

# A GENERALIZATION IN SPACE OF JUNG'S THEOREM

Florentin Smarandache  
University of New Mexico  
200 College Road  
Gallup, NM 87301, USA  
E-mail: [smarand@unm.edu](mailto:smarand@unm.edu)

In this short note we will prove a generalization of Jung's theorem in space.

**Theorem.** Let us have  $n$  points in space such that the maximum distance between any two points is  $a$ . Prove that there exists a sphere of radius  $r \leq a \frac{\sqrt{6}}{4}$  that contains in its interior or on its surface all these points.

*Proof:*

Let  $P_1, \dots, P_n$  be the points. Let  $S_1(O_1, r_1)$  be a sphere of center  $O_1$  and radius  $r_1$ , which contains all these points. We note  $r_2 = \max_{1 \leq i \leq n} P_i O_1 = P_1 O_1$  and construct the sphere  $S_2(O_1, r_2)$ ,  $r_2 \leq r_1$ , with  $P_1 \in Fr(S_2)$ , where  $Fr(S_2) =$  frontier (surface) of  $S_2$ .

We apply a homothety  $H$  in space, of center  $P_1$ , such that the new sphere  $H(S_2) = S_3(O_3, r_3)$  has the property:  $Fr(S_3)$  contains another point, for example  $P_2$ , and of course  $S_3$  contains all points  $P_i$ .

1) If  $P_1, P_2$  are diametrically opposite in  $S_3$  then  $r_{\min} = \frac{a}{2}$ .

If no, we do a rotation  $R$  so that  $R(S_3) = S_4(O_4, r_4)$  for which  $\{P_3, P_2, P_1\} \subset Fr(S_4)$  and  $S_4$  contains all points  $P_i$ .

2) If  $\{P_1, P_2, P_3\}$  belong to a great circle of  $S_4$  and they are not included in an open semicircle, then  $r_{\min} \leq \frac{a}{\sqrt{3}}$  (**Jung's theorem**).

If no, we consider the fascicle of spheres  $S$  for which  $\{P_1, P_2, P_3\} \subset Fr(S)$  and  $S$  contains all points  $P_i$ . We choose a sphere  $S_5$  such that  $\{P_1, P_2, P_3, P_4\} \subset Fr(S_5)$ .

3) If  $\{P_1, P_2, P_3, P_4\}$  are not included in an open semisphere of  $S_5$ , then the tetrahedron  $\{P_1, P_2, P_3, P_4\}$  can be included in a regulated tetrahedron of side  $a$ , whence we find that the radius of  $S_5$  is  $\leq a \frac{\sqrt{6}}{4}$ .

If no, let's note  $\max_{1 \leq i < j \leq 4} P_i P_j = P_1 P_4$ . Does the sphere  $S_6$  of diameter  $P_1 P_4$  contain all points  $P_i$ ?

If yes, stop (we are in the case 1).

If no, we consider the fascicle of spheres  $S'$  such that  $\{P_1, P_4\} \subset Fr(S')$  and  $S'$  contains all the points  $P_i$ . We choose another sphere  $S_7$ , for which  $P_5 \notin \{P_1, P_2, P_3, P_4\}$  and  $P_5 \in Fr(S_7)$ .

With these new notations (the points  $P_1, P_4, P_5$  and the sphere  $S_7$ ) we return to the case 2.

This algorithm is finite; therefore it constructs the required sphere.

[Published in "GAZETA MATEMATICA", Nr. 9-10-11-12, 1992, Bucharest, Romania, p.352]