Derivation of the reflection integral equation of the zeta function by the complex analysis

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Abstract

This paper derives the reflection integral equation of the zeta function by the complex analysis.

Many researchers have attempted proof of Riemann hypothesis, but have not been successful. The proof of this Riemann hypothesis has been an important mathematical issue. In this paper, we attempt to derive the reflection integral equation by the complex analysis as preparation proving Riemann hypothesis.

We obtain a generating function of the inverse Mellin-transform. We obtain new generating function by multiplying the generating function with exponents and reversing the sign. We derive the reflection integral equation from inverse *Z*-transform of the generating function.

We derive the summation equation, the asymptotic expansion, Faulhaber's formula, and Nörlund–Rice integral from the reflection integral equation.

CONTENTS

1 Introduction

1.1 Issue

Many researchers have attempted proof of Riemann hypothesis, but have not been successful. The proof of this Riemann hypothesis has been an important mathematical issue. In this paper, we attempt to derive the reflection integral equation by the complex analysis as preparation proving Riemann hypothesis.

1.2 Importance of the issue

Proof of the Riemann hypothesis is one of the most important unsolved problems in mathematics.

For this reason, many mathematicians have tried the proof of Riemann hypothesis. However, those trials were not successful. One of the methods proving Riemann hypothesis is interpreting the zeros of the zeta function as the eigenvalues of a certain operator. However, the operator was not found until now. The reflection integral equation is considered to be one of the operators. For this reason, derivation of the reflection integral equation is an important issue.

1.3 Research trends so far

Leonhard Euler introduced the zeta function in 1737. Bernhard Riemann expanded the argument of the function to the complex number in 1859.

David Hilbert and George Polya[2] suggested that the zeros of the function were probably eigenvalues of a certain operator around 1914. This conjecture is called "Hilbert-Polya conjecture.
Zeev Rudnick and Peter Sarnak[3] are studying the distribution of zeros by random matrix theory in 1996. Shigenobu Kurokawa is studying the field with one element[4] around 1996. Alain Connes[5] showed the relation between noncommutative geometry and the Riemann hypothesis in 1998. Christopher Deninger[6] is studying the eigenvalue interpretation of the zeros in 1998.

1.4 New derivation method of this paper

We obtain a generating function of the inverse Mellin-transform. We obtain new generating function by multiplying the generating function with exponents and reversing the sign. We derive the reflection integral equation from inverse Z-transform of the generating function.

(Reflection integral equation)

$$\zeta(1-s)=\oint_C -B(s,t)\zeta(t)\frac{dt}{2\pi i} \tag{1.1}$$

We derive the summation equation, the asymptotic expansion, Faulhaber's formula, and Nörlund–Rice integral from the reflection integral equation.

(Summation equation)

$$\zeta(s)=\sum_{t=0}^{\infty}\frac{-1}{B(s,t-1)(t-1)}\zeta(1-t) \tag{1.2}$$

2 Confirmations of known results

In this chapter, we confirm known results.

2.1 Mellin transform

Hjalmar Mellin[7] published Mellin transform in 1904.
(Mellin transform)

$$f(s)=M[F(x)] \tag{2.1}$$

$$f(s)=\int_0^{\infty}x^{s-1}F(x)dx \tag{2.2}$$

If the function $f(s)$ is analytic in the strip $S = \{a < \mathrm{Re}(s) < b\}$, and if it tends to zero uniformly as $\mathrm{Im}(s) \to \pm\infty$ for any real value c between a and b the following line integral converges absolutely.
(Inverse Mellin transform)

$$F(z)=M^{-1}[f(s)] \tag{2.3}$$

$$F(z)=\int_{c-i\infty}^{c+i\infty}\frac{f(s)}{z^s}\frac{ds}{2\pi i} \tag{2.4}$$

$$S=\{a<\mathrm{Re}(s)<b\} \tag{2.5}$$

The real part of the strip S needs to be greater than the real part of all poles of the integrand. The strip S is shown below. The white circles mean poles.

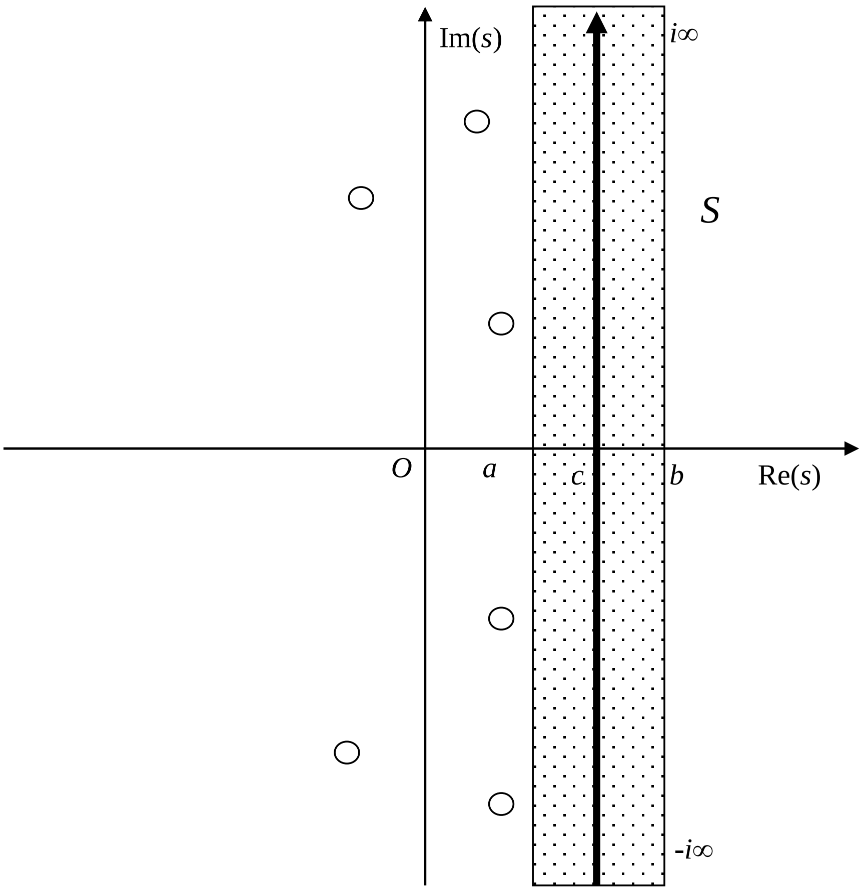

Figure 2.1: The strip *S* of the inverse Mellin transform

We can express the inverse Mellin transform by the following contour integration.
(Inverse Mellin transform)

$$F(z) = M^{-1}[f(s)] \tag{2.6}$$

$$F(z) = \oint_C \frac{f(s)}{z^s} \frac{ds}{2\pi i} \tag{2.7}$$

The circuit of integration *C* circles around all poles of the integrand. For example, we suppose the circuit of integration $C = C_I + C_R$ as follows. The white circles mean poles.

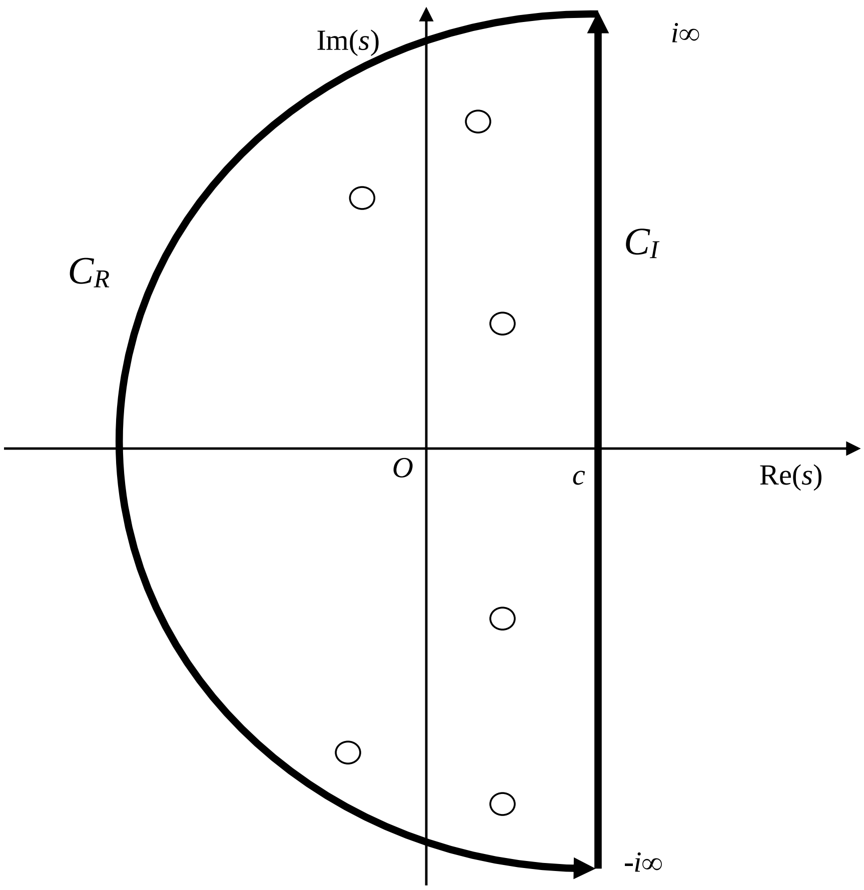

Figure 2.2: The circuit of integration of the inverse Mellin transform

If the line integration of the path C_R is 0, the contour integration of the circuit of integration C equals the line integration of the path C_I. Then we have the following formula.

$$\oint_C \frac{f(s)}{z^s}\frac{ds}{2\pi i} = \int_{c-i\infty}^{c+i\infty} \frac{f(s)}{z^s}\frac{ds}{2\pi i} \tag{2.8}$$

2.2 Hurewicz's *Z*-transform

Witold Hurewicz [8] published *Z*-transform in 1947. When the function $F(z)$ is holomorphic over the domain $D = \{0<|z|< R\}$, the function can be transformed to the series which converges uniformly in wider sense over the domain.

