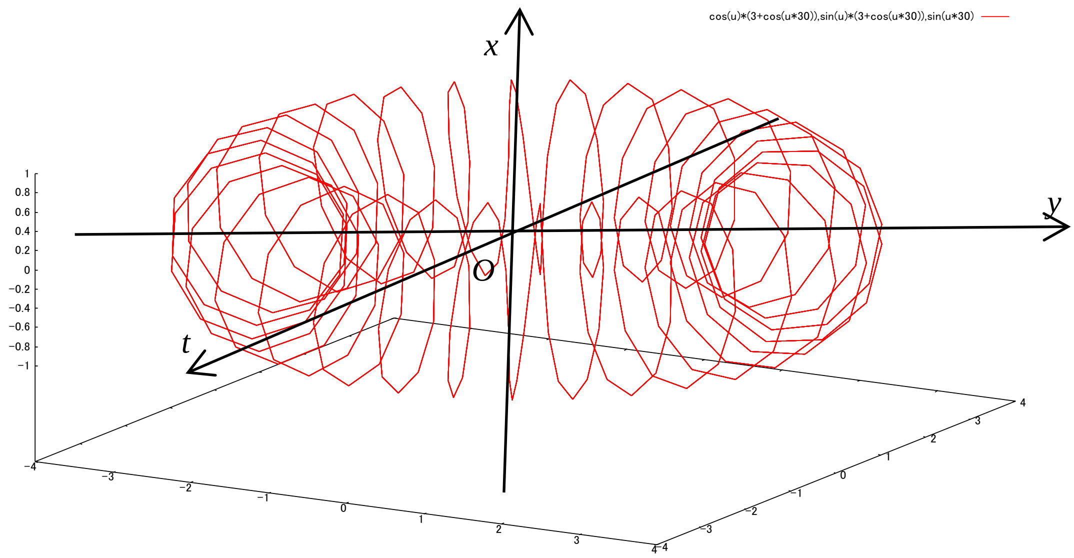

Figure 3.10: 1-dimensional helical circle

We can express 1-dimensional helical circle by the quaternion (1, i, j, k) as follows.

$$t + ix + jy + kz = (r\ \exp(in\phi) + R)\exp(j\phi) \tag{3.28}$$

Here, n is an integer. ϕ is the rotation angle of the major radius of helical circle. R is major radius of helical circle. r is minor radius of helical circle.

Is it possible to do same thing in 2-dimensional space? We consider it in the next section.

3.2.2 Construction of 2-dimensional helical space

We can construct 2-dimensional helical space as follows.

