A GENERALIZATION OF A THEOREM OF CARNOT

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Theorem of Carnot: Let $M$ be a point on the diagonal $AC$ of an arbitrary quadrilateral $ABCD$. Through $M$ one draws a line which intersects $AB$ in $\alpha$ and $BC$ in $\beta$. Let us draw another line, which intersects $CD$ in $\gamma$ and $AD$ in $\delta$. Then one has:

$$\frac{A\alpha}{B\beta} \cdot \frac{B\beta}{C\gamma} \cdot \frac{C\gamma}{D\delta} \cdot \frac{D\delta}{A\delta} = 1.$$ 

Generalization: Let $A_1...A_n$ be a polygon. On a diagonal $A_iA_k$ of this polygon one takes a point $M$ through which one draws a line $d_1$ which intersects the lines $A_1A_2, A_2A_3, ..., A_{k-1}A_k$ respectively in the points $P_1, P_2, ..., P_{k-1}$ and another line $d_2$ intersects the other lines $A_kA_{k+1}, ..., A_{n-1}A_n, A_nA_1$ respectively in the points $P_{k+1}, ..., P_n$. Then one has:

$$\prod_{i=1}^{n} \frac{A_iP_i}{A_{\phi(i)}P_i} = 1,$$

where $\phi$ is the circular permutation

$$\begin{pmatrix}
1 & 2 & \ldots & n-1 & n \\
2 & 3 & \ldots & n & 1
\end{pmatrix}.$$

Proof:

Let us have $1 \leq j \leq k-1$. One easily shows that:

$$\frac{A_iP_j}{A_{j+1}P_j} = \frac{D(A_i,d_j)}{D(A_{j+1},d_j)}$$

where $D(A,d)$ represents the distance from the point $A$ to the line $d$, since the triangles $P_jA_jA_j'$ and $P_{j+1}A_{j+1}A_{j+1}'$ are similar. (One notes with $A_j'$ and $A_{j+1}'$ the projections of the points $A_j$ and $A_{j+1}$ on the line $d_j$).

It results from it that:

$$\frac{A_1P_1}{A_2P_1} \cdot \frac{A_2P_2}{A_3P_2} \cdots \frac{A_{k-1}P_{k-1}}{A_kP_{k-1}} = \frac{D(A_1,d_1)}{D(A_2,d_1)} \cdot \frac{D(A_2,d_1)}{D(A_3,d_1)} \cdots \frac{D(A_{k-1},d_1)}{D(A_k,d_1)} = \frac{D(A_1,d_1)}{D(A_k,d_1)}$$

In a similar way, for $k \leq h \leq n$ one has:
\[
\frac{A_h P_h}{A_{\varphi(h)} P_h} = \frac{D(A_h, d_2)}{D(A_{\varphi(h)}, d_2)}
\]

and

\[
\prod_{h=1}^{n} \frac{A_h P_h}{A_{\varphi(h)} P_h} = \frac{D(A_1, d_2)}{D(A_1, d_2)},
\]

The product of the theorem is equal to:

\[
\frac{D(A_1, d_1)}{D(A_k, d_1)} \cdot \frac{D(A_k, d_2)}{D(A_1, d_2)},
\]

but

\[
\frac{D(A_1, d_1)}{D(A_k, d_1)} = \frac{A_1 M}{A_k M},
\]

since the triangles \(MA_1 A'_1\) and \(MA_k A'_k\) are similar. In the same way, because the triangles \(MA_1 A'_1\) and \(MA_k A'_k\) are similar (one notes with \(A'_i\) and \(A'_k\) the respective projections of \(A_i\) and \(A_k\) on the line \(d_2\)), one has:

\[
\frac{D(A_k, d_2)}{D(A_1, d_2)} = \frac{A_k M}{A_1 M}.
\]

The product from the statement is therefore equal to 1.

Remark: If one replaces \(n\) by 4 in this theorem, one finds the theorem of Carnot.