(*Z*-transform)

$$F(z) = Z[f(n)] \tag{2.9}$$

$$F(z) = \sum_{n=-\infty}^{\infty} \frac{f(n)}{z^n} \tag{2.10}$$

$$D = \{0 < |z| < R\} \tag{2.11}$$

Therefore, when the minimum distance between origin and poles is R the domain D of the Z-transform is shown below.

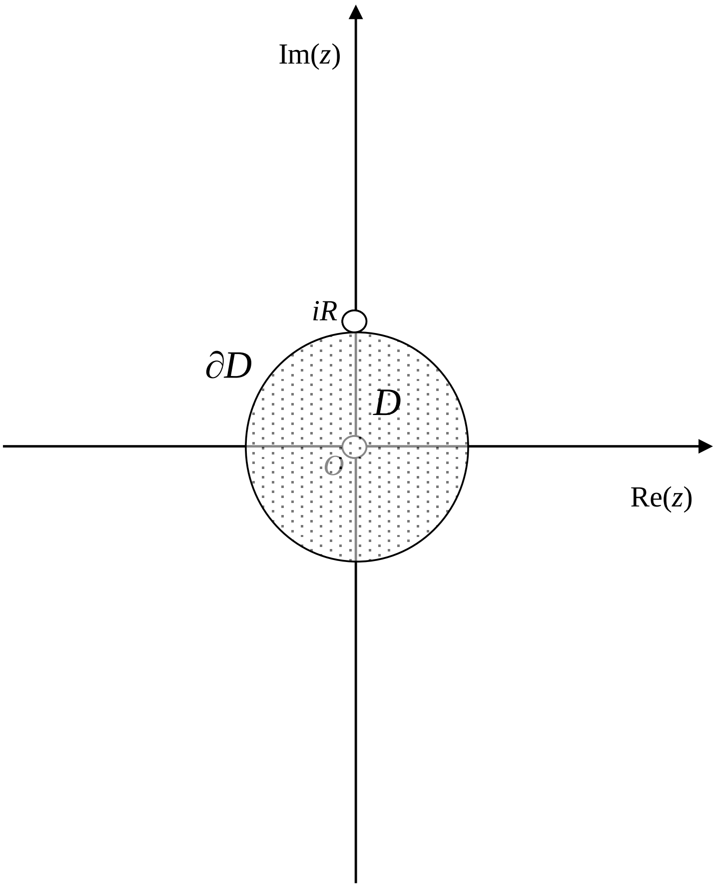

Figure 2.3: The domain of the Z-transform

The inverse Z-transform is the contour integration along the circuit of integration ∂D circles the domain D.

(Inverse Z-transform)

$$f(n) = Z^{-1}[F(z)] \tag{2.12}$$

$$f(n) = \oint_{\partial D} z^{n-1} F(z) \frac{dz}{2\pi i} \tag{2.13}$$

2.3 Cauchy's residue theorem

Augustin-Louis Cauchy published residue theorem[9] in 1831.

We suppose that the function $F(z)$ has isolated singularities c_k on the domain D inside of the simple closed curve ∂D and is holomorphic on both the domain D and the closed curve ∂D except for the isolated singularities. Then, we have the following formula.

(Residue theorem)

$$\oint_{\partial D} F(z)\frac{dz}{2\pi i} = \sum_{k=1}^{n} \operatorname*{Res}_{z=c_k} F(z)dz \tag{2.14}$$

2.4 Euler's gamma function

Leonhard Euler[10] introduced the gamma function as a generalization of the factorial in 1729.
The gamma function is defined by the following equation.
(Definitional integral formula of the gamma function)

$$\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}dx \tag{2.15}$$

Hermann Hankel published the following integral representation[11] in 1863.
(Contour integration of gamma function)

$$\frac{1}{\Gamma(1-s)} = \oint_\gamma z^{s-1}e^z \frac{dz}{2\pi i} \tag{2.16}$$

The integral path of gamma function is the path γ in the following figure. The white circles mean poles.

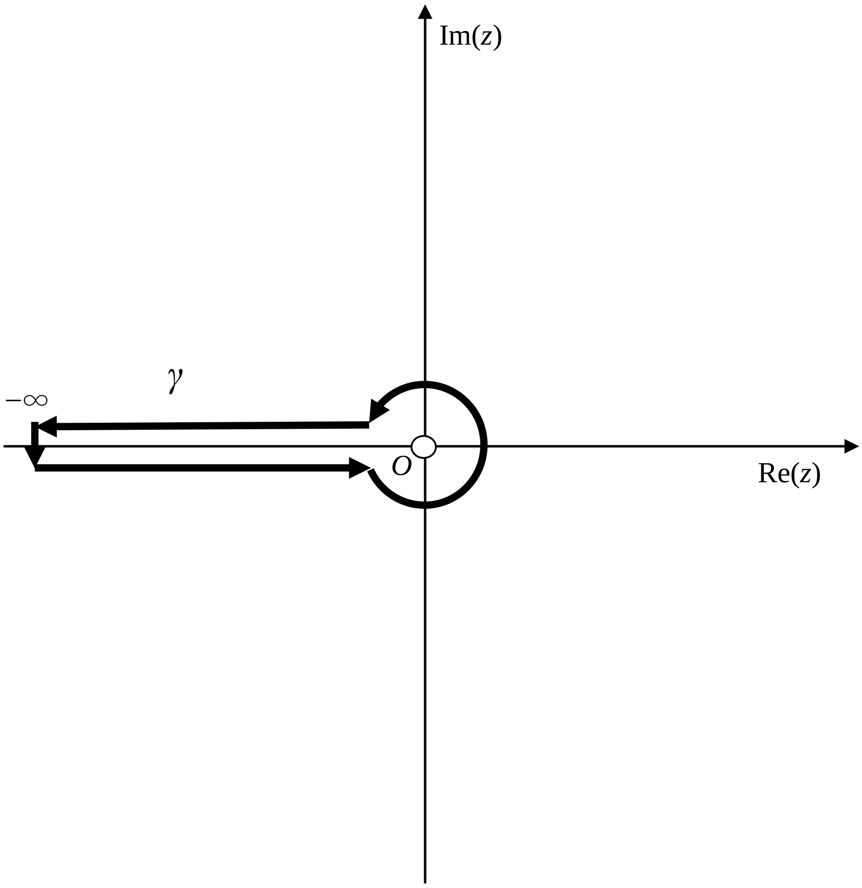

Figure 2.4: The integral path of gamma function

The gamma function has the following formula.
(Euler's reflection formula)

$$\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} \frac{1}{\Gamma(s)} \tag{2.17}$$

The gamma function satisfies the following equations.

$$n! = \Gamma(n+1) \tag{2.18}$$

$$\Gamma(s+1) = s\Gamma(s) \tag{2.19}$$

The gamma function has the following definitional integral formula.

$$\frac{\Gamma(s)}{q^s} = \int_0^\infty x^{s-1} e^{-qx} dx \tag{2.20}$$

2.5 Euler's beta function

Leonhard Euler introduced the beta function in 1768 in his book[12]. We can express the Beta function by using the gamma functions.
(Definitional formula of the beta function)

$$B(s,t) = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)} \tag{2.21}$$

We can obtain the following reflection formula of Beta function from reflection formula of the gamma function.

(Reflection formula of Beta function)

$$B(1-s,1-t) = \frac{\pi \sin(\pi(s+t-1))}{\sin(\pi s)\sin(\pi t)} \frac{1}{B(s,t-1)(t-1)} \tag{2.22}$$

2.6 Riemann zeta function

Bernhard Riemann[13] expanded the argument of the zeta function to the complex number in 1859.

The definitional series of the function is shown below.

(The definitional series)

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots = \sum_{k=1}^{\infty} \frac{1}{k^s} \tag{2.23}$$

The function is also defined by the following formula.

(Definitional integral formula)

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} \frac{e^{-x}}{1-e^{-x}} dx \tag{2.24}$$

We can interpret the above formula as the following Mellin transform.

(Mellin transform)

$$g(s) = M[G(z)] \tag{2.25}$$

$$g(s) = \int_0^{\infty} x^{s-1} G(x) dx \tag{2.26}$$

$$G(z) = \frac{e^{-z}}{1-e^{-z}} \tag{2.27}$$

$$g(s) = \zeta(s)\Gamma(s) \tag{2.28}$$

The inverse Mellin transform of the function is shown below.

(Inverse Mellin transform)

$$G(z) = M^{-1}[g(s)] \tag{2.29}$$

$$G(z) = \oint_C \frac{g(s)}{z^s} \frac{ds}{2\pi i} \tag{2.30}$$

$$G(z) = \frac{e^{-z}}{1 - e^{-z}} \tag{2.31}$$

$$g(s) = \zeta(s)\Gamma(s) \tag{2.32}$$

The contour integration of the function is shown below.
(The contour integration)

$$\frac{\zeta(s)}{\Gamma(1-s)} = \oint_\gamma z^{s-1} \frac{e^z}{1-e^z} \frac{dz}{2\pi i} \tag{2.33}$$

We can interpret the above formula as the following the inverse *Z*-transform.
(Inverse *Z*-transform)

$$h(s) = Z^{-1}[H(z)] \tag{2.34}$$

$$h(s) = \oint_\gamma z^{s-1} H(z) \frac{dz}{2\pi i} \tag{2.35}$$

$$H(z) = \frac{e^z}{1-e^z} \tag{2.36}$$

$$h(s) = \frac{\zeta(s)}{\Gamma(1-s)} \tag{2.37}$$

The integral path γ is shown in the following figure. The white circles mean poles.