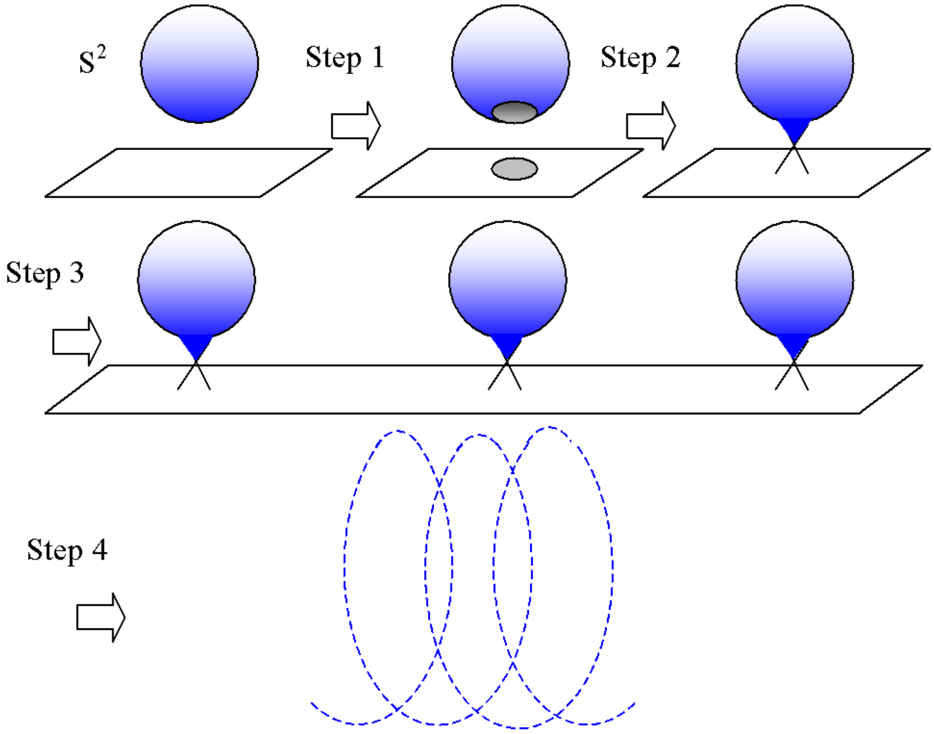

Figure 3.11: Construction of 2-dimensional helical space

We explain the transformation of each step in the following table.

Table 3.5: Construction of 2-dimensional helical space

Step	Method of construction
1	If we remove one point from a 2-sphere, we can get 2-dimensional disk. On the other hand, if we remove one point from a plane, we can get a boundary like a circle.
2	We connect their boundaries.
3	If we repeat this process, we can connect many 2-sphere.
4	If we change the orientation of the 2-sphere, we can construct 2-dimensional helical space.

We cannot express 2-dimensional helical space by the trigonometric functions. We cannot express 2-dimensional helical space by the complex function, too. Therefore, I guess 2-dimensional helical space can does not exist. However, 3- dimensional helical space might exist. We consider it in the next section.

3.2.3 Construction of 3-dimensional helical space

We can construct 3-dimensional helical space as follows.

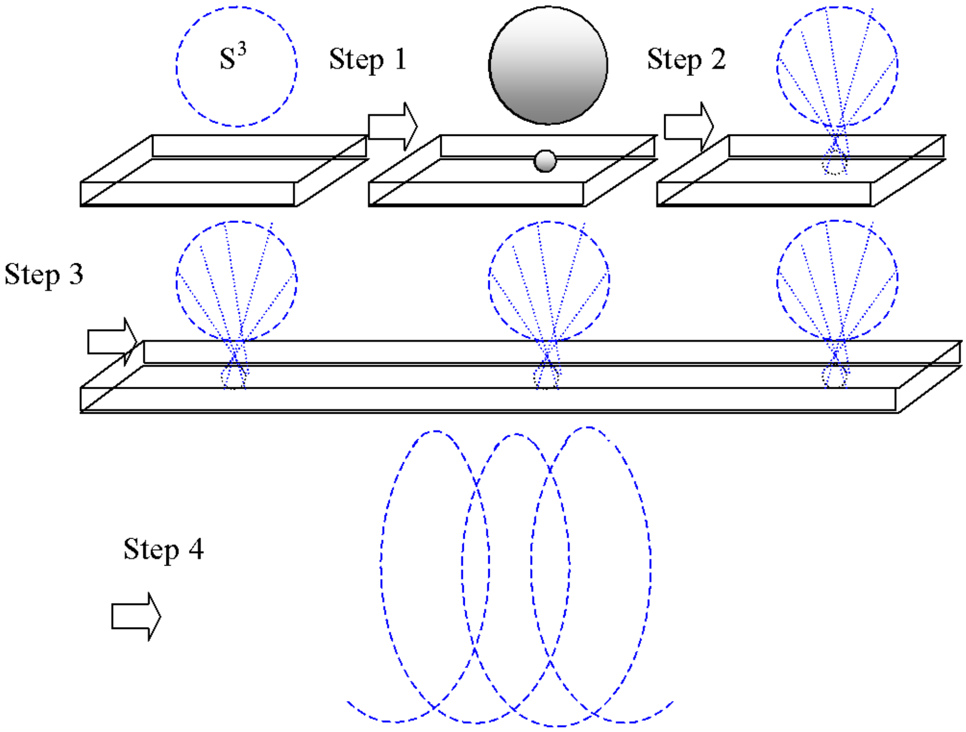

Figure 3.12: Construction of 3-dimensional helical space

We explain the transformation of each step in the following table.

Table 3.6: Construction of 3-dimensional helical space

Step	Method of construction
1	If we remove one point from a 3-sphere, we can get 3-dimensional solid sphere. On the other hand, if we remove one point from 3-dimensional space, we can get a boundary like 2-sphere.
2	We connect their boundaries.
3	If we repeat this process, we can connect many 3-sphere.
4	If we change the orientation of the 3-sphere, we can construct 3-dimensional helical space.