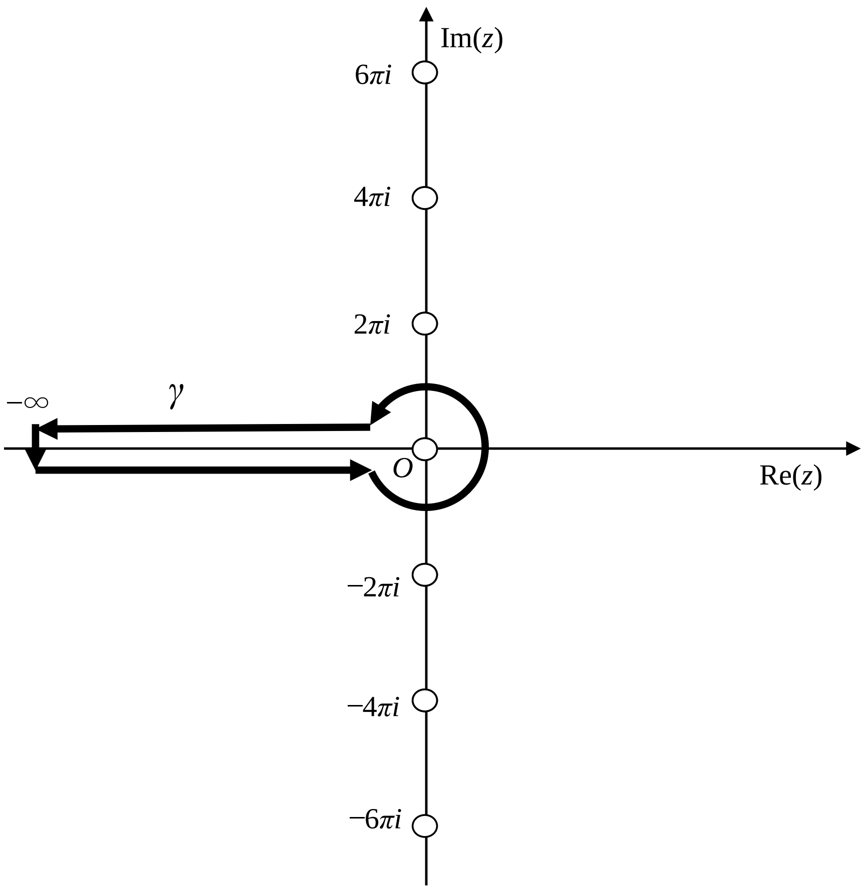

Figure 2.5: The integral path of zeta function

The *Z*-transform of the function as follows.
(*Z*-transform)

$$H(z) = Z[h(s)] \tag{2.38}$$

$$H(z) = \sum_{s=-\infty}^{\infty} \frac{h(s)}{z^s} \tag{2.39}$$

$$H(z) = \frac{e^z}{1-e^z} \tag{2.40}$$

$$h(s) = \frac{\zeta(s)}{\Gamma(1-s)} \tag{2.41}$$

The generating functions of Mellin transform and *Z*-transform have the following relations.

$$H(z) = -e^z G(z) \tag{2.42}$$

$$H(z) = G(-z) \tag{2.43}$$

Riemann showed the following formula.
(Riemann's reflection formula)

$$\zeta(1-s) = \frac{2}{(2\pi)^s}\Gamma(s)\cos\left(\frac{\pi}{2}s\right)\zeta(s) \tag{2.44}$$

Riemann proposed the following conjecture.
(Riemann hypothesis)
Nontrivial zeros all have real part 1/2.

We express the examples of nontrivial zeros ρ_1 and ρ_2 in the following figure and equation. The black circles are zeros and the white circle means a pole.

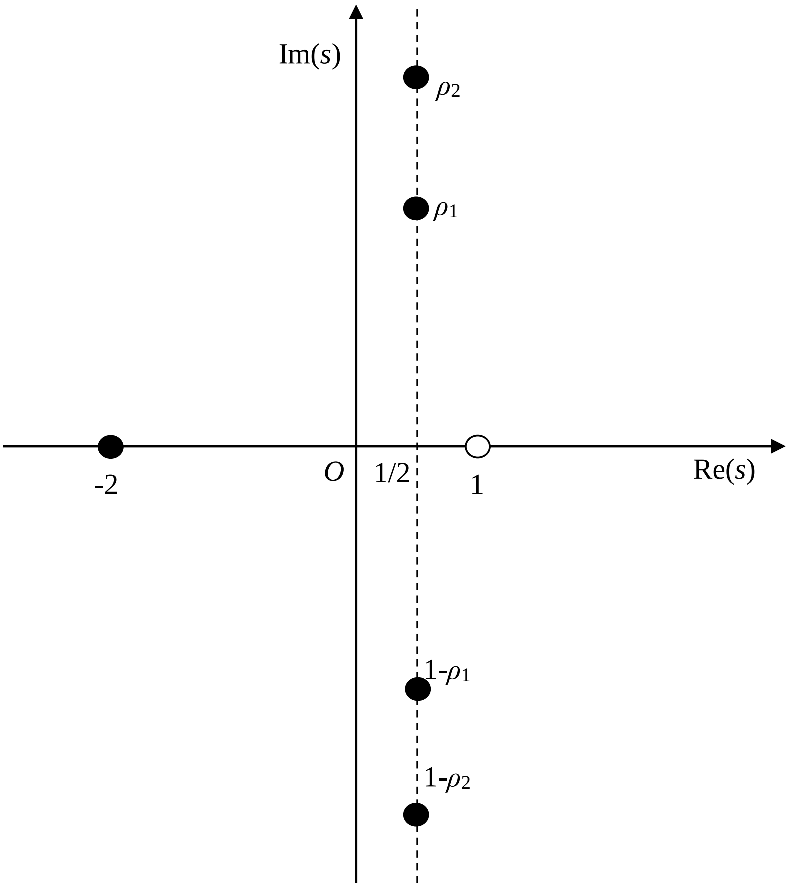

Figure 2.6: Nontrivial zeros of zeta function

$$\rho_1 = \frac{1}{2} + i\,(14.13\cdots) \tag{2.45}$$

$$\rho_2 = \frac{1}{2} + i\,(21.02\cdots) \tag{2.46}$$

Since the proof of the Riemann hypothesis has not been successful, it has been an important mathematical issue.

2.7 Bernoulli polynomials

Jakob Bernoulli introduced Bernoulli numbers in 1713 in his book[14]. Seki Takakazu also introduced Bernoulli numbers in 1712 in his book[15] independently. Bernoulli numbers are defined by Bernoulli polynomials. The definition of Bernoulli polynomials is shown below.

(Bernoulli polynomials)

$$\frac{xe^{qx}}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n(q)}{n!} x^n \tag{2.47}$$

The above series are called "formal power series" because it does not converge over the whole domain. The convergent radius is 2π because the minimum distance between origin and poles is 2π for the generating function.

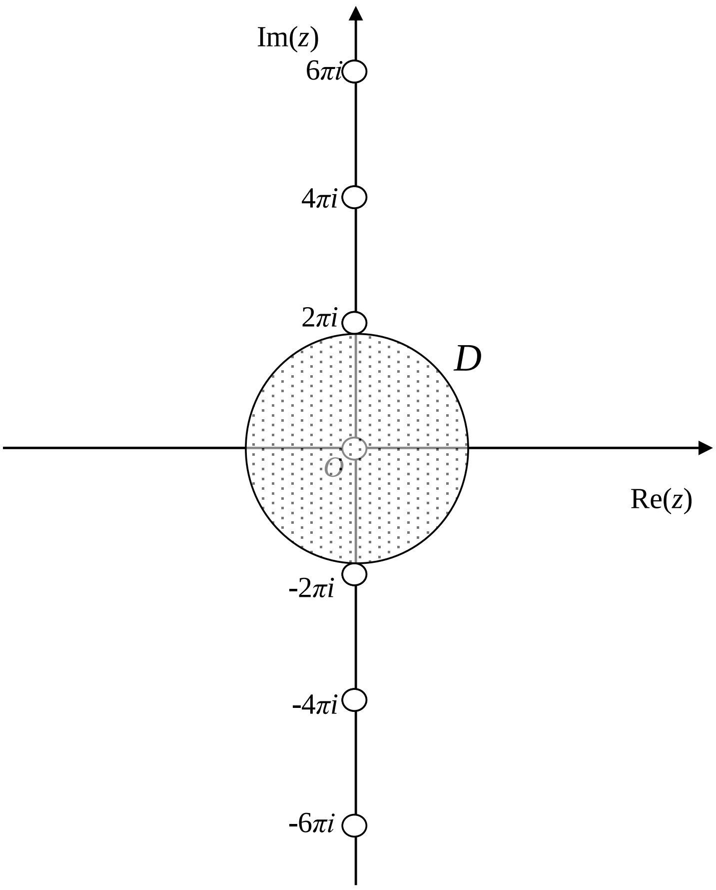

Figure 2.7: The convergent radius of Bernoulli polynomials

2.8 Bernoulli numbers

We suppose that $B_n(q)$ is Bernoulli polynomials. There are the following two kinds of definitions of Bernoulli numbers B_n.

$$B_n = B_n(0) \tag{2.48}$$

$$B_n = B_n(1) \tag{2.49}$$

In this paper, in order to unite with the definition of Bernoulli function explained later, the latter definition is adopted. At the former and the latter, there is the following difference by $n = 1$.

$$B_n(0) = -1/2 \tag{2.50}$$

$$B_n(1) = 1/2 \tag{2.51}$$

Bernoulli polynomials $B_n(1)$ equals to $B_n(0)$ except $n = 1$. The definition of Bernoulli numbers is shown below.

(Definitional series of Bernoulli numbers)

$$\frac{xe^x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n \tag{2.52}$$

Bernoulli numbers has the following formula for even positive integer n.

(Reflection formula of Bernoulli numbers)

$$\zeta(n) = \frac{(2\pi)^n}{2} \frac{1}{n!} (-1)^{\frac{n}{2}+1} B_n \tag{2.53}$$

Bernoulli numbers has the following formula for natural number n.

(Formula of Bernoulli numbers)

$$\zeta(-n) = -\frac{B_{n+1}}{n+1} \tag{2.54}$$

According to Vich's book[16], we can express Bernoulli numbers by Z-transform as follows.

$$\frac{(1/z)e^{1/z}}{e^{1/z} - 1} = Z\left[\frac{B_n}{n!}\right] \tag{2.55}$$

In this paper, we express the Z-transform of Bernoulli numbers as shown below.

$$H(z) = Z[h(s)] \tag{2.56}$$

$$H(z) = \sum_{s=-\infty}^{\infty} \frac{h(s)}{z^s} \tag{2.57}$$

$$H(z) = \frac{e^z}{1-e^z} \tag{2.58}$$

$$h(s) = \frac{-B_{1-s}}{\Gamma(-s)} \tag{2.59}$$

3 Derivation of the reflection integral equation

3.1 The framework of the method to derivation

The inverse Mellin transform of zeta function is shown below.

$$G(z) = M^{-1}[g(s)] \tag{3.1}$$

$$G(z) = \frac{g(s)}{z^s}\frac{ds}{2\pi i} \tag{3.2}$$

$$G(z) = \frac{e^{-z}}{1-e^{-z}} \tag{3.3}$$

$$g(s) = \zeta(s)\Gamma(s) \tag{3.4}$$

The inverse *Z*-transform of the function is shown below.

$$h(s) = Z^{-1}[H(z)] \tag{3.5}$$

$$h(s) = \oint_{\gamma} z^{s-1} H(z) \frac{dz}{2\pi i} \tag{3.6}$$

$$H(z) = \frac{e^z}{1-e^z} \tag{3.7}$$

$$h(s) = \frac{\zeta(s)}{\Gamma(1-s)} \tag{3.8}$$

The generating functions of Mellin transform and *Z*-transform have the following relations.

$$H(z) = -e^{z} G(z) \tag{3.9}$$

$$H(z) = G(-z) \tag{3.10}$$

The framework of the method to derivation is shown below.

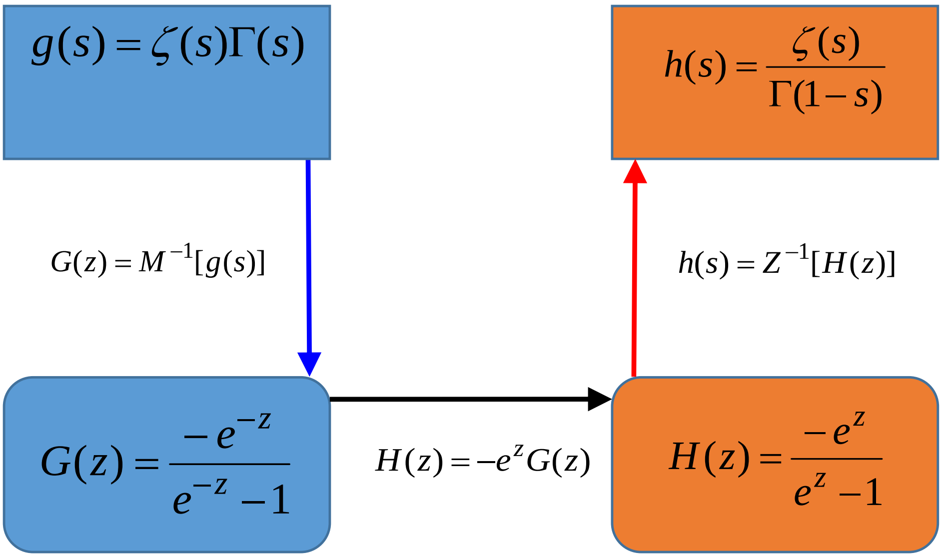

Figure 3.1: The framework of the method to derivation

We can obtain the reflection integral equation by the anticlockwise path.
(Reflection integral equation)

$$\zeta(1-s) = \oint_C -B(s,t)\zeta(t)\frac{dt}{2\pi i} \tag{3.11}$$

This paper explains this derivation method.

3.2 Derivation of the reflection integral equation from the inverse Mellin transform

Inverse Mellin transform of the zeta function is shown below.

$$G(z) = M^{-1}[g(t)] \tag{3.12}$$

$$G(z) = \oint_C \frac{g(t)}{z^t} \frac{dt}{2\pi i} \tag{3.13}$$

$$G(z) = \frac{e^{-z}}{1-e^{-z}} \tag{3.14}$$

$$g(t) = \zeta(t)\Gamma(t) \tag{3.15}$$

The circuit of integration *C* of the inverse Mellin transform needs to circle around all poles of the integrand. Then we adopt the following circuit of integration *C*. The white circles mean poles.

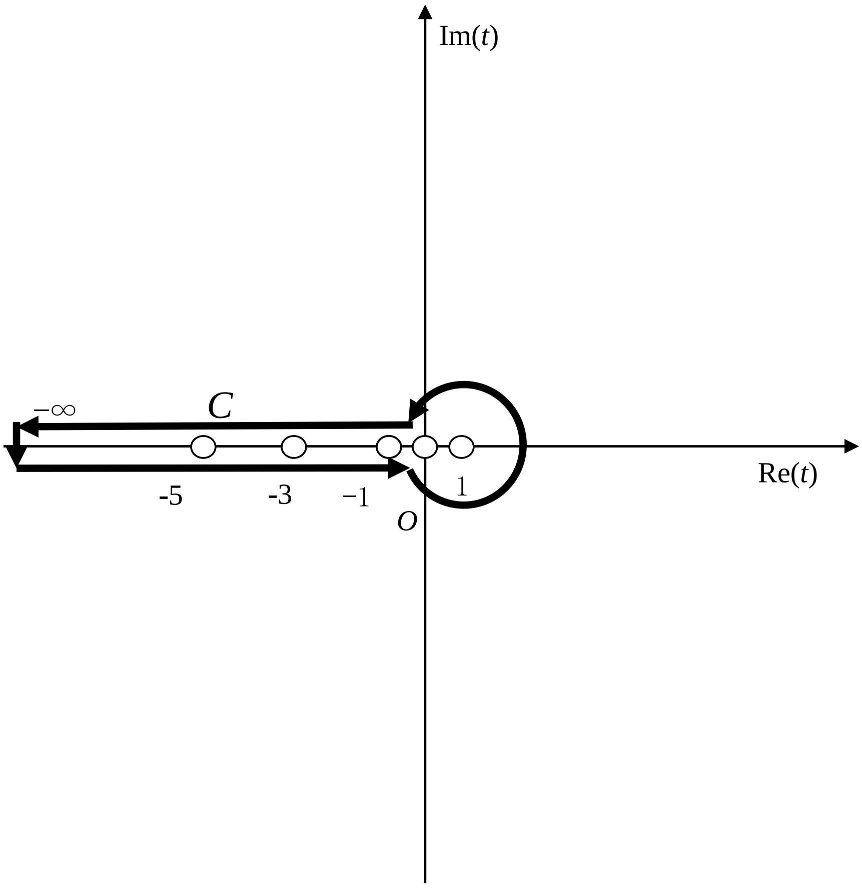

Figure 3.2: The integral path of the inverse Mellin transform

On the other hand, Inverse *Z*-transform of the function is shown below.
(Inverse *Z*-transform)

$$h(s) = Z^{-1}[H(z)] \tag{3.16}$$

$$h(s) = \oint_{\gamma} z^{s-1} H(z) \frac{dz}{2\pi i} \tag{3.17}$$

$$H(z) = \frac{e^z}{1-e^z} \tag{3.18}$$

$$h(s) = \frac{\zeta(s)}{\Gamma(1-s)} \tag{3.19}$$

We can deform the equation of the inverse Z-transform as follows.

$$h(s) = \oint_{\gamma} z^{s-1} \left\{ -e^z G(z) \right\} \frac{dz}{2\pi i} \tag{3.20}$$

We obtain the following equation by substituting the equation of the inverse Mellin transform.

$$h(s) = \oint_{\gamma} z^{s-1} \left\{ -e^z \oint_{C} \frac{g(t)}{z^t} \frac{dt}{2\pi i} \right\} \frac{dz}{2\pi i} \tag{3.21}$$

In order to integrate the above equation for the variable z, we deform the above equation as follows.

$$h(s) = \oint_{C} -\left(\oint_{\gamma} z^{s-t-1} e^z \frac{dz}{2\pi i} \right) g(t) \frac{dt}{2\pi i} \tag{3.22}$$

We apply the following formula to the above equation.
(Reflection integral formula of the gamma function)

$$\frac{1}{\Gamma(1-s)} = \oint_{\gamma} z^{s-1} e^z \frac{dz}{2\pi i} \tag{3.23}$$

Then we can get the following equation.

$$\frac{\zeta(s)}{\Gamma(1-s)} = \oint_{C} -\left(\frac{1}{\Gamma(1-s+t)} \right) \Gamma(t) \zeta(t) \frac{dt}{2\pi i} \tag{3.24}$$

Here, we simplify the above equation by using the following the beta function.

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{3.25}$$

As the result, we can obtain the following equation.
(Reflection integral equation)

$$\zeta(1-s)=\oint_C -B(s,t)\zeta(t)\frac{dt}{2\pi i} \tag{3.26}$$

Here, we replaced *s* with 1-*s*.
The integral path is the path *C* in the following figure. The white circles mean poles.

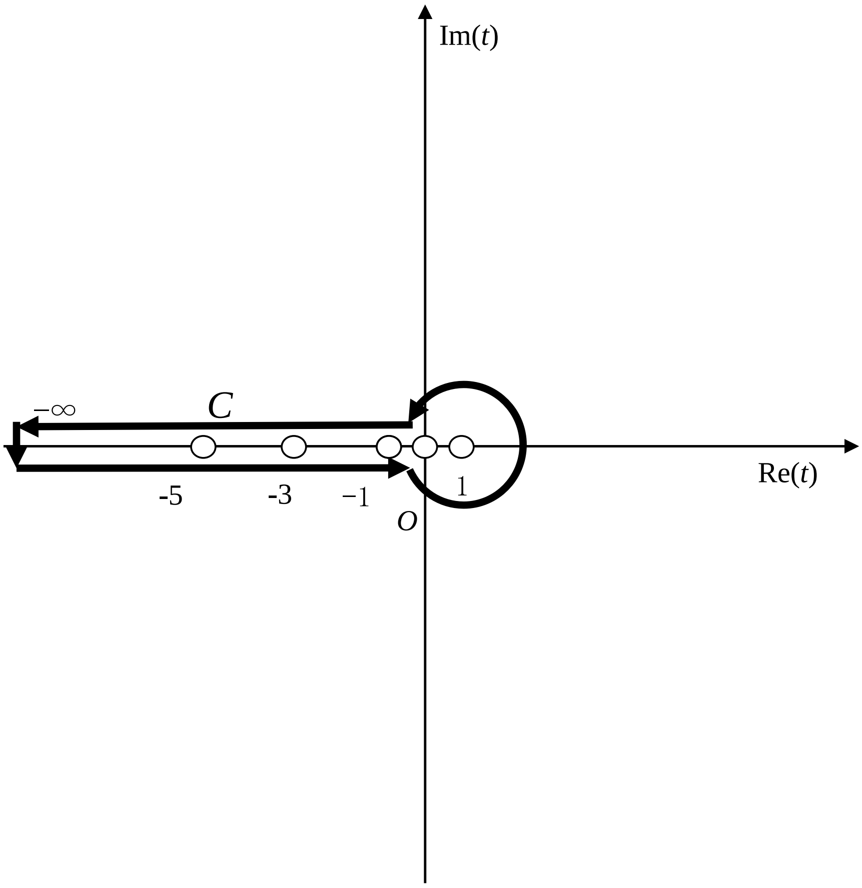

Figure 3.3: The integral path of the reflection integral equation of zeta function

4 Derivation of summation equation

We derive the summation equation from the reflection integral equation and residue theorem.