We can express 3-dimensional helical space by the quaternion (1, i, j, k) as follows.

$$t + ix + jy + kz = r\exp(i\theta_1 + j\theta_2 + k\theta_3) \tag{3.29}$$

Here, θ_m is the rotation angle of the helical space. r is a positive real number.

If we combine the both ends, we can get 3-dimensional helical sphere.

$$\begin{aligned} & t + ix + jy + kz \\ & \quad = (r\ \exp(in_1\phi_1 + jn_2\phi_2 + kn_3\phi_3) + R)\exp(j\phi_1 + k\phi_2 + i\phi_3) \end{aligned} \tag{3.30}$$

Here, n_m is integer. ϕ_m is the rotation angle of the major radius of helical circle. R is major radius of helical circle. r is minor radius of helical circle.

We can express the 3-dimensional helical sphere symbolically in the following figure.

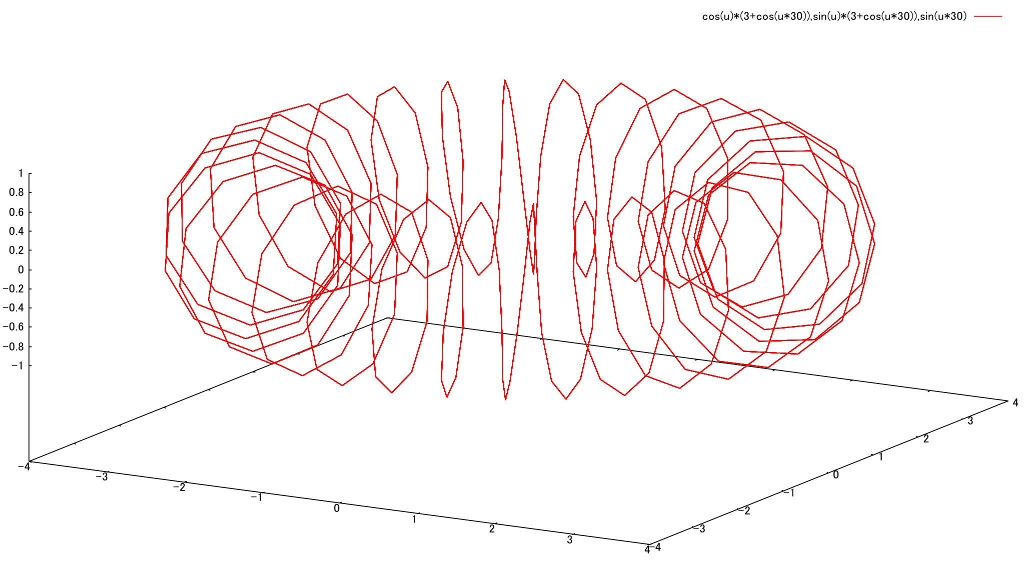

Figure 3.13: 3-dimensional helical sphere

3.2.4 Consideration of 3-dimensional helical space

1-dimensional helical space corresponded to complex. On the other hand, 3-dimensional helical space corresponded to quaternion. I guess 2-dimensional helical space does not exist because triples of numbers do not exist.

We can interpret a position in 3-dimensional helical sphere as the position in normal 3-dimensional space. Therefore, we can interpret an angular momentum in 3-dimensional helical sphere as an angular momentum in normal 3-dimensional space. In other words, we can interpret the spin of the quantum mechanics as the rotation of a particle.

4 Conclusion

In this paper, we derived the following property of the spin.
(1) Two-valuedness of a spin
(2) Angular momentum of a spin

5 Future Issues

Future issues are shown as follows.
- Derivation of Dirac equation

6 Arrangement of Terms

Table 6.1: Spin, etc.

Term	Explanation
Spin	Rotation of the object that contains the axis of rotation.
Spheric spin	Rotation that does not include the inside out circle.
Toric spin	Rotation, including the inside out circle.

Table 6.2: Helical space, etc.

Category	Term
Space	Helical space
Circle	Helical circle
Sphere	Helical sphere

7 References

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