The reflection integral equation is shown below.
(Reflection integral equation)

$$\zeta(1-s)=\oint_C -B(s,t)\zeta(t)\frac{dt}{2\pi i} \tag{4.1}$$

We replace the variable *s* to 1-*s* and
replace the variable *t* to 1-*t* in the above equation.

$$\zeta(s)=\oint_{C_0} B(1-s,1-t)\zeta(1-t)\frac{dt}{2\pi i} \tag{4.2}$$

The integral path C_0 is shown in the following figure. The white circles mean poles.

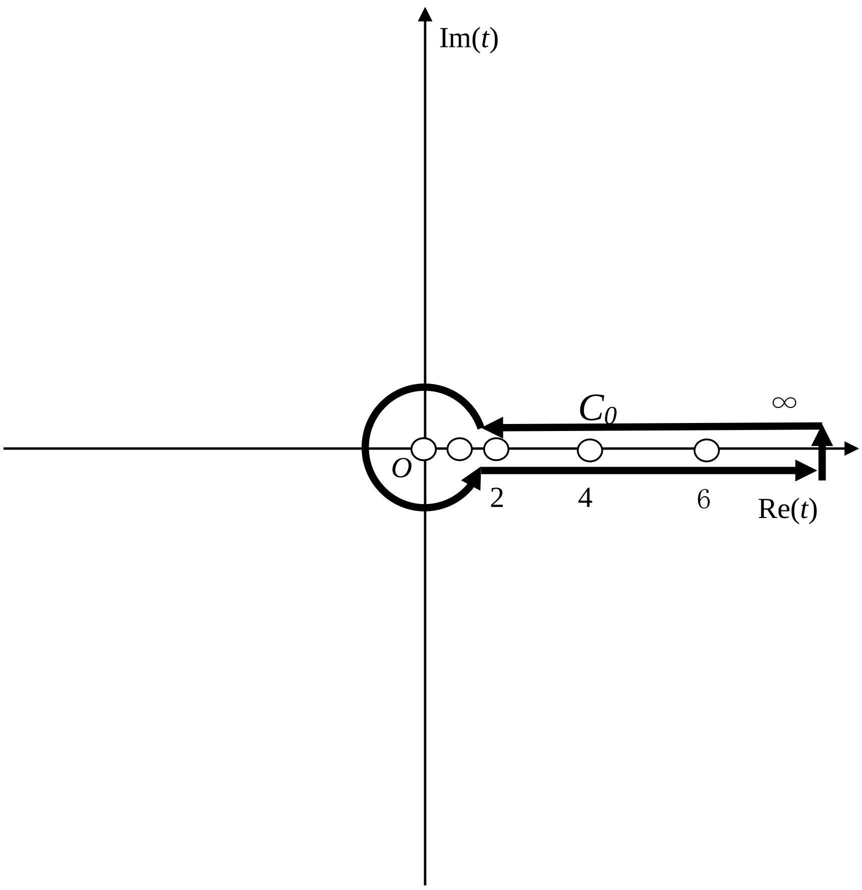

Figure 4.1: The integral path of the reflection integral equation of zeta function

We can obtain the following reflection formula of Beta function from reflection formula of the gamma function.

(Reflection formula of Beta function)

$$B(1-s,1-t)=\frac{\pi\sin(\pi(s+t-1))}{\sin(\pi s)\sin(\pi t)}\frac{1}{B(s,t-1)(t-1)} \tag{4.3}$$

Therefore, we can deform the reflection integral equation as follows.

$$\zeta(s)=\oint_{C_0}\frac{\pi\sin(\pi(s+t-1))}{\sin(\pi s)\sin(\pi t)}\frac{\zeta(1-t)}{B(s,t-1)(t-1)}\frac{dt}{2\pi i} \tag{4.4}$$

We can calculate the above integration by residue theorem as follows.

$$\zeta(s)=\sum_{k=1}^{\infty}\operatorname*{Res}_{t=c_k}\frac{\pi\sin(\pi(s+t-1))}{\sin(\pi s)\sin(\pi t)}\frac{\zeta(1-t)dt}{B(s,t-1)(t-1)} \tag{4.5}$$

Here, c_k is k-th pole. The singularities are 0, 1, 2, 4, 6, …

We have the following equation for integer n.

$$\operatorname*{Res}_{t=n} \frac{\pi \sin(\pi(s+t-1))}{\sin(\pi s)\sin(\pi t)} dt = -1 \tag{4.6}$$

We can express derive the following equation since all singularities are integer and the function ζ (1-t) is 0 for t=3, 5, 7, ···.
(Summation equation)

$$\zeta(s) = \sum_{t=0}^{\infty} \frac{-1}{B(s,t-1)(t-1)} \zeta(1-t) \tag{4.7}$$

We cannot calculate by the above equation because it is divergent. In order to solve the problem, we derive asymptotic expansion of Hurwitz zeta function.

5 Confirmations of known results (Part 2)

In this chapter, we confirm known results.

5.1 Hurwitz zeta function

Adolf Hurwitz[17] introduced the following generalized zeta function in 1882.
(Definitional series)

$$\zeta(s,q) = \frac{1}{(q+0)^s} + \frac{1}{(q+1)^s} + \frac{1}{(q+2)^s} + \cdots = \sum_{k=0}^{\infty} \frac{1}{(q+k)^s} \tag{5.1}$$

The relation between Hurwitz and Riemann zeta function is shown below.

$$\zeta(s) = \left\{ \frac{1}{1^s} + \frac{1}{2^s} + \cdots + \frac{1}{(q-1)^s} \right\} + \left\{ \frac{1}{(q+0)^s} + \frac{1}{(q+1)^s} + \cdots \right\} \tag{5.2}$$

$$\zeta(s) = \sum_{k=1}^{q-1} \frac{1}{k^s} + \zeta(s,q) \tag{5.3}$$

Hurwitz zeta function becomes Riemann zeta function when $q = 1$.

$$\zeta(s,1) = \zeta(s) \tag{5.4}$$

Hurwitz zeta function is also defined by the following formula.
(Definitional integral formula)

$$\zeta(s,q) = \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} \frac{-e^{-qx}}{e^{-x}-1} dx \tag{5.5}$$

We can interpret the above formula as the following Mellin transform.
(Mellin transform)

$$g(s,q) = M[G(x,q)] \tag{5.6}$$

$$g(s,q) = \int_0^\infty x^{s-1} G(x,q)\,dx \tag{5.7}$$

$$G(x,q) = \frac{e^{-qx}}{1-e^{-x}} \tag{5.8}$$

$$g(s,q) = \zeta(s,q)\Gamma(s) \tag{5.9}$$

Hurwitz zeta function has the following integration.
(Contour integration)

$$\frac{\zeta(s,q)}{\Gamma(1-s)} = \oint_\gamma z^{s-1} \frac{e^{qz}}{1-e^z} \frac{dz}{2\pi i} \tag{5.10}$$

We can interpret the above formula as the following inverse *Z*-transform.
(Inverse *Z*-transform)

$$h(s,q) = Z^{-1}[H(z,q)] \tag{5.11}$$

$$h(s,q) = \oint_\gamma z^{s-1} H(z,q) \frac{dz}{2\pi i} \tag{5.12}$$

$$H(z,q) = \frac{e^{qz}}{1-e^z} \tag{5.13}$$

$$h(s,q) = \frac{\zeta(s,q)}{\Gamma(1-s)} \tag{5.14}$$

Bernoulli polynomials have the following formula for natural number n.
(Formula of Bernoulli polynomials)

$$\zeta(-n,q) = -\frac{B_{n+1}(q)}{n+1} \tag{5.15}$$

5.2 Euler–Maclaurin formula

Euler[18] discovered the following formula in 1738. After that, Maclaurin[19] also discovered the same formula in 1742 independently.
(Euler–Maclaurin formula)

$$I = \int_p^q f(x)dx \tag{5.16}$$

$$S = \frac{1}{2} f(p) + f(p+1) + \cdots + f(q-1) + \frac{1}{2} f(q) \tag{5.17}$$

$$S - I = \sum_{k=2}^{r} \frac{B_k}{k!} \left(f^{(k-1)}(q) - f^{(k-1)}(p) \right) + R \tag{5.18}$$

In the above formula, R is an error term.

5.3 Asymptotic expansion

Euler[20] calculated the value of the zeta function by Euler–Maclaurin formula in 1755.
(Asymptotic expansion)

$$\zeta(s) = \sum_{k=1}^{q-1} \frac{1}{k^s} + \frac{q^{1-s}}{s-1} + \sum_{k=1}^{r} \frac{B_k}{k!} \frac{(s+k-2)!}{(s-1)! q^{s+k-1}} + R \tag{5.19}$$

In the above formula, R is an error term. Detail method to derive the above formula is shown in the book[21] written by Edwards in 1974.

We can reform the above formula as follows.

$$\zeta(s) = \sum_{k=1}^{q-1} \frac{1}{k^s} + \sum_{k=0}^{r} \frac{B_k}{k!} \frac{\Gamma(s+k-1)}{\Gamma(s) q^{s+k-1}} + R \tag{5.20}$$

Here we used the following equation.

$$\frac{q^{1-s}}{s-1} = \sum_{k=0}^{0} \frac{B_k}{k!} \frac{\Gamma(s+k-1)}{\Gamma(s) q^{s+k-1}} \tag{5.21}$$

5.4 Faulhaber's formula

Johann Faulhaber[22] published the formula of the sum of powers in 1631. We can express the formula for natural number n as below.
(Faulhaber’s formula)

$$\sum_{k=1}^{q} k^n = \frac{1}{n+1} \sum_{k=0}^{n} (-1)^k \binom{n+1}{k} B_k(0) q^{n+1-k} \tag{5.22}$$

We can express the above formula by using Bernoulli polynomial $B_k(1)$ as follows.
(Faulhaber’s formula)

$$\sum_{k=1}^{q-1} k^n = \frac{1}{n+1}\sum_{k=0}^{n}(-1)^k \binom{n+1}{k} B_k(1) q^{n+1-k} \tag{5.23}$$

In this paper, we express the above formula by Bernoulli number B_k as follows.
(Faulhaber's formula)

$$\sum_{k=1}^{q-1} k^n = \frac{1}{n+1}\sum_{k=0}^{n}(-1)^k \binom{n+1}{k} B_k q^{n+1-k} \tag{5.24}$$

5.5 Ramanujan master theorem

Srinivasa Ramanujan obtained the following theorem[23] about 1910.
(Ramanujan master theorem)

$$F(x) = \sum_{n=0}^{\infty} \frac{f(n)}{n!}(-x)^n \tag{5.25}$$

$$f(-s) = \frac{1}{\Gamma(s)}\int_0^{\infty} x^{s-1} F(x)dx \tag{5.26}$$

We have the above equations for the following the Bernoulli number and zeta function.

$$F(x) = \frac{-x}{e^{-x}-1} \tag{5.27}$$

$$f(n) = B_n \tag{5.28}$$

$$f(-s) = s\zeta(s+1) \tag{5.29}$$

This theorem suggests that the following relation.

$$\zeta(1-s) = -\frac{B_s}{s} \tag{5.30}$$

5.6 Woon's introduction of continuous Bernoulli numbers

S. C. Woon[24] introduced continuous Bernoulli numbers in 1997.
Bernoulli numbers has the following formula for natural number *n*.
(Formula of Bernoulli numbers)

$$\zeta(-n) = -\frac{B_{n+1}}{n+1} \tag{5.31}$$

Woon extended these Bernoulli numbers to the continuous Bernoulli function, and showed the following formula for the complex number *s*.

(Formula of Bernoulli function)

$$\zeta(1-s) = -\frac{B(s)}{s} \tag{5.32}$$

In this paper we use the following notation for Bernoulli function based on the notation for Bernoulli numbers.

(Formula of Bernoulli function)

$$\zeta(1-s) = -\frac{B_s}{s} \tag{5.33}$$

We can obtain the following equation by substituting the above formula to "Riemann's reflection formula".

$$-\frac{B_s}{s} = \frac{2}{(2\pi)^s}\Gamma(s)\cos\left(\frac{\pi}{2}s\right)\zeta(s) \tag{5.34}$$

We can obtain the following equation by deforming the above formula.

(Reflection formula of Bernoulli function)

$$\zeta(s) = -\frac{(2\pi)^s}{2}\frac{1}{\Gamma(s+1)}\frac{1}{\cos(\pi s/2)}B_s \tag{5.35}$$

The above formula becomes the following formula for even positive integer *s*.

$$\zeta(s) = \frac{(2\pi)^s}{2}\frac{1}{s!}(-1)^{\frac{s}{2}+1}B_s \tag{5.36}$$

The above formula equals to the following formula for even positive integer *n*.

(Reflection formula of Bernoulli numbers)

$$\zeta(n) = \frac{(2\pi)^n}{2}\frac{1}{n!}(-1)^{\frac{n}{2}+1}B_n \tag{5.37}$$

The above result suggests that the validity of "Formula of Bernoulli function."

5.7 Nörlund–Rice integral

Niels Erik Nörlund[25] published Nörlund–Rice integral in 1924.

(Nörlund–Rice integral)

$$\sum_{k=c}^{n}\binom{n}{k}(-1)^k f(k) = \oint_C B(n+1,-t)\,f(t)\frac{dt}{2\pi i} \tag{5.38}$$

Here the path *C* circles around poles *c*, …, *n* for positive integer *c*. *B*(*x*, *y*) is Euler's Beta function.

Philippe Flajolet[26] published Poisson–Mellin–Newton cycle in 1985 for Nörlund–Rice integral. (Poisson–Mellin–Newton cycle)

$$F(x) = \frac{1}{e^x}\sum_{n=0}^{\infty} a_n x^n \tag{5.39}$$

$$f(s) = \int_0^{\infty} x^{s-1} F(x) dx \tag{5.40}$$

$$a_n = (-1)^n \oint_C \frac{n!\Gamma(t-n)}{\Gamma(t+1)} \frac{f(t)}{\Gamma(-t)} \frac{dt}{2\pi i} \tag{5.41}$$

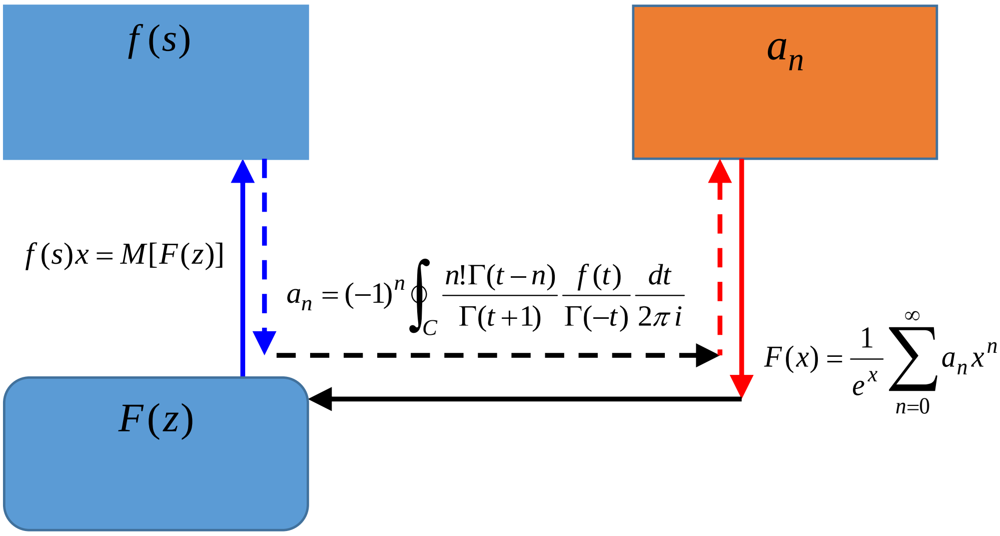

Figure 5.1: Poisson–Mellin–Newton cycle

6 Derivation of asymptotic expansion

We cannot calculate by the summation equation of zeta function because it is divergent. In order to solve the problem, we derive the asymptotic expansion from summation equation of Hurwitz zeta function.

6.1 Relation between Riemann and Hurwitz zeta function

We show the relation between the Riemann and Hurwitz zeta function.

$$\zeta(s)=\sum_{k=1}^{q-1}\frac{1}{k^s}+\zeta(s,q) \tag{6.1}$$

We derive the asymptotic expansion from the above relation.

6.2 Derivation of summation equation of Hurwitz zeta function

Hurwitz zeta function is defined by the following formula.
(Definitional integral formula)

$$\zeta(s,q)=\frac{1}{\Gamma(s)}\int_0^\infty x^{s-1}\frac{-e^{-qx}}{e^{-x}-1}dx \tag{6.2}$$

The formula can be expressed by the generating function of Mellin transform.

$$\zeta(s,q)=\frac{1}{\Gamma(s)}\int_0^\infty x^{s-1}G(x,q)dx \tag{6.3}$$

We can obtain the following equation by deforming the above formula.

$$\zeta(s,q)=\frac{1}{\Gamma(s)}\int_0^\infty x^{s-1}\left\{-e^{-qx}H(x)\right\}dx \tag{6.4}$$

We can obtain the following equation by substituting the equation of *Z*-transform.

$$\zeta(s,q)=\frac{1}{\Gamma(s)}\int_0^\infty x^{s-1}\left\{-e^{-qx}\sum_{t=0}^{\infty}\frac{\zeta(1-t)}{\Gamma(t)}x^{t-1}\right\}dx \tag{6.5}$$

$$D=\{0<|x|<2\pi\} \tag{6.6}$$

The *Z*-transform converges over the domain *D*. Therefore, we can commute the order of the integration and the summation over the domain.

In order to integrate the above equation for the variable *x*, we deform the above equation as follows.

$$\zeta(s,q)=\frac{1}{\Gamma(s)}\sum_{t=0}^{\infty}\frac{-\zeta(1-t)}{\Gamma(t)}\int_0^\infty x^{s+t-2}e^{-qx}dx \tag{6.7}$$

We apply the following equation to the above equation.

$$\frac{\Gamma(s)}{q^s} = \int_0^{\infty} x^{s-1} e^{-qx} dx \tag{6.8}$$

As the result, we can obtain the following equation.

$$\zeta(s,q) = \frac{1}{\Gamma(s)} \sum_{t=0}^{\infty} \frac{-\zeta(1-t)}{\Gamma(t)} \frac{\Gamma(s+t-1)}{q^{s+t-1}} \tag{6.9}$$

Here, we simplify the above equation by using the following the beta function.

$$\frac{1}{B(x,y)} = \frac{\Gamma(x+y)}{\Gamma(x)\Gamma(y)} \tag{6.10}$$

As the result, we can obtain the following equation.
(Summation equation)

$$\zeta(s,q) = \sum_{t=0}^{\infty} \frac{-1}{B(s,t-1)(t-1)} \frac{\zeta(1-t)}{q^{s+t-1}} \tag{6.11}$$

Therefore, we can express the summation equation Riemann zeta function as follows.
(Summation equation)

$$\zeta(s) = \sum_{k=1}^{q-1} \frac{1}{k^s} + \sum_{t=0}^{\infty} \frac{-1}{B(s,t-1)(t-1)} \frac{\zeta(1-t)}{q^{s+t-1}} \tag{6.12}$$

The solution of the above equation reaches an infinite value because the convergent radius of "definitional series of Bernoulli function" is 2π.

However, the numerical calculation of the summation equation is possible if we choose appropriate q and integration domain.

6.3 Derivation of asymptotic expansion

In this section we derive the asymptotic expansion from the summation equation.

We can express the summation equation Riemann zeta function as follows.
(Summation equation)

$$\zeta(s) = \sum_{k=1}^{q-1} \frac{1}{k^s} + \sum_{k=0}^{\infty} \frac{-1}{B(s,k-1)(k-1)} \frac{\zeta(1-k)}{q^{s+k-1}} \tag{6.13}$$

We can obtain the following equation by replacing beta function to gamma function.

$$\zeta(s)=\sum_{k=1}^{q-1}\frac{1}{k^s}+\sum_{k=0}^{\infty}\frac{-\Gamma(s+k-1)}{\Gamma(s)\Gamma(k-1)(k-1)}\frac{\zeta(1-k)}{q^{s+k-1}} \tag{6.14}$$

We can deform the above equation by Formula of Bernoulli polynomials as follows.

$$\zeta(s)=\sum_{k=1}^{q-1}\frac{1}{k^s}+\sum_{k=0}^{\infty}\frac{-\Gamma(s+k-1)}{\Gamma(s)\Gamma(k-1)(k-1)}\frac{-kB_k}{q^{s+k-1}} \tag{6.15}$$

We can deform the above equation as follows.

$$\zeta(s)=\sum_{k=1}^{q-1}\frac{1}{k^s}+\sum_{k=0}^{\infty}\frac{B_k}{k!}\frac{\Gamma(s+k-1)}{\Gamma(s)q^{s+k-1}} \tag{6.16}$$

The solution of the above equation reaches an infinite value because the convergent radius of "definitional series of Bernoulli function" is 2π. Therefore, we change the upper limit of summation to a variable r which depends on the variable q.

$$\zeta(s)=\sum_{k=1}^{q-1}\frac{1}{k^s}+\sum_{k=0}^{r}\frac{B_k}{k!}\frac{\Gamma(s+k-1)}{\Gamma(s)q^{s+k-1}} \tag{6.17}$$

The above equation equals to asymptotic expansion.

7 Derivation of Faulhaber's formula

In this section we derive Faulhaber's formula from the summation equation.

We can express Riemann zeta function as follows for natural number n and integer k. (Summation equation)

$$\zeta(n)=\sum_{k=1}^{q-1}\frac{1}{k^n}+\sum_{k=0}^{\infty}\frac{-1}{B(n,k-1)(k-1)}\frac{\zeta(1-k)}{q^{n+k-1}} \tag{7.1}$$

We can derive the following asymptotic expansion as shown in the previous section. (Asymptotic expansion)

$$\zeta(n)=\sum_{k=1}^{q-1}\frac{1}{k^n}+\sum_{k=0}^{r}\frac{B_k}{k!}\frac{\Gamma(n+k-1)}{\Gamma(n)q^{n+k-1}} \tag{7.2}$$

We replace the variable n to $-n$ in the above equation.

$$\sum_{k=1}^{q-1}k^n=\zeta(-n)-\sum_{k=0}^{r}\frac{B_k}{k!}\frac{\Gamma(-n+k-1)}{\Gamma(-n)}q^{n+1-k} \tag{7.3}$$

We can deform the above equation by Euler's reflection formula as follows.

$$\sum_{n=1}^{q-1} k^n = \zeta(-n) - \sum_{k=0}^{r} \frac{B_k}{k!} \frac{\sin(\pi(n+1))}{\sin(\pi(n-k+2))} \frac{\Gamma(n+1) q^{n+1-k}}{\Gamma(n-k+2)} \tag{7.4}$$

We have the following equation for natural number *n* and integer *k*.

$$\frac{\sin(\pi(n+1))}{\sin(\pi(n-k+2))} = -(-1)^k \tag{7.5}$$

Therefore, we can obtain the following equation.

$$\sum_{n=1}^{q-1} k^n = \zeta(-n) + \sum_{k=0}^{r} \frac{B_k}{k!} (-1)^k \frac{\Gamma(n+1) q^{n+1-k}}{\Gamma(n-k+2)} \tag{7.6}$$

We have the following equation for natural number *n* and integer $k \geq n+2$.

$$\frac{1}{\Gamma(n-k+2)} = 0 \tag{7.7}$$

Therefore, we can obtain the following equation.

$$\sum_{k=n+2}^{r} \frac{B_k}{k!} (-1)^k \frac{\Gamma(n+1) q^{n+1-k}}{\Gamma(n-k+2)} = 0 \tag{7.8}$$

According to the above result, we can change the upper limit of summation to n+1 of the equation (7.6).

$$\sum_{n=1}^{q-1} k^n = \zeta(-n) + \sum_{k=0}^{n+1} \frac{B_k}{k!} (-1)^k \frac{\Gamma(n+1) q^{n+1-k}}{\Gamma(n-k+2)} \tag{7.9}$$

We can express the following equation by using the factorial.

$$\sum_{k=1}^{q-1} k^n = \zeta(-n) + \sum_{k=0}^{n+1} (-1)^k \frac{n! B_k q^{n+1-k}}{k!(n-k+1)!} \tag{7.10}$$

We can express the following equation by using the binomial coefficient.

$$\sum_{k=1}^{q-1} k^n = \zeta(-n) + \frac{1}{n+1} \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} B_k q^{n+1-k} \tag{7.11}$$

We can deform the above equation by Formula of Bernoulli polynomials as follows.

$$\sum_{k=1}^{q-1} k^n = -\frac{B_{n+1}}{n+1} + \frac{1}{n+1}\sum_{k=0}^{n+1}(-1)^k \binom{n+1}{k} B_k q^{n+1-k} \tag{7.12}$$

We have the following equation for natural number n and integer $k = n+1$.

$$\frac{B_{n+1}}{n+1} = \frac{1}{n+1}(-1)^k \binom{n+1}{k} B_k q^{n+1-k} \tag{7.13}$$

Therefore, we can obtain the following equation.

$$-\frac{B_{n+1}}{n+1} + \frac{1}{n+1}\sum_{k=n+1}^{n+1}(-1)^k \binom{n+1}{k} B_k q^{n+1-k} = 0 \tag{7.14}$$

According to the above result, we can deform the equation as follows.

$$\sum_{k=1}^{q-1} k^n = \frac{1}{n+1}\sum_{k=0}^{n}(-1)^k \binom{n+1}{k} B_k q^{n+1-k} \tag{7.15}$$

The above equation equals to Faulhaber's formula.

8 Derivation of Nörlund–Rice integral

8.1 Derivation of the reflection integral formula

We suppose new function $H(z)$ for arbitrary function $G(z)$ as follows.

$$H(z) = -e^z G(z) \tag{8.1}$$

We obtain new function $g(s)$ from Mellin transform of the function $G(z)$.

$$g(s) = M[G(z)] \tag{8.2}$$

The inverse Mellin transform is shown below.

$$G(z) = M^{-1}[g(s)] \tag{8.3}$$

We obtain new function $h(s)$ from the inverse Z-transform of the function $H(z)$.

$$h(s) = Z^{-1}[H(z)] \tag{8.4}$$

The relation of the above functions is shown below.

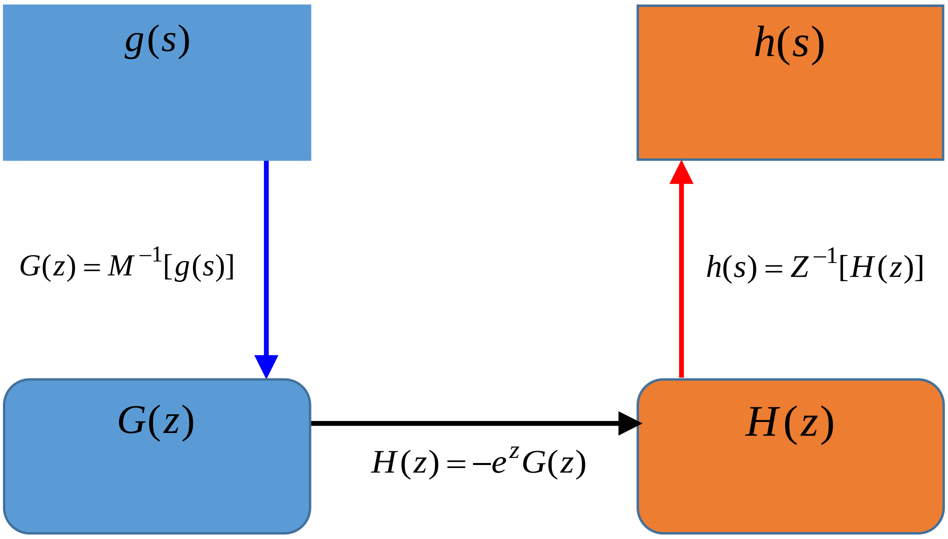

Figure 8.1: The inverse Mellin transform and the inverse *Z*-transform

The formula of the inverse *Z*-transform is shown below.

$$h(s)=\oint_{\gamma} z^{s-1}H(z)\frac{dz}{2\pi i} \tag{8.5}$$

We can deform the formula of the inverse *Z*-transform as follows.

$$h(s)=\oint_{\gamma} z^{s-1}\left\{-e^zG(z)\right\}\frac{dz}{2\pi i} \tag{8.6}$$

We substitute the formula of the inverse Mellin transform into the above formula.

$$h(s)=\oint_{\gamma} z^{s-1}\left\{-e^z\oint_C \frac{g(t)}{z^t}\frac{dt}{2\pi i}\right\}\frac{dz}{2\pi i} \tag{8.7}$$

We deform the above formula in order to integrate it by the variable *z*.

$$h(s)=\oint_C -\left\{\oint_{\gamma} z^{s-t-1}e^z\frac{dz}{2\pi i}\right\}g(t)\frac{dt}{2\pi i} \tag{8.8}$$

We apply the following contour integration of gamma function to the above equation.

$$\frac{1}{\Gamma(1-s)}=\oint_{\gamma} z^{s-1}e^z\frac{dz}{2\pi i} \tag{8.9}$$

As the result, we can obtain the following equation.

$$h(s) = \oint_C \frac{-1}{\Gamma(1-s+t)} g(t) \frac{dt}{2\pi i} \tag{8.10}$$

We define new functions $\phi(s)$ and $\chi(s)$ as follows.

$$\phi(s) = \frac{g(s)}{\Gamma(s)} \tag{8.11}$$

$$\chi(s) = h(s)\Gamma(1-s) \tag{8.12}$$

Then we can deform the above formula as follows.

$$\chi(s) = \oint_C \frac{-\Gamma(1-s)\Gamma(t)}{\Gamma(1-s+t)} \phi(t) \frac{dt}{2\pi i} \tag{8.13}$$

We can express the above formula by Euler's Beta function as follows.
(Reflection integral formula)

$$\chi(s) = \oint_C -B(1-s,t)\phi(t) \frac{dt}{2\pi i} \tag{8.14}$$

8.2 Derivation of the summation formula

We can obtain the summation formula by adopting residue theorem to the contour integral
(Summation formula)

$$\chi(s) = \sum_{t=c+1}^{m+1} \frac{-1}{B(s,t-1)(t-1)} \phi(1-t) \tag{8.15}$$

Here the path C circles around poles c+1, …, m+1 for positive integer d. $B(x, y)$ is Euler's Beta function.

8.3 Derivation of Nörlund–Rice integral

We can derive Nörlund–Rice integral from the reflection integral formula and the summation formula.

The summation formula is shown below.
(Summation formula)

$$\chi(s) = \sum_{t=c+1}^{m+1} \frac{-1}{B(s,t-1)(t-1)} \phi(1-t) \tag{8.16}$$

We replace the variable s to $-n$ and the variable t to k +1 in the above equation.

$$\chi(-n) = \sum_{k=c}^{m} \frac{-1}{B(-n,k)(k)} \phi(-k) \tag{8.17}$$

Then we introduce the following new function $f(k)$.

$$f(k) = \phi(-k) \tag{8.18}$$

We can express the formula by the function $f(k)$.

$$\chi(-n) = \sum_{k=c}^{m} \frac{-1}{B(-n,k)(k)} f(k) \tag{8.19}$$

We deform the above equation by Euler's gamma function.

$$\chi(-n) = \sum_{k=c}^{m} \frac{-\Gamma(-n+k)}{\Gamma(-n)\Gamma(k+1)} f(k) \tag{8.20}$$

We obtain the following formula by using Euler's reflection formula.

$$\chi(-n) = \sum_{k=c}^{m} \frac{-\Gamma(n+1)}{\Gamma(n-k+1)\Gamma(k+1)} \frac{\sin\left(\pi(n+1)\right)}{\sin\left(\pi(n-k+1)\right)} f(k) \tag{8.21}$$

We have the following equation for natural number n and integer k.

$$\frac{\sin\left(\pi(n+1)\right)}{\sin\left(\pi(n-k+1)\right)} = (-1)^k \tag{8.22}$$

We can obtain the following formula.

$$\chi(-n) = \sum_{k=c}^{m} \frac{-\Gamma(n+1)}{\Gamma(n-k+1)\Gamma(k+1)} (-1)^k f(k) \tag{8.23}$$

We have the following equation for natural number n and integer $k \geq n+1$

$$\frac{1}{\Gamma(n-k+1)} = 0 \tag{8.24}$$

Therefore we can change the variable m to n.

$$\chi(-n) = \sum_{k=c}^{n} \frac{-\Gamma(n+1)}{\Gamma(n-k+1)\Gamma(k+1)} (-1)^k f(k) \tag{8.25}$$

We can express the following equation by using the factorial.

$$\chi(-n)=\sum_{k=c}^{n}(-1)^k\,\frac{-n!}{(n-k)!k!}f(k) \tag{8.26}$$

We can express the following equation by using the binomial coefficient.

$$-\chi(-n)=\sum_{k=c}^{n}\binom{n}{k}(-1)^k f(k) \tag{8.27}$$

On the other hand, the reflection integral formula is shown below.
(Reflection integral formula)

$$\chi(s)=\oint_C -B(1-s,t)\phi(t)\frac{dt}{2\pi i} \tag{8.28}$$

We replace the variable *s* to *-n* and the variable *t* to *-t* in the above equation.

$$-\chi(n)=\oint_C -B(1+n,-t)\phi(-t)\frac{dt}{2\pi i} \tag{8.29}$$

We use the following function.

$$f(t)=\phi(-t) \tag{8.30}$$

Then, we obtain the following equation.

$$-\chi(-n)=\oint_C B(n+1,-t)f(t)\frac{dt}{2\pi i} \tag{8.31}$$

Therefore, we can obtain the following equation.

$$\sum_{k=c}^{n}\binom{n}{k}(-1)^k f(k)=\oint_C B(n+1,-t)f(t)\frac{dt}{2\pi i} \tag{8.32}$$

The above equation equals to Nörlund–Rice integral.

9 Conclusion

We obtained the following results in this paper.
- We derived reflection integral equation.
- We derived summation equation.
- We derived asymptotic expansion.
- We derived Faulhaber’s formula.
- We derived Nörlund–Rice integral.

10 Future issues

The future issues are shown below.

- To study the relation between the generating function of *Z*-transform and zeros.
- To study the eigenvalues of integral equation.

11 Appendix

11.1 Table of *Z*-transform

Table of *Z*-transform is shown below.

$f(s)$	$F(z)=Z[f(s)]$	Num.
$f(s)=\dfrac{\zeta(s)}{\Gamma(1-s)}$	$F(z)=\dfrac{e^z}{1-e^z}$	(11.1)
$f(s)=\dfrac{\eta(s)}{\Gamma(1-s)}$	$F(z)=\dfrac{e^z}{1+e^z}$	(11.2)
$f(s,q)=\dfrac{\zeta(s,q)}{\Gamma(1-s)}$	$F(z,q)=\dfrac{e^{qz}}{1-e^z}$	(11.3)
$f(s,q)=\dfrac{\eta(s,q)}{\Gamma(1-s)}$	$F(z,q)=\dfrac{e^{qz}}{1+e^z}$	(11.4)
$f(s,q,w)=\dfrac{\Phi(w,s,q)}{\Gamma(1-s)}$	$F(z,q,w)=\dfrac{e^{qz}}{1-we^z}$	(11.5)
$f(s,q,\lambda)=\dfrac{L(\lambda,q,s)}{\Gamma(1-s)}$	$F(z,q,\lambda)=\dfrac{e^{qz}}{1-\exp(2\pi i\lambda)e^z}$	(11.6)

The definition of the functions is shown below.
(Definitional integral formula of Riemann zeta function)

$$\zeta(s)=\frac{1}{\Gamma(s)}\int_0^\infty x^{s-1}\frac{e^{-x}}{1-e^{-x}}dx \tag{11.7}$$

(Definitional integral formula of Dirichlet[27] eta)

$$\eta(s)=\frac{1}{\Gamma(s)}\int_0^\infty x^{s-1}\frac{e^{-x}}{1+e^{-x}}dx \tag{11.8}$$

(Definitional integral formula of Hurwitz zeta function)

$$\zeta(s,q)=\frac{1}{\Gamma(s)}\int_0^\infty x^{s-1}\frac{e^{-qx}}{1-e^{-x}}dx \tag{11.9}$$

(Definitional integral formula of Hurwitz eta function)

$$\eta(s,q)=\frac{1}{\Gamma(s)}\int_0^\infty x^{s-1}\frac{e^{-qx}}{1+e^{-x}}dx \tag{11.10}$$

(Definitional integral formula of Lerch transcendent[28])

$$\Phi(w,s,q)=\frac{1}{\Gamma(s)}\int_0^\infty x^{s-1}\frac{e^{-qx}}{1-we^{-x}}dx \tag{11.11}$$

(Definitional integral formula of Lerch zeta function)

$$L(\lambda,q,s)=\frac{1}{\Gamma(s)}\int_0^\infty x^{s-1}\frac{e^{-qx}}{1-\exp(2\pi i\lambda)e^{-x}}dx \tag{11.12}$$

The formulas of the polynomials are shown below.

$$\zeta(1-n)=-\frac{B_n(1)}{n} \tag{11.13}$$

$$\eta(-n)=\frac{E_n(1)}{2} \tag{11.14}$$

$$\zeta(1-n,q)=-\frac{B_n(q)}{n} \tag{11.15}$$

$$\eta(-n,q)=\frac{E_n(q)}{2} \tag{11.16}$$

$$\Phi(w,1-n,q)=-\frac{\beta_n(q,w)}{n} \tag{11.17}$$

The definitions of the polynomials are shown below.
(Bernoulli polynomials)

$$\frac{xe^{qx}}{e^x-1}=\sum_{n=0}^{\infty}\frac{B_n(q)}{n!}x^n \tag{11.18}$$

(Euler polynomials)

$$\frac{2e^{qx}}{e^x+1}=\sum_{n=0}^{\infty}\frac{E_n(q)}{n!}x^n \tag{11.19}$$

(Apostol-Bernoulli polynomials [29])

$$\frac{xe^{qx}}{we^x-1}=\sum_{n=0}^{\infty}\frac{\beta_n(q,w)}{n!}x^n \tag{11.20}$$

11.2 Derivation of the reflection integral equation from the reflection integral formula

The reflection integral formula is shown below.
(Reflection integral formula)

$$\chi(s)=\oint_C -B(1-s,t)\phi(t)\frac{dt}{2\pi i} \tag{11.21}$$

We add the following condition.

$$H(-z)=G(z) \tag{11.22}$$

Then we can obtain the following relation.

$$\phi(s)=\chi(s) \tag{11.23}$$

In other words, the following two equations are equivalent.

$$G(-z)=-e^z G(z) \tag{11.24}$$

$$\phi(1-s)=\oint_C -B(s,t)\phi(t)\frac{dt}{2\pi i} \tag{11.25}$$

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